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Parabolic Equations with Measure Data and Three Unbounded Nonlinearities in Weighted Sobolev Spaces

Y. Akdim [1], J. Bennouna [2], M. Mekkour [1*] and H. Redwane [3]

[1] *Sidi Mohamed Ben Abdellah University, Poly-disciplinary Faculty of Taza, Laboratory LSI, Department MPI, P.O. Box 1223 Taza Gare, Morocco*
[2] *Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Laboratory LAMA, Department of Mathematics, B.P. 1796, Atlas Fez, Morocco.*
[3] *Faculté des Sciences Juridiques, Economiques et Sociales. Université Hassan 1, B.P. 784, Settat. Morocco*

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Abstract: In this work, we study the degenerated problem

$$\begin{gathered}\frac{\partial b(x,u)}{\partial t} + \operatorname{div}(a(x,t,u,Du)) + H(x,t,u,Du) = \mu \quad \text{in } Q,\\ u = 0 \quad \text{on } \partial\Omega \times (0,T),\\ b(x,u)(t=0) = b(x,u_0) \quad \text{on } \Omega,\end{gathered} \tag{1}$$

in the framework of weighted Sobolev space. The main contribution of our work is to prove the existence of a renormalized solution without the sign condition and the coercivity condition on $H(x,t,u,Du)$. The critical growth condition on H is with respect to Du and no growth with respect to u. The datum μ is assumed in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and $b(x,u_0) \in L^1(\Omega)$.

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* Corresponding author: mailto:mekkour.mounir@yahoo.fr

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$, $Q = \Omega \times [0,T]$ and $w = \{w_i(x) : 0 \leq i \leq N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable almost everywhere strictly positive function on Ω), satisfying some integrability conditions (see Section 2). Let $Au = -\operatorname{div}(a(x,t,u,Du))$ be a Leray-Lions operator defined from the weighted Sobolev space $L^p(0,T;W_0^{1,p}(\Omega,w))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$.

Now, we consider the degenerated parabolic problem associated with the differential equation

$$\begin{gathered} \frac{\partial b(x,u)}{\partial t} + Au + H(x,t,u,Du) = \mu \quad \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega \times]0,T[, \\ b(x,u)(t=0) = b(x,u_0) \quad \text{on } \Omega. \end{gathered} \tag{2}$$

In problem (2), the data μ and $b(x,u_0)$ are in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,Du))$ is a Leray-Lions operator which is coercive, $b(x,u)$ is unbounded function on u, H is a nonlinear lower order term and $\mu = f - \operatorname{div} F$ with $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$.

Problem (2) is studied in [2] with $\mu \in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ and under the strong hypothesis relatively to H, more precisely they supposed that $b(x,u) = u$ and the non-linearity H satisfying the sign condition

$$H(x,t,s,\xi)s \geq 0, \tag{3}$$

and the growth condition of the form

$$|H(x,t,s,\xi)| \leq b(s)\Big(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x,t)\Big). \tag{4}$$

In the case where the second member $f \in L^1(Q)$, (2) is studied in [2].

It is our purpose to prove the existence of renormalized solution for (2) in the setting of the weighted Sobolev space without the sign condition (3), and without the following coercivity condition

$$|H(x,t,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i(x)|\xi_i|^p \quad \text{for } |s| \geq \gamma, \tag{5}$$

our growth condition on H is simpler than (4) it is a growth with respect to Du and no growth condition with respect to u (see assumption (H3) below), the second term μ belongs to $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. Note that our paper generalizes [2].

In the case of $H(x,t,u,Du) = div(\phi(u))$ is studied by H. Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz spaces see [18,20].

The notion of renormalized solution was introduced by DiPerna and Lions [11] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (2) by Boccardo et al [7] when the right hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [18] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso, Murat, Orsina and Prignet [10] for the case of the right hand side being general measure data. Our paper can be considered as a continuation of [3–5] in the case where $F = 0$.

2 Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $2 < p < \infty$ and $w = \{w_i(x),\ 0 \le i \le N\}$ be a vector of weight functions; i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that, there exists

$$r_0 > \max(N,p) \quad \text{such that } w_i^{\frac{-r_0}{r_0-p}} \in L^1_{\mathrm{loc}}(\Omega), \tag{6}$$

$$w_i \in L^1_{\mathrm{loc}}(\Omega), \tag{7}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega), \tag{8}$$

for any $0 \le i \le N$. We denote by $W^{1,p}(\Omega,w)$ the space of real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \quad \text{for } i = 1,\dots,N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{9}$$

Condition (7) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $V = W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (9). Moreover, condition (8) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0,\dots,N\}$ and where p' is the conjugate of p; i.e., $p' = \frac{p}{p-1}$, (see [13]).

3 Basic Assumptions

Assumption (H1)

For $2 \le p < \infty$, we assume that the expression

$$|||u|||_V = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{10}$$

is a norm defined on V which is equivalent to the norm (9), and there exists a weight function σ on Ω such that, $\sigma \in L^1(\Omega)$ and $\sigma^{-1} \in L^1(\Omega)$. We assume also the Hardy inequality

$$\Big(\int_\Omega |u(x)|^p \sigma\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{11}$$

holds for every $u \in V$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$W^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega,\sigma), \tag{12}$$

expressed by the inequality (11) is compact. Notice that $(V, |||\cdot|||_V)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1 If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, +\infty[\cap[\frac{1}{p-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \quad \text{and} \quad w_i^{\frac{N}{N-1}} \in L^1_{\text{loc}}(\Omega) \quad \text{for all } i = 1, \ldots, N. \tag{13}$$

Notice that the assumptions (7) and (13) imply

$$|||u||| = \Big(\sum_{i=1}^{N} \int_{\Omega} |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\Big)^{1/p}, \tag{14}$$

which is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (9) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega) \tag{15}$$

is compact for all $1 \leq q \leq p_1^*$ if $p\nu < N(\nu+1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu+1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of p_1; see [12, pp. 30-31].

Assumption (H2)

$$b : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function} \tag{16}$$

such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1-function with $b(x, 0) = 0$. Next, for any $k > 0$, there exists $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \Big|D_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \leq B_k(x) \tag{17}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq k$, we denote by $D_x\big(\frac{\partial b(x,s)}{\partial s}\big)$ the gradient of $\frac{\partial b(x,s)}{\partial s}$ defined in the sense of distributions. For $i = 1, \ldots, N$,

$$|a_i(x,t,s,\xi)| \leq \beta w_i^{1/p}(x)[k(x,t) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}], \tag{18}$$

for a.e. $(x,t) \in Q$,all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, some function $k(x,t) \in L^{p'}(Q)$ and $\beta > 0$, here σ and q are as in (H1).

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi - \eta) > 0 \quad \text{for all } \xi \neq \eta, \tag{19}$$

$$a(x,t,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{20}$$

where α is a strictly positive constant.

Assumption (H3)

Furthermore, let $H(x,t,s,\xi) : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq \gamma(x,t) + g(s) \sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{21}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x,t)$ belongs to $L^1(Q)$.

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

4 Some Technical Results

Characterization of the time mollification of a function u.

In order to deal with time derivative, we introduce a time mollification of a function u belonging to a some weighted Lebesgue space. Thus we define for all $\mu \ge 0$ and all $(x,t) \in Q$,

$$u_\mu = \mu \int_\infty^t \tilde{u}(x,s)\exp(\mu(s-t))ds \quad where \quad \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s).$$

Proposition 4.1 *[2]*
1) if $u \in L^p(Q,w_i)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and

$$\|u_\mu\|_{L^p(Q,w_i)} \le \|u\|_{L^p(Q,w_i)}.$$

2) If $u \in W_0^{1,p}(Q,w)$, then $u_\mu \to u$ in $W_0^{1,p}(Q,w)$ as $\mu \to \infty$.
3) If $u_n \to u$ in $W_0^{1,p}(Q,w)$, then $(u_n)_\mu \to u_\mu$ in $W_0^{1,p}(Q,w)$.

Some weighted embedding and compactness results.

In this section we establish some embedding and compactness results in weighted Sobolev spaces, some trace results, Aubin's and Simon's results [21].

Let $V = W_0^{1,\,p}(\Omega,w)$, $H = L^2(\Omega,\sigma)$ and let $V^* = W^{-1,p'}$ with $(2 \le p < \infty)$.

Let $X = L^p(0,T;W_0^{1,\,p}(\Omega,w))$. The dual space of X is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : \ v' \in X^*\}$ endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

which is a Banach space. Here u' stands for the generalized derivative of u, i.e.,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad for \ all \quad \varphi \in C_0^\infty(0,T).$$

Lemma 4.1 *[19]*
1)The evolution triple $V \subseteq H \subseteq V^*$ is verified.
2) The imbedding $W_p^1(0,T,V,H) \subseteq C(0,T,H)$ is continuous.
3)The imbedding $W_p^1(0,T,V,H) \subseteq L^p(Q,\sigma)$ is compact.

Lemma 4.2 *[2] Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \le C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$*

Lemma 4.3 *[2]. Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad in \quad D'(Q),$$

where α_n and β_n are bounded respectively in X^* and in $L^1(Q)$. If v_n is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$, then $v_n \to v$ in $L^p_{loc}(Q,\sigma)$. Further $v_n \to v$ strongly in $L^1(Q)$.

Definition 4.1 Let $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ and $b(x,u_0) \in L^1(\Omega)$. A real-valued function u defined on Q is a renormalized solution of problem (2) if

$$T_k(u) \in L^p(0,T;W_0^{1,\,p}(\Omega,w)) \text{ for all } k \geq 0 \ \ and \ \ b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{22}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt \to 0 \quad as \quad m \to \ +\infty, \tag{23}$$

$$\begin{aligned} &\frac{\partial B_S(x,u)}{\partial t} - div\,(S'(u)a(x,t,u,Du)) + S''(u)a(x,t,u,Du)Du \\ &+ H(x,t,u,Du)S'(u) = fS'(u) - div\,(S'(u)F) + S''(u)FDu \quad in \quad D'(Q), \end{aligned} \tag{24}$$

for all functions $S \in W^{2,\,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr$ and

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad in \quad \Omega. \tag{25}$$

Remark 4.1 Equation (24) is formally obtained through pointwise multiplication of equation (2) by $S'(u)$. However, while $a(x,t,u,Du)$ and $H(x,t,u,Du)$ do not in general make sense in (2), all the terms in (2) have a meaning in $D'(Q)$. Indeed, if M is such that $suppS' \subset [-M,M]$, the following identifications are made in (24):

- $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.
- $S'(u)a(x,t,u,Du)$ identifies with $S'(u)a(x,t,T_M(u),DT_M(u))$ a.e in Q.

Since $|T_M(u)| \leq M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (18) and (22) that

$$S'(u)a(x,t,T_M(u),DT_M(u)) \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*).$$

- $S''(u)a(x,t,u,Du)Du$ identifies with $S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u)$ and

$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \in L^1(Q).$$

- $S'(u)H(x,t,u,Du)$ identifies with $S'(u)H(x,t,T_M(u),DT_M(u))$ a.e in Q. Since $|T_M(u)| \leq M$ a.e in Q and $S'(u) \in L^\infty(Q)$, we obtain from (18) and (21) that

$$S'(u)H(x,t,T_M(u),DT_M(u)) \in L^1(Q).$$

- $S'(u)f$ belongs to $L^1(Q)$ while $S'(u)F$ belongs to $\prod_{i=1}^{N} L^{p'}(Q,w_i^*)$.
- $S''(u)FDu$ identifies with $S''(u)FDT_k(u)$ which belongs to $L^1(Q)$.

The above considerations show that equation (24) holds in $D'(Q)$ and that

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w^*)) + L^1(Q).$$

Due to the properties of S and (24), $\frac{\partial S(u)}{\partial t} \in L^{p'}(0,T;W^{-1,\,p'}(\Omega,w^*)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (25) makes sense, since, due to the properties of S (increasing) and (17), we have

$$|B_S(x,r) - B_S(x,r')| \le A_k(x)\,|S(r) - S(r')| \quad for \quad all \quad r,r' \in \mathbb{R}. \tag{26}$$

5 Existence Results

In this section we establish the following existence theorem.

Theorem 5.1 *Let $f \in L^1(Q)$, $F \in \prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ and u_0 is a measurable function such that $b(x,u_0) \in L^1(\Omega)$. Assume that (H1) and (H2) hold true. Then, there exists at least a renormalized solution u of the problem (2) in the sense of Definition 4.1.*

Proof. **Step 1: Approximate problem and a priori estimates.**
For $n > 0$, let us define the following approximation of b, H, f and u_0;

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r \quad \text{for } n > 0. \tag{27}$$

In view of (27), b_n is a Carathéodory function and satisfies (17), there exist $\lambda_n > 0$ and functions $A_n \in L^1(\Omega)$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \quad \text{and} \quad \left|D_x\Big(\frac{\partial b_n(x,s)}{\partial s}\Big)\right| \le B_n(x)$$

a.e. in Ω, $s \in \mathbb{R}$.

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1 + \frac{1}{n}|H(x,t,s,\xi)|}\chi_{\Omega_n}.$$

Note that Ω_n is a sequence of compacts covering the bounded open set Ω and χ_{Ω_n} is its characteristic function.

$$f_n \in L^{p'}(Q), \quad \text{and} \quad f_n \to f \quad \text{a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to +\infty, \tag{28}$$

$$u_{0n} \in D(\Omega), \quad \|b_n(x,u_{0n})\|_{L^1} \le \|b(x,u_0)\|_{L^1}, \tag{29}$$

$$b_n(x,u_{0n}) \to b(x,u_0) \quad \text{a.e. in } \Omega \text{ and strongly in } L^1(\Omega). \tag{30}$$

Let us now consider the approximate problem:

$$\begin{gathered} \frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,Du_n)) + H_n(x,t,u_n,Du_n) = f_n - div(F) \quad \text{in } D'(Q), \\ u_n = 0 \quad \text{in } (0,T)\times\partial\Omega, \\ b_n(x,u_n(t=0)) = b_n(x,u_{0n}). \end{gathered} \tag{31}$$

Note that $H_n(x,t,s,\xi)$ satisfies the following conditions

$$|H_n(x,t,s,\xi)| \le H(x,t,s,\xi) \quad \text{and} \quad |H_n(x,t,s,\xi)| \le n.$$

For all $u, v \in L^p(0,T;W_0^{1,p}(\Omega,w))$,

$$|\int_Q H_n(x,t,u,Du)v\,dx\,dt| \le \Big(\int_Q |H_n(x,t,u,Du)|^{q'}\sigma^{-\frac{q'}{q}}\,dx\,dt\Big)^{1/q'}\Big(\int_Q |v|^q\sigma\,dx\,dt\Big)^{1/q}$$

$$\le n\int_0^T\Big(\int_{\Omega_n}\sigma^{1-q'}dx\Big)^{1/q'}dt\,\|v\|_{L^q(Q,\sigma)} \le C_n\,\|v\|_{L^p(0,T;W_0^{1,p}(\Omega,w))}\,.$$

Moreover, since $f_n \in L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega,w))$ of (31) is an easy task (see e.g. [15], [2]).

Let $\varphi \in L^p(0,T;W_0^{1,\ p}(\Omega,w)) \cap L^\infty(Q)$ with $\varphi > 0$, choosing $v = \exp(G(u_n))\varphi$ as test function in (31) where $G(s) = \int_0^s \frac{g(r)}{\alpha}dr$ (the function g appears in (21)), we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)D(\exp(G(u_n))\varphi)dxdt$$

$$+\int_Q H_n(x,t,u_n,Du_n)\exp(G(u_n))\varphi dxdt = \int_Q f_n\exp(G(u_n))\varphi dxdt$$

$$+\int_Q FD(\exp(G(u_n))\varphi)dxdt.$$

In view of (21) and (20) we obtain

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi dxdt$$

$$\le \int_Q \gamma(x,t)\exp(G(u_n))\varphi dxdt + \int_Q f_n\exp(G(u_n))\varphi dxdt$$

$$+\int_Q F\exp(G(u_n))D\varphi dxdt + \int_Q FD(\exp(G(u_n)))\varphi dxdt, \tag{32}$$

for all $\varphi \in L^p(0,T;W_0^{1,\ p}(\Omega,w)) \cap L^\infty(Q)$ with $\varphi > 0$.

On the other hand, taking $v = \exp(-G(u_n))\varphi$ as test function in (31) we deduce as in (32) that,

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))\varphi dxdt + \int_Q a(x,t,u_n,Du_n)\exp(-G(u_n))D\varphi dxdt$$

$$+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi dxdt \ge \int_Q f_n\exp(-G(u_n))\varphi dxdt$$

$$+\int_Q F\exp(-G(u_n))D\varphi dxdt + \int_Q FD(\exp(-G(u_n)))\varphi dxdt, \tag{33}$$

for all $\varphi \in L^p(0,T;W_0^{1,\ p}(\Omega,w)) \cap L^\infty(Q)$ with $\varphi > 0$.

For every $\tau \in]0,T[$, let $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$ in (32) we have

$$\int_\Omega B^n_{k,G}(x,u_n(\tau))dx + \int_{Q_\tau} a(x,t,u_n,Du_n)\exp(G(u_n))DT_k(u_n)^+dxdt$$

$$\leq \int_{Q_\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+ dxdt + \int_{Q_\tau} f_n \exp(G(u_n))T_k(u_n)^+ dxdt$$
$$+ \int_Q FD(T_k(u_n)^+)\exp(G(u_n))dxdt + \int_Q FT_k(u_n)^+ \exp(G(u_n))Du_n \frac{g(u_n)}{\alpha} dxdt \quad (34)$$
$$+ \int_\Omega B^n_{k,G}(x,u_{0n})dx,$$

where $B^n_{k,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} T_k(s)^+ \exp(G(s))ds$. Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$ we have

$$0 \leq \int_\Omega B^n_{k,G}(x,u_{0n})dx \leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\|b(x,u_0)\|_{L^1(\Omega)}. \quad (35)$$

Using (35), $B^n_{k,G}(x,u_n) \geq 0$, Young's inequality and (20) we obtain

$$\alpha\Big(\frac{p-1}{p}\Big)\int_{Q_\tau}\sum_{i=1}^N \left|\frac{\partial T_k(u_n)^+}{\partial x_i}\right|^p w_i \exp(G(u_n))dxdt \quad (36)$$
$$\leq k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big(\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + c\,\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)} + \|b_n(x,u_{0n})\|_{L^1(\Omega)}\Big)$$
$$+\frac{1}{\alpha}\int_{Q_\tau} Fg(u_n)\exp(G(u_n))Du_n\chi_{\{u_n>0\}}dxdt.$$

Let us observe that, if we take $\varphi = \rho(u_n) = \int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (32) and using (20) we obtain

$$\left[\int_\Omega B^n_g(x,u_n)dx\right]_0^T + \alpha\int_Q\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt$$
$$\leq \left(\int_0^\infty g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\Big)$$
$$+\int_Q FDu_n g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))dxdt$$
$$+\left(\int_0^\infty g(s)ds\right)\int_Q\Big|FDu_n\Big|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}dxdt,$$

where $B^n_g(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds$, which implies, since $B^n_g(x,r) \geq 0$ and Young's inequality,

$$\alpha\int_{\{u_n>0\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\exp(G(u_n))dxdt$$
$$\leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\Big(\|\gamma\|_{L^1(Q)} + \|f\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}\Big)$$

$$+C_1 \|g\|_\infty \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_Q \sum_{i=1}^N |F_i|^{p'} w_i^* \, dx \, dt$$

$$+\frac{\alpha}{2p} \int_Q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \frac{g(u_n)}{\alpha} \exp(G(u_n))\chi_{\{u_n>0\}} dxdt$$

$$+C_2 \int_0^\infty g(s)\, ds \, \|g\|_\infty \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_Q |F|^{p'} w^* \, dx \, dt$$

$$+\frac{\alpha}{2p} \int_Q \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \frac{g(u_n)}{\alpha} \exp(G(u_n))\chi_{\{u_n>0\}} dxdt$$

we obtain

$$\int_{\{u_n>0\}} g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \le C_3.$$

Similarly, let $\varphi = \displaystyle\int_{u_n}^0 g(s)\chi_{\{s<0\}} ds$ as a test function in (33), we conclude that

$$\int_{\{u_n<0\}} g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \le C_4.$$

Consequently,

$$\int_Q g(u_n) \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp(G(u_n)) dxdt \le C_5. \tag{37}$$

where $C_1, \cdots, C_5$ are constants independent of n. We deduce that

$$\int_Q \sum_{i=1}^N \left|\frac{\partial T_k(u_n)^+}{\partial x_i}\right|^p w_i dxdt \le C_6 \ k. \tag{38}$$

Similarly to (38) we take $\varphi = T_k(u_n)^- \chi_{(0,\tau)}$ in (33) we deduce that

$$\int_Q \sum_{i=1}^N \left|\frac{\partial T_k(u_n)^-}{\partial x_i}\right|^p w_i dxdt \le C_7 \ k. \tag{39}$$

Combining (38) and (39) we conclude that

$$\|T_k(u_n)\|^p_{L^p(0,T;W_0^{1,\ p}(\Omega,w))} \le C_8 \ k, \tag{40}$$

where $C_6, \ C_7, \ C_8$ are constants independent of n.

Then, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and $T_k(u_n)$ converges to v_k weakly in $L^p(0,T;W_0^{1,p}(\Omega,w))$, and by the compact imbedding (15) gives

$$T_k(u_n) \to v_k \quad \text{strongly in } L^p(Q,\sigma) \text{ and a.e. in } Q.$$

We deduce from the above inequalities (34), (35) and (40) that

$$\int_\Omega B^n_{k,G}(x, u_n(\tau)) dx \le C \ k. \tag{41}$$

Let $k>0$ be large enough and B_R be a ball of Ω, we have

$$\begin{aligned}
&k\,\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T])\\
&=\int_0^T\int_{\{|u_n|>k\}\cap B_R}|T_k(u_n)|\,dx\,dt\\
&\le\int_0^T\int_{B_R}|T_k(u_n)|\,dx\,dt\\
&\le\Big(\int_Q|T_k(u_n)|^p\sigma\,dx\,dt\Big)^{1/p}\Big(\int_0^T\int_{B_R}\sigma^{1-p'}\,dx\,dt\Big)^{1/p'}\\
&\le Tc_R\Big(\int_Q\sum_{i=1}^N w_i(x)\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^p\,dx\,dt\Big)^{1/p}\\
&\le ck^{1/p},
\end{aligned}$$

which implies

$$\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T])\le\frac{c_1}{k^{1-\frac{1}{p}}},\quad\forall k\ge1.$$

So, we have

$$\lim_{k\to+\infty}(\text{meas}(\{|u_n|>k\}\cap B_R\times[0,T]))=0.$$

Now we turn to prove the almost everywhere convergence of u_n and $b_n(x,u_n)$. Consider now a function non decreasing $g_k\in C^2(\mathbb{R})$ such that $g_k(s)=s$ for $|s|\le\frac{k}{2}$ and $g_k(s)=k$ for $|s|\ge k$. Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\begin{aligned}
&\frac{\partial B_k^n(x,u_n)}{\partial t}-div(a(x,t,u_n,Du_n)g_k'(u_n))+a(x,t,u_n,Du_n)g_k''(u_n)Du_n\\
&+H_n(x,t,u_n,Du_n)g_k'(u_n)=f_ng_k'(u_n)-div(Fg_k'(u_n))+Fg_k''(u_n)Du_n,
\end{aligned}\tag{42}$$

where $B_k^n(x,z)=\displaystyle\int_0^z\frac{\partial b_n(x,s)}{\partial s}g_k'(s)ds$.

As a consequence of (40), we deduce that $g_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,\,p}(\Omega,w))$ and $\frac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. Due to the properties of g_k and (17), we conclude that $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.

Hence Lemma 4.3 allows us to conclude that $g_k(u_n)$ is compact in $L^p_{loc}(Q,\sigma)$. Thus, for a subsequence, it also converges in measure and almost everywhere in Q (since we have, for every $\lambda>0$,)

$$\begin{aligned}
&meas(\{|u_n-u_m|>\lambda\}\cap B_R\times[0,T])\le meas(\{|u_n|>k\}\cap B_R\times[0,T])\\
&+meas(\{|u_m|>k\}\cap B_R\times[0,T])+meas(\{|g_k(u_n)-g_k(u_m)|>\lambda\}).
\end{aligned}$$

Let $\varepsilon>0$, then, there exist $k(\varepsilon)>0$ such that,

$$meas(\{|u_n-u_m|>\lambda\}\cap B_R\times[0,T])\le\varepsilon\quad for\quad all\quad n,m\ge n_0(k(\varepsilon),\lambda,R).$$

This proves that (u_n) is a Cauchy sequence in measure in $B_R\times[0,T])$, thus converges almost everywhere to some measurable function u. Then for a subsequence denoted again u_n, we have

$$u_n \to u \quad a.e \quad in \quad Q, \tag{43}$$

and from (40) we deduce

$$b_n(x, u_n) \to b(x, u) \quad a.e \quad in \quad Q, \tag{44}$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad weakly \quad in \quad L^p(0, T; W_0^{1,\,p}(\Omega, w)) \tag{45}$$

and then, the compact imbedding (12) gives,

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^q(Q, \sigma) \text{ and a.e in } Q.$$

Which implies, by using (18), for all $k > 0$ that there exists a function $\Lambda_k \in \prod_{i=1}^{N} L^{p'}(Q, w_i^*)$, such that

$$a(x, t, T_k(u_n), DT_k(u_n)) \rightharpoonup \Lambda_k \quad weakly \quad in \quad \prod_{i=1}^{N} L^{p'}(Q, w_i^*). \tag{46}$$

We now establish that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$. Using (43) and passing to the limit-inf in (41) as n tends to $+\infty$, we obtain that $\frac{1}{k}\int_\Omega B_{k,G}(x, u(\tau))dx \leq C$, for almost any τ in $(0, T)$. Due to the definition of $B_{k,G}(x, s)$ and the fact that $\frac{1}{k}B_{k,G}(x, u)$ converges pointwise to $\displaystyle\int_0^u sgn(s)\frac{\partial b(x, s)}{\partial s}\exp(G(s))ds \geq |b(x, u)|$, as k tends to $+\infty$, shows that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$.

Lemma 5.1 *Let u_n be a solution of the approximate problem (31). Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x, t, u_n, Du_n)Du_n dxdt = 0. \tag{47}$$

Proof. Considering the following function $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (32) this function is admissible since $\varphi \in L^p(0, T; W_0^{1,\,p}(\Omega, w))$ and $\varphi \geq 0$. Then by Young's inequality, we have

$$\int_\Omega B^m_{n,G}(x, u_n)(T)dx + \int_{\{m\leq u_n\leq m+1\}} a(x, t, u_n, Du_n)Du_n \exp(G(u_n))dxdt$$

$$\leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\int_{\{|u_n|>m\}}|f_n|\,dxdt + \int_{\{|u_n|>m\}}|\gamma|\,dxdt + \int_{\{|u_{n0}|>m\}}|b_n(x, u_{0n})|\,dx\right]$$

$$+C_1\int_{\{u_n\geq m\}}\sum_{i=1}^N|F_i|^{p'}w_i^*dxdt + \frac{\alpha}{p}\int_{\{m\leq u_n\leq m+1\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i\exp(G(u_n))dxdt$$

$$+C_2\int_{\{u_n\geq m\}}\sum_{i=1}^N|F_i|^{p'}w_i^*dxdt + C_3\int_{\{u_n\geq m\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\exp(G(u_n))dxdt,$$

where $B^m_{n,G}(x, r) = \displaystyle\int_0^r \frac{\partial b_n(x, s)}{\partial s}\exp(G(s))\alpha_m(s)ds.$

Using (20) and since $B^m_{n,G}(x,u_n)(T) > 0$, we obtain

$$\left(\frac{p-1}{p}\right)\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\exp(G(u_n))dxdt$$

$$\le \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left[\int_{\{|u_n|>m\}}(|f_n|+|\gamma|)dxdt+\int_{\{|u_{n0}|>m\}}|b_n(x,u_{0n})|\,dx\right]$$

$$+C_4\int_{\{u_n\ge m\}}\sum_{i=1}^N|F_i|^{p'}w_i^*dxdt+C_5\int_{\{u_n>m\}}g(u_n)\exp(G(u_n))\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_idxdt. \quad (48)$$

Take $\varphi=\rho_m(u_n)=\displaystyle\int_0^{u_n}g(s)\chi_{\{s>m\}}ds$ as test function in (32), we obtain

$$\left[\int_\Omega B^n_m(x,u_n)dx\right]_0^T+\int_Q a(x,t,u_n,Du_n)Du_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))dxdt$$

$$\le\left(\int_m^\infty g(s)\chi_{\{s>m\}}ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}\right)$$

$$+\int_Q FDu_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))dxdt$$

$$+\left(\int_m^\infty g(s)ds\right)\int_Q FDu_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>m\}}dxdt,$$

where $B^n_m(x,r)=\displaystyle\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))ds$, which implies, since $B^n_m(x,r)\ge 0$, (20) and Young's inequality,

$$\frac{\alpha(p-1)}{p}\int_{\{u_n>m\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_ig(u_n)\exp(G(u_n))dxdt \quad (49)$$

$$\le\left(\int_m^\infty g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$$

$$\cdot\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}+\|b_n(x,u_{0n})\|_{L^1(\Omega)}+C\,\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)}\right).$$

Using (49) and the strong convergence of f_n in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$, $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$ and $F\in\displaystyle\prod_{i=1}^N L^{p'}(Q,w_i^*)$, by Lebesgue's theorem, passing to the limit in (48), we conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}}a(x,t,u_n,Du_n)Du_ndxdt=0. \quad (50)$$

On the other hand, let $\varphi=T_1(u_n-T_m(u_n))^-$ as test function in (33) and reasoning as in the proof of (50) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}}a(x,t,u_n,Du_n)Du_ndxdt=0. \quad (51)$$

Thus (47) follows from (50) and (51).

Step 2: Almost everywhere convergence of the gradients.

This step is devoted to introduce for $k \geq 0$ a fixed time regularization of the function $T_k(u)$ in order to perform the monotonicity method. Let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad \left|w^i_\mu\right| \leq k, \tag{52}$$

$$w^i_\mu \to T_k(u) \ \text{ in } \ L^p(0,T;W_0^{1,\,p}(\Omega,w)), \ \text{as } \mu \to \infty. \tag{53}$$

We will introduce the following function of one real variable s, which is defined as:

$$h_m(s) = \begin{cases} 1, & \text{if } |s| \leq m, \\ 0, & \text{if } |s| \geq m+1, \\ m+1+|s|, & \text{if } m \leq |s| \leq m+1. \end{cases}$$

For $m > k$, let $\varphi = (T_k(u_n) - w^i_\mu)^+ h_m(u_n) \in L^p(0,T;W_0^{1,\,p}(\Omega,w)) \cap L^\infty(Q)$ and $\varphi \geq 0$, then taking this function in (32), we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&- \int_{\{m\leq |u_n| \leq m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w^i_\mu)^+ dxdt \\
&\quad \leq \int_Q (\gamma(x,t) + f_n)\exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)dxdt \\
&\quad + \int_Q FDu_n \frac{g(u_n)}{\alpha}\exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)dxdt \\
&\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \exp(G(u_n))FD(T_k(u_n) - w^i_\mu)h_m(u_n)dxdt \\
&\quad - \int_{\{m\leq |u_n| \leq m+1\}} \exp(G(u_n))FDu_n(T_k(u_n) - w^i_\mu)^+ dxdt.
\end{aligned} \tag{54}$$

Observe that

$$\left|\int_{\{m\leq |u_n| \leq m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n) - w^i_\mu)^+ dxdt\right|$$

$$\leq 2k\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{\{m\leq u_n \leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt,$$

and

$$\left|\int_{\{m\leq |u_n| \leq m+1\}} \exp(G(u_n))FDu_n(T_k(u_n) - w^i_\mu)^+ dxdt\right|$$

$$\leq 2k \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \frac{\|F\|_{\prod_{i=1}^{N} L^{p'}(Q,w_i^*)}}{\alpha^{\frac{1}{p}}} \left(\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt\right)^{\frac{1}{p}}.$$

Thanks to (47) the third integral and fourth integral of the right hand side tend to zero as n and m tend to infinity, and by Lebesgue's theorem and $F \in \prod_{i=1}^{N} L^{p'}(Q, w_i^*)$, we deduce that the right hand side converges to zero as n, m and μ tend to infinity. Since

$$(T_k(u_n) - w_\mu^i)^+ h_m(u_n) \rightharpoonup (T_k(u) - w_\mu^i)^+ h_m(u) \quad weakly* \quad in \quad L^\infty(Q), \quad as \quad n \to \infty$$

and strongly in $L^p(0,T; W_0^{1,\,p}(\Omega, w))$ and $(T_k(u) - w_\mu^i)^+ h_m(u) \rightharpoonup 0$ weakly* in $L^\infty(Q)$ and strongly in $L^p(0,T; W_0^{1,\,p}(\Omega, w))$ as $\mu \to \infty$. Let $\varepsilon_l(n, m, \mu, i): \quad l = 1, ...,$ are various functions tending to zero as n, m, i and μ tend to infinity.

The very definition of the sequence w_μ^i makes it possible to establish the following lemma.

Lemma 5.2 *For $k \geq 0$ we have*

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w_\mu^i) h_m(u_n) dxdt \geq \varepsilon(n, m, \mu, i). \quad (55)$$

Proof. (see [19]).

Similarly to [3, 4] for the second term of the left hand side of (54) we conclude

$$\lim_{n\to\infty} \int_Q [a(x,t,T_k(u_n), DT_k(u_n)) - a(x,t,T_k(u_n), DT_k(u))] \times [DT_k(u_n) - DT_k(u)]\, dxdt = 0. \quad (56)$$

Which implies that

$$T_k(u_n) \to T_k(u) \quad strongly \quad in \quad L^p(0,T; W_0^{1,\,p}(\Omega, w)) \quad \forall k. \quad (57)$$

Now, observe that we have, for every $\sigma > 0$

$$\begin{aligned} meas\Big\{(x,t) \in \Omega \times [0,T] : |Du_n - Du| > \sigma\Big\} &\leq meas\Big\{(x,t) \in \Omega \times [0,T] : |Du_n| > k\Big\} \\ &+ meas\Big\{(x,t) \in \Omega \times [0,T] : |u| > k\Big\} \\ &+ meas\Big\{(x,t) \in \Omega \times [0,T] : |DT_k(u_n) - DT_k(u)| > \sigma\Big\} \end{aligned}$$

then as a consequence of (57) we also have, that Du_n converges to Du in measure and therefore, always reasoning for subsequence,

$$Du_n \to Du \quad a.e \quad in \quad Q. \quad (58)$$

Which implies that

$$a(x,t,T_k(u_n), DT_k(u_n)) \rightharpoonup a(x,t,T_k(u), DT_k(u)) \quad in \quad \prod_{i=1}^{N} L^{p'}(Q, w_i^*). \quad (59)$$

Step 3: Equi-integrability of the nonlinearity sequence.
We shall now prove that $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ $a.e$ in Q, consider now $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as test function in (32), we obtain

$$\left[\int_\Omega B_h^n(x,u_n)dx\right]_0^T + \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$\leq \left(\int_h^\infty g(s)\chi_{\{s>h\}}ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}\right)$$
$$+ \int_Q FDu_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))dxdt$$
$$+ \left(\int_h^\infty g(s)\chi_{\{s>h\}}ds\right)\int_Q |FDu_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>h\}}dxdt,$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))ds$, which implies, since $B_h^n(x,r) \geq 0$, (20) and Young's inequality,

$$\frac{\alpha(p-1)}{p}\int_{\{u_n>h\}}\sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)\exp(G(u_n))dxdt$$
$$\leq \left(\int_h^\infty g(s)ds\right)\exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)$$
$$\left(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} + \|b_n(x,u_{0n})\|_{L^1(\Omega)} + C\|F\|^{p'}_{\prod_{i=1}^N L^{p'}(Q,w_i^*)}\right),$$

we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}}\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i g(u_n)dxdt = 0.$$

Consequently,

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt = 0,$$

which implies, for h large enough and for a subset E of Q,

$$\lim_{meas(E)\to 0}\int_E g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt \leq \|g\|_\infty \lim_{meas(E)\to 0}\int_E\sum_{i=1}^N\left|\frac{\partial T_h(u_n)^+}{\partial x_i}\right|^p w_i dxdt$$
$$+ \int_{\{|u_n|>h\}} g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i dxdt$$

then we deduce that $g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i$ is equi-integrale. Thus we have obtained that $g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i$ converge to $g(u)\sum_{i=1}^N\left|\frac{\partial u}{\partial x_i}\right|^p w_i$ strongly in $L^1(Q)$. Consequently, by using (21), we conclude that

$$H_n(x,t,u_n,Du_n) \to \ H(x,t,u,Du) \ \ strongly \ \ in \ \ L^1(Q). \tag{60}$$

Step 4: In this step we prove that u satisfies (23).
Observe that for any fixed $m \geq 0$ one has

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n = \int_Q a(x,t,u_n,Du_n)(DT_{m+1}(u_n) - DT_m(u_n))$$

$$= \int_Q a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n) - \int_Q a(x,t,T_m(u_n),DT_m(u_n))DT_m(u_n).$$

According to (59) and (57), one is at liberty to pass to the limit as $n \to \ +\infty$ for fixed $m \geq 0$ and to obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,Du_n)Du_n dxdt\\ &=\int_Q a(x,t,T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)dxdt - \int_Q a(x,t,T_m(u),DT_m(u))DT_m(u)dxdt\\ &=\int_{\{m\leq|u|\leq m+1\}} a(x,t,u,Du)Dudxdt.\end{aligned} \tag{61}$$

Taking the limit as $m \to +\infty$ in (61) and using the estimate (47) show that u satisfies (24).

Step 5: In this step we show that u satisfies (24) and (25). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that supp$(S') \subset [-M, M]$. Pointwise multiplication of the approximate equation (31) by $S'(u_n)$ leads to

$$\begin{aligned}&\frac{\partial B_S^n(x,u_n)}{\partial t} - div[S'(u_n)a(x,t,u_n,Du_n)] + S''(u_n)a(x,t,u_n,Du_n)Du_n\\ &+ S'(u_n)H_n(x,t,u_n,Du_n = fS'(u_n) - div(FS'(u)) + S''(u)FDu \ \ in \ \ D'(Q).\end{aligned} \tag{62}$$

In what follows we pass to the limit as in (62) n tends to $+\infty$.
• Limit of $\frac{\partial B_S^n(x,u_n)}{\partial t}$.
Since S is bounded and continuous, $u_n \to u \ \ a.e \ \ in \ \ Q$ implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e in Q and $L^\infty \ \ weak - *$. Then $\frac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\frac{\partial B_S(x,u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.
• Limit of $-div[S'(u_n)a_n(x,t,u_n,Du_n)]$.
Since supp$(S') \subset [-M, M]$, we have for $n \geq M$

$$S'(u_n)a_n(x,t,u_n,Du_n) = S'(u_n)a(x,t,T_M(u_n),DT_M(u_n)) \ \ a.e \ \ in \ \ Q.$$

The pointwise convergence of $u_n \ \ to \ \ u$ and (59) as n tends to $+\infty$ and the bounded character of S' permit us to conclude that

$$S'(u_n)a_n(x,t,u_n,Du_n) \rightharpoonup S'(u)a(x,t,T_M(u),DT_M(u)) \ \ in \ \ \prod_{i=1}^N L^{p'}(Q,w_i^*), \tag{63}$$

as n tends to $+\infty$. $S'(u)a(x,t,T_M(u),DT_M(u))$ has been denoted by $S'(u)a(x,t,u,Du)$ in equation (24).

• Limit of $S''(u_n)a(x,t,u_n,Du_n)Du_n$.
As far as the 'energy' term

$$S''(u_n)a(x,t,u_n,Du_n)Du_n = S''(u_n)a(x,t,T_M(u_n),DT_M(u_n))DT_M(u_n) \ \ a.e \ \ in \ \ Q.$$

The pointwise convergence of $S'(u_n)$ *to* $S'(u)$ and (59) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a_n(x,t,u_n,Du_n)Du_n \rightharpoonup S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \ \ weakly \ \ in \ \ L^1(Q). \tag{64}$$

Recall that $S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) = S''(u)a(x,t,u,Du)Du \ \ a.e \ \ in \ \ Q.$

• Limit of $S'(u_n)H_n(x,t,u_n,Du_n)$.
Since supp$(S') \subset [-M,M]$ and (60), we have

$$S'(u_n)H_n(x,t,u_n,Du_n) \to S'(u)H(x,t,u,Du) \ \ strongly \ \ in \ \ L^1(Q), \tag{65}$$

as n tends to $+\infty$.

• Limit of $S'(u_n)f_n$.
Since $u_n \to u$ a.e in Q, we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$ as $n \to +\infty$.

• Limit of $div(S'(u_n)F)$.
The fact that $S'(u_n)$ is bounded and converges to $S'(u)$ a.e in Q as n tends to $+\infty$ makes it possible to obtain that $div(S'(u_n)F) \to div(S'(u)F)$ strongly in $L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$ as $n \to +\infty$.

• Limit of $S''(u_n)FDu_n$.
This term is equal to $FDS'(u_n)$. Since $DS'(u_n)$ converges to $DS'(u_n)$ weakly in $\prod_{i=1}^{N} L^{p'}(Q,w_i^*)$ as n tends to $+\infty$, we obtain $S''(u_n)FDu_n = FDS'(u_n) \rightharpoonup FDS'(u)$ weakly in $L^1(Q)$ as $n \to +\infty$. The term $FDS'(u)$ identifies with $S''(u)FDu$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (62) and to conclude that u satisfies (24). It remains to show that $B_S(x,u)$ satisfies the initial condition (25). To this end, firstly remark that, S being bounded, $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, (62) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega,w^*))$. As a consequence, an Aubin's type lemma (see, e.g, [21]) implies that $B_S^n(x,u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_S^n(x,u_n)(t=0) = B_S^n(x,u_0^n)$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that $B_S(x,u)(t=0) = B_S(x,u_0)$ in Ω. As a conclusion of step 1 to step 5, the proof of Theorem 5.1 is complete.

6 Example

Let us consider the following special case: $b(x,s) = Z(x)C(s)$ where $Z \in W^{1,\,p}(\Omega, w)$, $Z(x) \geq \alpha > 0$ and $C \in C^1(\mathbb{R})$ such that $\forall \;\; k > 0 \;:\;\; 0 < \lambda_k \equiv \inf\limits_{|s|\leq k} C'(s)$ and $C(0) = 0$.

$$0 < \lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left| \nabla_x \left(\frac{\partial b(x,s)}{\partial s} \right) \right| \leq B_k(x) \quad \forall \; |s| \leq k, \tag{66}$$

$$H(x,t,s,\xi) = \frac{-2s}{1+s^4} \sum_{i=1}^{N} w_i(x) \, |\xi_i|^p \quad \text{and} \quad a_i(x,t,s,d) = w_i(x) \, |d_i|^{p-2} \, d_i, \quad i = 1, ..., N, \tag{67}$$

with $w_i(x)$ a weight function strictly positive. Then, we can consider the Hardy inequality in the form

$$\left(\int_\Omega |u(x)|^p \, \sigma(x) dx \right)^{\frac{1}{p}} \leq c \left(\int_\Omega |Du(x)|^p \, w(x) dx \right)^{\frac{1}{p}}.$$

It is easy to show that the $a_i(t,x,s,d)$ are Caratheodory functions satisfying the growth condition (18), the coercivity (20) and the monotonicity condition.

While the Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (21), indeed $|H(x,t,s,\xi)| \leq \frac{2|s|}{1+s^4} \sum\limits_{i=1}^{N} w_i(x) \, |\xi_i|^p = g(s) \sum\limits_{i=1}^{N} w_i(x) \, |\xi_i|^p$ where $g(s) = \frac{2|s|}{1+s^4}$ is a function bounded positive continuous which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition (3) and the coercivity condition. In particular, let us use special weight function, w, expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x) = dist(x, \partial\Omega)$ and set $w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x)$. Finally, the hypotheses of Theorem 5.1 are satisfied. Therefore, the following problem:

$$\begin{cases} b(x,u) \in L^\infty([0,T]; L^1(\Omega)) \quad \text{and} \quad T_k(u) \in L^p(0,T; W_0^{1,\,p}(\Omega, w)), \\ \lim\limits_{m \to +\infty} \displaystyle\int_{\{m \leq |u| \leq m+1\}} a(x,t,u,Du) Du dx dt = 0, \\ \dfrac{\partial B_S(x,u)}{\partial t} - div \left[S'(u) a(x,t,u,Du) \right] + S''(u) \displaystyle\sum_{i=1}^{N} w_i \left| \frac{\partial u}{\partial x_i} \right|^p, \\ -\dfrac{2u}{1+u^4} \displaystyle\sum_{i=1}^{N} w_i \left| \frac{\partial u}{\partial x_i} \right|^p S'(u) = f S'(u) - div(S'(u)F) + F S''(u) Du, \\ B_S(x,u)(t=0) = B_S(x,u_0) \;\; in \;\; \Omega, \\ \forall \;\; S \in W^{2,\infty}(\mathbb{R}) \;\; with \;\; S' \;\; has \;\; a \;\; compact \;\; support \;\; in \;\; \mathbb{R}, \\ and \; B_S(x,r) = \displaystyle\int_0^r \frac{\partial b(x,\sigma)}{\partial \sigma} S'(\sigma) d\sigma, \end{cases} \tag{68}$$

has at least one renormalised solution.

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