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Constrained Motion of Mechanical Systems and Tracking Control of Nonlinear Systems: Connections and Closed-form Results

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Abstract: This paper aims to expose the connections between the determination of the equations of motion of constrained systems and the problem of tracking control of nonlinear mechanical systems. The duality between the imposition of constraints on a mechanical system and the trajectory requirements for tracking control is exposed through the use of a simple example. It is shown that given a set of constraints, d'Alembert's principle corresponds to the problem of finding the optimal tracking control of a mechanical system for a specific control cost function that Nature seems to choose. Furthermore, the general equations for constrained motion of mechanical systems that do not obey d'Alembert's principle yield, through this duality, the entire set of continuous controllers that permit exact tracking of the trajectory requirements. The way Nature seems to handle the tracking control problem of highly nonlinear systems suggests ways in which we can develop new control methods that do not make any approximations and/or linearizations related to the nonlinear system dynamics, or its controllers. More general control costs are used and Nature's approach is thereby extended to general control problems.

Keywords: *nonlinear mechanical systems; constraints; tracking control; closed form controllers; d'Alembert's principle; nonideal constraints.*

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1 Introduction

Sir Isaac Newton described the field of mechanics in his preface to the Principia in the following words [1]:

"*In this sense rational mechanics will be the science of motions resulting from any forces whatsoever, and the forces required to produce any motions, accurately proposed and demonstrated.*"

Today, while the first part of Newton's definition of mechanics has become our usual understanding of this field, the second part is usually relegated primarily to the field of control theory. Indeed, the problem that Newton famously solved was a control problem: the determination of the forces required to be acting on the planets so that their motions obey the observed motions described by Kepler's first two laws.

To illustrate the view point of Newton, let us consider an elementary example, the problem of finding the equations of motion of a spherical pendulum like the one shown in Figure 1. The problem of finding the equation of motion of this simple system, which consists of a particle of mass m constrained to move so that it is always at a fixed distance, L, from its fixed point of support, O, in a nonuniform gravitational field, can alternatively be looked at from the dual stand-point of tracking control.

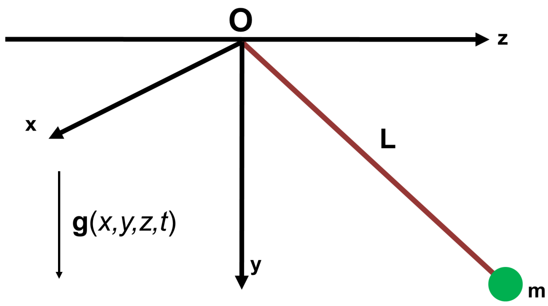

Figure 1: A spherical pendulum.

Consider a particle of mass m moving in a nonuniform gravitational field; it is now required to determine the control force that needs to be applied to this particle so that it is constrained to lie, at each instant of time t, on the sphere S^2 defined by the relation

$$\varphi(x, y, z, t) := x^2(t) + y^2(t) + z^2(t) - L^2 = 0. \tag{1}$$

We will show that this control problem can be handily approached using the theory of constrained motion of mechanical systems. Let us denote the 3 by 1 vector (the 3-vector) $q := [x \;\; y \;\; z]^T$. Clearly, the equation of motion of the particle as it freely moves in the nonuniform gravitational field in which the acceleration due to gravity at any point is $g(x, y, z, t)$ (see Figure 1), is simply given by the equation

$$M\ \ddot{q}(t) \ := \ m\ I_3\ \ddot{q}^T \ = \ [0 \;\; mg(x, y, z, t) \;\; 0]^T \ := \ Q, \tag{2}$$

where I_3 is the 3 by 3 identity matrix. The acceleration of the particle at any time t, can be written as the 3-vector $a(q,t) = [0 \;\; g \;\; 0]^T$. [From here on, we shall drop the arguments of the various quantities, unless needed for clarity.] We shall refer to equation (2) as the *unconstrained* (or uncontrolled) *equation of motion* for the mechanical system. A control theorist may prefer to call the equation a description of the 'plant' whose trajectories need to be controlled so that they satisfy the control requirement stated in (1). In order to achieve this, an additional force will need to be applied to the particle so that its acceleration is altered from $a(q,t)$, and its equation of motion now becomes

$$M\,\ddot{q} = Q + Q^C. \tag{3}$$

This additional force, Q^C, which is a 3-vector, that needs to be applied to the constrained system can be viewed as the force of constraint that ensures that equation (1) is satisfied. It can also, from a dual perspective, be seen as the control force that must be applied to the system described by (2), so that it satisfies the trajectory requirement (1) that is imposed on it.

The initial conditions $q(0)$, and $\dot{q}(0)$ whose components could be chosen arbitrarily in the case of system (2) can no longer be chosen arbitrarily. Instead, the components of $q(t)$ must satisfy relation (1) at each instant of time (and hence also at the initial time); also, the components of $\dot{q}(t)$ must satisfy the relation

$$x(t)\,\dot{x}(t) + y(t)\,\dot{y}(t) + z(t)\,\dot{z}(t) = 0, \tag{4}$$

at each instant of time (and hence also at the initial time). Equation (4) is obtained by differentiating equation (1) with respect to time. One may want to further differentiate equation (4) to obtain the relation

$$x(t)\,\ddot{x}(t) + y(t)\,\ddot{y}(t) + z(t)\,\ddot{z}(t) = -\dot{x}^2(t) - \dot{y}^2(t) - \dot{z}^2(t), \tag{5}$$

which can be written in matrix-vector form as

$$A\,\ddot{q} = b, \tag{6}$$

where $A := [x \;\; y \;\; z]^T$, and $b = -\dot{x}^2(t) - \dot{y}^2(t) - \dot{z}^2(t)$. We note that for a given set of initial conditions that satisfy equations (1) and (4) at $t = 0$, equation (6) is equivalent to equation (1). This simple example thus illustrates the connections between the problem of constrained motion and the problem of tracking control. Specifically, we find the following analogous concepts given in Table 1. As we go along, we will extend and refine

Analytical Dynamics	**Control Theory**
Unconstrained System	Uncontrolled System, or Plant
Constrained System	Controlled System
Constraints	Trajectory Requirements
Constraint Force	Control Force, or Control

Table 1: Analogous Concepts in Analytical Dynamics and Control Theory.

this preliminary table. In what follows, we will move back and forth between these dual concepts, allowing ourselves to be aided in our understanding of constrained motion to expose new insights into trajectory control, and vice versa.

2 General Constrained Mechanical Systems and the Trajectory Control Problem

Our spherical pendulum problem is an illustrative 'toy problem' created simply to provide some insights into the connections that we are trying to establish. The problem could, of course, have been made considerably more challenging by requiring that the point of support, O, move over a surface say $\phi(q, \dot{q}, t) = 0$, and/or requiring that the pendulum's length varies in a prescribed manner so that $L(t) = f(q, \dot{q}, t)$. We can now frame the general problem of constrained motion in analytical dynamics as follows:

1. Consider an unconstrained (uncontrolled) nonlinear nonautonomous mechanical system described by the equation

$$\begin{gathered} M(q,t)\ \ddot{q} = Q(q, \dot{q}, t), \\ q(0) = q_0 \text{ and } \dot{q}(0) = \dot{q}_0, \end{gathered} \tag{7}$$

   where M is a positive definite n by n matrix, and q is an n-vector,

2. We require this system to satisfy the m consistent constraints (trajectory requirements) given by the relations

$$\phi_i(q,t) = 0, \quad i = 1, 2, ..., h, \tag{8}$$

   and

$$\psi_i(q, \dot{q}, t) = 0, \quad i = h+1, h+2, ..., m. \tag{9}$$

3. We need to find a constraint (control) force, Q^C, so that the constrained (controlled) system described by

$$\begin{gathered} M(q,t)\,\ddot{q} = Q(q, \dot{q}, t) + Q^C(q, \dot{q}, t), \\ q(0) = q_0 \text{ and, } \dot{q}(0) = \dot{q}_0, \end{gathered} \tag{10}$$

   exactly satisfies trajectory requirements (8) and (9).

We shall assume that q_0 and $\dot{q}_0$ satisfy the trajectory requirements (8) and (9) at time $t = 0$. Later on, we will relax this condition. We define the acceleration of the uncontrolled (unconstrained) system by

$$a(q, \dot{q}, t) = M^{-1}(q,t)Q(q, \dot{q}, t). \tag{11}$$

Also, assuming sufficient smoothness, we can differentiate the h equations in the set (8) twice with respect to time (as we just did in our toy problem, see (5)), and the $(m-h)$ equations in the set (9) once with respect to time, to obtain the relation

$$A(q, \dot{q}, t)\,\ddot{q} = b(q, \dot{q}, t), \tag{12}$$

where A is an m by n matrix of rank k. Each row of the matrix A corresponds to one of the trajectory requirements in the sets (8) or (9).

3 The Control Force Q^C

Having now laid out some of the underlying concepts relevant to the duality between the problem of constrained motion and the problem of tracking control, let us concentrate in this section on how one might determine the control force Q^C. Before we embark on this, it might be worthwhile going back to our toy problem and investigating if such a force Q^C indeed exists, so that the trajectory requirement (1) is always satisfied, and if so, whether it can be uniquely found. That such a force Q^C exists, is obvious, because we know the equation of motion of a pendulum and so we know that a right hand side for equation (3) exists so that the constraint (1) is *exactly* satisfied for all time, given that the initial conditions satisfy the constraints. So there most-likely exists a control that is Lipschitz continuous, as we require in mechanics so that the solution of (3) is unique and it concurs with practical observations of the motions of a pendulum. Our next question is then, can Q^C be uniquely found ?

Unfortunately, not ! For the spherical pendulum, at each instant of time, we have the following six unknowns: the three components of the 3-vector $\ddot{q}$, and the three components of the 3-vector Q^C. At each instant of time, starting with a given state $(q, \dot{q})$ of the system, we have the three equations given by the set in (3) and an additional equation of constraint (1) (or alternately (6)) – a total of 4 equations. The number of unknowns exceeds the number of equations by two, and hence, at each instant of time, the problem of finding the 6 unknowns (accelerations and control forces) of the system is underdetermined ! To get them uniquely we would need to have two more independent equations. Moving to our dual vision of the problem as one of trajectory control, there must then be an infinity of control forces (controllers) Q^C that can *exactly* track the trajectory expressed by equation (1) !

However, the equation of motion of a spherical pendulum, which satisfies the constraint (trajectory requirement), *is* unique – hence Q^C is unique – and its motion pretty well agrees with what is in fact physically observed. So clearly, Nature must then be picking the constraint force (control force) Q^C in such a manner so as to satisfy some additional criterion – one which somehow yields the (additional) two missing equations, and yields a unique answer for the control force !

3.1 D'Alembert's and Guass's principle, and the cost function

Flipping back to our understanding of constrained motion, we may then ask, how does Nature pick the constraint force Q^C so that the motion of our spherical pendulum matches our physical observations ? This is a problem that was first attacked by d'Alembert, and later on, more generally, by Lagrange [2]. Lagrange came up with the precise statement of what is today called d'Alembert's principle or prescription. D'Alembert's prescription is as follows:

The constraint force Q^C is such that for all vectors $v(t) \neq 0$ that satisfy the relation $Av = 0$, Nature seems to require that $v^T Q^C = 0$.

The nonzero vectors v that satisfy the relation $A\,v = 0$ are called virtual displacements, and the quantity $W^C = v^T Q^C$ is referred to as the total work done by the forces of constraint under virtual displacements. And this prescription, somewhat miraculously – for any general mechanical system – generates the correct number of additional equations so that the constraint force Q^C in equation (10) can be uniquely found at each instant of time!

To see how this works for our spherical pendulum, observe that the rank of our matrix A in (6) is 1, and so the null space of this 1 by 3 matrix is 2. Thus at each time t, there are two linearly independent 3-vectors v_1 and v_2 that satisfy the relation $Av = 0$ which we can find. D'Alembert's prescription then requires that ${v_1}^T Q^C = 0$, and ${v_2}^T Q^C = 0$. These two additional equations used with the four equations (the three equations in set (3) and equation (6)) that we had previously, yield the six equations needed for finding the six unknowns – $\ddot{q}$ and Q^C – at each instant of time. What is more astonishing is that d'Alembert's prescription yields the constraint force Q^C which when used in equation (10) yields the motion, $q(t)$, of the mechanical system that is fairly close, in numerous situations, to what is actually observed in the physical world; hence, its enormous value in modeling physical systems.

To summarize, we cannot, in general, determine the constraint force Q^C uniquely. D'Alembert's principle generates additional equations (exactly the right number) to give us a unique Q^C at each instant of time, which causes the constrained system to move in a manner that is in concert with physical observations. It turns out that this prescription of d'Alembert regarding the constraint force Q^C is exactly the same as the following condition on the constraint (control) force Q^C from the dual viewpoint [3]. This condition, called Gauss's Principle, is the following: From all those control (constraint) forces Q^C that can exactly satisfy the trajectory requirements (8) and (9), Nature chooses that control force Q^C that minimizes the control cost given by

$$J(t) = \left[Q^C(q, \dot{q}, t)\right]^T M^{-1}(q, t)\, Q^C(q, \dot{q}, t) = \left\|Q^C\right\|^2_{M^{-1}} \tag{13}$$

at *each instant of time*. As seen from (13), $J(t)$ is simply the square of the weighted L_2 norm of the control force, Q^C.

So we see that d'Alembert's prescription in mechanics – a prescription that causes mathematical models of constrained mechanical system to suitably predict the physically observed motions of these systems – has a dual that says that Nature appears to be constantly solving an optimal control problem, minimizing the cost function $J(t)$ given in (13). But unlike most control engineers today, who would prefer to minimize $\int_0^T J(t)dt$, where T is some final time over which the control is executed, Nature seems to do this minimization *at each instant of time*. Also, the so-called weighting matrix that she uses in the cost function is M^{-1}. This is indeed clever! For example, imagine a multi-body system, with several masses, that is described by equation (7). Say we want to control this system so that it satisfies some given trajectory requirements given by relations (8) and (9). Realizing that the larger masses require larger forces to be exerted on them to cause them to move, Nature attempts to satisfy these requirements (constraints) on this multi-body system, by being in favor of applying forces to the smaller masses – hence, the weighting by the matrix M^{-1}.

We have so far only considered the properties of the constraint force Q^C, without answering the question: what *is* it? Can one find it explicitly, in closed form? We do that next.

3.2 Closed form solution to the optimal tracking control problem for nonlinear, nonautonomous mechanical systems using the theory of constrained motion

The problem of finding the constraint force Q^C that Nature uses has a long and varied history. The problem was first formulated by Lagrange [2], and has been worked on by numerous scientists and engineers [3–9]. A simple expression for the explicit form of the control force was obtained in 1992, and it is given by [10]

$$Q^C = -M^{1/2}(AM^{-1/2})^+(Aa-b), \tag{14}$$

where X^+ denotes the Moore-Penrose inverse of the matrix X [11,12]. The equation of motion of the constrained system, which may be thought of as the fundamental equation of mechanics, can thus be explicitly written *in extensio*, using relation (10), as

$$M(q,t)\,\ddot{q} = Q(q,\dot{q},t) + Q^C(q,\dot{q},t), \tag{15}$$

where

$$Q^C(q,\dot{q},t) = -M^{1/2}(q,t)[A(q,\dot{q},t)M^{-1/2}(q,t)]^+[A(q,\dot{q},t)a(q,\dot{q},t) - b(q,\dot{q},t)].$$

What now might be gleaned from a controls point of view from relation (15)? First, we observe that $a(q,\dot{q},t)$ (see equation (11)) is the acceleration of the uncontrolled (unconstrained) system. However, to track the given trajectory described by the set of equations (8) and (9), the acceleration of the system needs to satisfy the trajectory requirement (12). Hence, the extent to which the acceleration, a, of the uncontrolled system does not satisfy this trajectory requirement is simply

$$e(q,\dot{q},t) := [A(q,\dot{q},t)a(q,\dot{q},t) - b(q,\dot{q},t)]. \tag{16}$$

This is in fact the error in the satisfaction of the trajectory constraint at time t by the acceleration (at that time) of the uncontrolled system. The expression for Q^C above says that this error signal is fed back to the system (7), just the way a modern-day control engineer might want to do negative feedback control! We also observe that Nature seems to choose a control gain matrix whose elements are, in general, highly nonlinear functions of q, $\dot{q}$, and t. It is given explicitly by

$$K(q,\dot{q},t) := M^{1/2}(q,t)[A(q,\dot{q},t)M^{-1/2}(q,t)]^+. \tag{17}$$

Thus the control methodology used by Nature, so that the uncontrolled system (7) exactly tracks the trajectory requirements stated in sets (8) and (9), can be encapsulated by the relation

$$M(q,t)\,\ddot{q} = Q(q,\dot{q},t) - K(q,\dot{q},t)e(q,\dot{q},t), \tag{18}$$

where K is the gain matrix and e is the error signal. Lastly, we point out that Nature appears to use an error signal for its feedback control law that is related to accelerations, and not to displacements, nor to velocities, or to integrals of the displacement, as is commonly done in control theory. She appears to be basing her feedback on ensuring that the *accelerations* of the controlled system satisfy the trajectory requirement given in (12); and yet, cleverly enough, as seen from the expression for the feedback error e in (16), she involves only the state $(q,\dot{q})$ of the mechanical system. The tracking controller given by equation (18) is not only optimal in that it minimizes the cost $J(t)$ given in (13), but it yields *exact* tracking; for, the set of equations (8) and (9) are the integrals of motion of the nonlinear system described by (15) (or, (18)). The minimal control cost at each instant of time is explicitly given by

$$J(t) = [Q^C]^T M^{-1} Q^C = \left\| (AM^{-1/2})^+ (Aa - b) \right\|^2. \tag{19}$$

As mentioned before, the closed form expression in equation (14) for the control force Q^C that nature uses satisfies the trajectory requirements. She gets this unique control force by minimizing the control cost $J(t)$ given in (13), which is simply the square of the weighted L_2 norm of control force, Q^C. Nature picks the weighting matrix to be the positive definite matrix $M^{-1}(q,t)$ and thereby produces control forces that are in conformity with the physically observed motions of constrained systems. However, what if the control engineer wants to use a different weighting matrix in his cost function? Namely, suppose (s)he wants to minimize at each instant of time the cost

$$J(t) \; = \; [Q^C(q,\dot{q},t)]^T \; N(q,t) \; Q^C(q,\dot{q},t) \; = \; \| \, Q^{\,C} \|_N^2 \, , \tag{20}$$

where $N(q,t)$ is a positive definite matrix. Using our dual perspective, this may also be thought of as a generalization of Gauss's Principle (in mechanics), wherein we use a weighting matrix in our control cost minimization that may be different from M^{-1}. It turns out that the unique control that minimizes this control cost is given (instead of equation (14)) by [14]

$$Q^C = -N(q,t)^{-1/2} A_N^+ (Aa - b) = -N^{-1} M^{-1} A^T [A(MNM)^{-1} A^T]^+ (Aa - b), \tag{21}$$

where $A_N = A(q,\dot{q},t) M(q,t)^{-1} N(q,t)^{-1/2}$. There is one last point that is worth mentioning. We had assumed that the initial conditions of the controlled system satisfy the trajectory requirements (8) and (9). What if the initial conditions do not lie on the so-called manifold described by the trajectory requirements? If one is close to the trajectory manifold, then instead of thinking of the trajectory requirements (8) and (9) as $\phi_i(q,t) = 0$ and $\psi_i(q,\dot{q},t) = 0$, one could consider the trajectory requirement as [13]

$$\ddot{\phi} + \Sigma\dot{\phi} + K\phi = 0, \quad \text{and} \quad \dot{\psi} = -\Lambda\psi, \tag{22}$$

where ϕ and ψ are h- and $(m-h)$-vectors that contain the ϕ_i's and ψ_j's respectively. The matrices Σ, K, and Λ can be chosen so that the solutions ϕ and ψ to the equations (22) tend to zero asymptotically as $t \to \infty$, so that the constraints $\phi_i = 0$ and $\psi_i = 0$ are ultimately satisfied. These equations lead to trajectory requirements which can again be stated in the form of Equation (12), and the control force is again given explicitly by equation (21)! The parameters that are used in the matrices Σ, K, and Λ control the rate and nature of convergence of the trajectories of the dynamical system towards the manifolds, $\phi_i(q,t) = 0$ and $\psi_i(q,\dot{q},t) = 0$.

To illustrate the nature of this control force, let us go back to our toy problem of controlling a mass m in a time varying gravity field so that it lies on the surface $\varphi(x,y,z,t) := x^2(t) + y^2(t) + z^2(t) - L^2 = 0$. The uncontrolled equation of motion is given by (2) in which M and Q are defined, and $a = M^{-1}Q = [0 \;\; g(x,y,z,t) \;\; 0]^T$. We use the constraint equation

$$\ddot{\varphi} + c\dot{\varphi} + k\varphi = 0, \qquad c > 0, \quad k > 0, \tag{23}$$

whose solution as $t \to \infty$ is $\varphi = 0$. Denoting, as before, $q := [x \;\; y \;\; z]^T$, this constraint can be rewritten as

$$A\ddot{q} = [x \;\; y \;\; z]\, \ddot{q}^T = -\dot{q}^T \dot{q} - c\dot{q}^T q - (k/2)(q^T q - L^2) := b. \tag{24}$$

Knowing M, Q, A, b, equation (21) then gives

$$Q^C = -\frac{mN^{-1}}{(AN^{-1}A^T)} q^T \{gy + \dot{q}^T \dot{q} + c\dot{q}^T q + \frac{k}{2}(q^T q - L^2)\}, \tag{25}$$

where N is a user-specified positive definite matrix. We note that the control is nonlinear and no approximations related to the nonlinear nature of the 'plant' are made. No *a priori* assumptions (such as a linear PD controller) are made about the controller either, and the control minimizes the control cost given in (20) at each instant of time.

Flipping back to analytical dynamics, our closed form equation given by (15) for the constrained motion of the system (10) presupposes that d'Alembert's prescription is valid for every mechanical system. What if it isn't? Constraint forces that do not obey d'Alembert's prescription are called nonideal, and often such systems are referred to as systems with nonideal constraints.

3.3 Mechanical systems with nonideal constraints and the set of controllers for exact trajectory control

The difficulty of incorporating systems with nonideal constraints into the framework of Lagrangian mechanics – though such systems are fairly commonplace in the physical world – arises because of the following two main reasons:

1. We need to have the specification of constraints to be general enough so as to encompass problems of practical utility.

2. The specification must, in order to comply with physical observations, yield the accelerations of the constrained systems uniquely when using the math-ware of analytical dynamics that has been developed over the last 250 years.

It is for this reason that most texts and treatises on mechanics summarily dispatch these systems beyond their boundaries, early on in their treatments of analytical dynamics (see [15] and [16]).

The main problem is how to modify and extend d'Alembert's principle. One way of doing this would be to extend d'Alembert's prescription to say that *at each instant of time, the work done by the force of constraint is prescribed <u>for the specific system at hand</u>.* Such a principle would then state that [17]:

$$\begin{array}{l} \textit{For any virtual displacement } v(t) \textit{ at time } t, \textit{ the work done by the} \\ \textit{force of constraint } W^C := v^T Q^C \text{ is prescribed to be equal to } v^T C(q, \dot{q}, t), \end{array} \tag{26}$$

where the n-vector $C(q, \dot{q}, t)$ is prescribed by the mechanician for the given, specific system being modeled. The prescription of C can be done through experimentation, and/or by analogy with other systems, or otherwise. At any given instant of time t, W^C can be positive, negative, or zero; this allows the possibility that energy can be fed into the system at the constraint, or it can be removed at the constraint. When $C \equiv 0$ for all time t, this extension of d'Alembert's principle reverts to d'Alembert's prescription.

For any sufficiently smooth C, one can find the explicit equation of motion for such a constrained system that satisfies exactly the constraint requirements (8) and (9) (or alternately (12)). Dropping the arguments of the various quantities, the equation is [17]

$$M\,\ddot{q} = Q - M^{1/2}B^{+}(Aa - b) + M^{1/2}(I - B^{+}B)M^{-1/2}C := Q + Q^{C}, \tag{27}$$

where $B(q, \dot{q}, t) = A(q, \dot{q}, t)M(q, t)^{-1/2}$. We notice that the first two terms on the right hand side of the first equality in equation (27) are identical to those on the right hand side of equation (15), the nonideal nature of the constraint force having simply added an additional term on the right hand side, for any given prescribed smooth function $C(q, \dot{q}, t)$. By choosing the Lipschitz continuous function $C(q, \dot{q}, t)$ arbitrarily, equation (27) provides all the possible Lipschitz continuous controllers [17] that can make the uncontrolled system (7) exactly track the trajectory requirements specified by equations (8) and (9). Clearly, the second and third members on the right hand side in the first equality of (27) are M-orthogonal, and so

$$J(t) = \left\|B^{+}(Aa - b)\right\|^2 + \left\|(I - B^{+}B)M^{-1/2}C\right\|^2.$$

The addition of the second term on the right hand side increases the cost from its optimal value of $\left\|B^{+}(Aa - b)\right\|^2$ to that now provided. As before, more generally, when the weighting matrix in the control cost is N instead of M^{-1} the explicit control that causes system (10) to exactly satisfy the trajectory requirements (8) and (9) is given in closed form by [14],

$$Q^{C} = -N(q, t)^{-1/2}A_N^{+}(Aa - b) + N^{-1/2}(I - A_N^{+}A_N)M^{-1/2}C \tag{28}$$

for arbitrary continuous functions $C(q, \dot{q}, t)$ and the equation of motion becomes

$$M\,\ddot{q} = Q - N^{-1/2}A_N^{+}(Aa - b) + N^{-1/2}(I - A_N^{+}A_N)M^{-1/2}C := Q + Q^{C}. \tag{29}$$

The second and third members in the first equality above are now N-orthogonal and the control cost now becomes

$$J(t) = \left\|Q^{C}\right\|_N^2 = \left\|A_N^{+}(Aa - b)\right\|^2 + \left\|(I - A_N^{+}A_N)M^{-1/2}C\right\|^2. \tag{30}$$

We can now expand Table 1 to expose the various analogous concepts that we have developed (see Table 2).

4 Example

In this section, we provide an example that utilizes the connections we have developed between analytical dynamics and control of nonlinear systems.

Energy control of nonlinear mechanical systems has become important nowadays and various energy harvesting schemes are being developed. We consider here the problem of energy control of a highly nonlinear mechanical system and approach it by using the connections that have been developed in the previous sections between analytical dynamics and control. The fundamental equation of mechanics (equations (14) and (15)) is used to obtain the explicit nonlinear control force required to achieve the desired energy control.

Analytical Dynamics	Control Theory
Unconstrained System $M(q,t)\ \ddot{q} = Q(q,\dot{q},t)$	Uncontrolled System, or Plant $M(q,t)\ \ddot{q} = Q(q,\dot{q},t)$
Constrained System $M(q,t)\,\ddot{q} = Q(q,\dot{q},t) + Q^C(q,\dot{q},t)$	Controlled System $M(q,t)\,\ddot{q} = Q(q,\dot{q},t) + Q^C(q,\dot{q},t)$
Constraints $\ddot{\phi} + \Sigma\dot{\phi} + K\phi = 0,$ $\dot{\psi} = -\Lambda\psi$	Trajectory Requirements $\phi_i(q,t) = 0,\ i = 1,2,\ldots,h$ $\psi_i(q,\dot{q},t) = 0,\ i = h+1, h+2,\ldots,m.$
Gauss's Principle (GP) $J(t) = [Q^C(q,\dot{q},t)]^T M^{-1}(q,t)\,Q^C(q,\dot{q},t)$	Control Cost $\int\limits_0^T [Q^C(q,\dot{q},t)]^T M^{-1}(q,t)\,Q^C(q,\dot{q},t)dt$
Constraint Force with GP $Q^C = -M^{1/2}(AM^{-1/2})^+(Aa-b)$	Control Force, or Control
Optimal at EACH Instant of time	Optimal over the interval of time [0,T]
Generalized Gauss's Principle $J(t) = [Q^C(q,\dot{q},t)]^T N(q,t)\,Q^C(q,\dot{q},t),$ where $N > 0$	$\int\limits_0^T [Q^C(q,\dot{q},t)]^T N(q,t)\,Q^C(q,\dot{q},t)dt,$ where $N > 0$
Equations of motion for Nonideal Constraints $M\ddot{q}\ = \{Q - N(q,t)^{-1/2}A_N^+(Aa-b)\ +$ $N^{-1/2}(I - A_N^+A_N)M^{-1/2}C(q,\dot{q},t)\}$	Full set of continuous controllers that satisfy trajectory requirements for arbitrary continuous $C(q,\dot{q},t)$.

Table 2: Analogous Concepts in Analytical Dynamics and Control Theory (detailed).

We consider a 3-DOF fixed-fixed Toda chain [18] as shown in Figure 2. Let m_i denote the mass of the i-th particle ($i = 1,2,3$) in the chain. The displacement of the mass m_i as measured from its equilibrium position is denoted by q_i, and its velocity is denoted by $\dot{q}_i$. Given any nonzero initial energy state, H_0 of the chain, our aim is to stabilize the chain at a different nonzero desired energy level, H^*. And to achieve this, control can be applied to one or more of these three masses.

In the present example, we control the energy of the chain by actuating the first mass, m_1, alone. We shall impose the requirement, that the energy of the system be increased to the desired value H^* as a constraint on the mechanical system, and the constraint force that will cause this constraint to be satisfied will then be the requisite control force that would need to be applied to the mass m_1. We begin with a description of the Toda

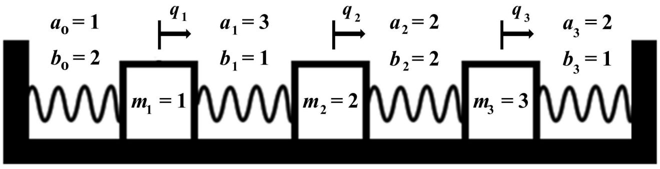

Figure 2: A 3-DOF fixed-fixed Toda chain.

potential.

(i) Toda Potential and Spring Force: The expression for the nonlinear potential of the Toda spring [18] is given by

$$u(q) \; = \; \frac{a}{b}\, e^{b\, q} - \; a\, q \; - \; \frac{a}{b}, \quad a > 0,\; b > 0, \tag{31}$$

whereas its exponential spring force $F_s(q)$ can be derived from its potential as

$$F_s(q) \; = -F_{restoring}(q) \; = \; \frac{\partial u(q)}{\partial q} \; = \; a\; \left(e^{b\, q} - \; 1\right). \tag{32}$$

A plot of the Toda spring potential and the Toda spring force is shown in Figures 3 and 4, respectively. For sufficiently small displacement, the spring force is approximately linear. However, the nonlinearity of the force gains prominence as the displacement increases. As can be inferred from Figure 4, a larger force is required to stretch the spring by a unit distance than is required to compress it. Hence, the Toda chain considered possesses spring elements that are stronger in tension than in compression. Such systems arise frequently in structural sub-systems such as the stringers in suspension bridges.

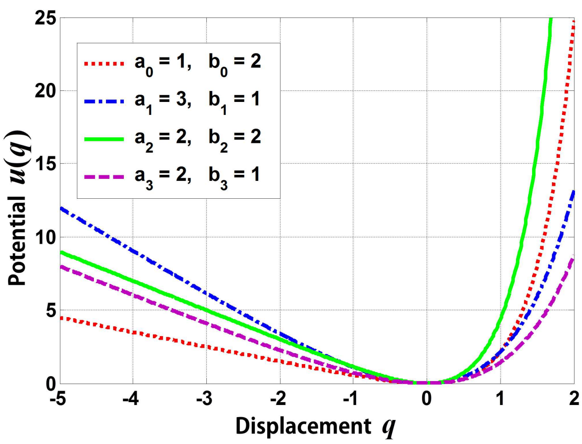

Figure 3: Toda Spring Potential.

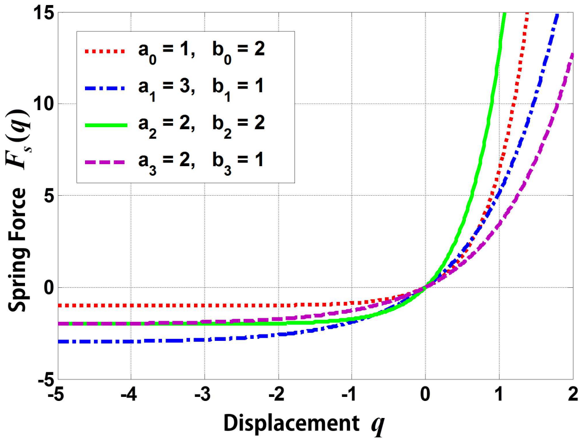

Figure 4: Toda Spring Force.

(ii) Unconstrained System: Consider the 3-DOF fixed-fixed Toda chain as shown in Figure 2. The total energy of the chain can be written down as

$$H(q,\dot{q}) = \sum_{i=1}^{3} \left[\frac{1}{2}\, m_i\, \dot{q}_i^2\right] + \sum_{i=0}^{3} \left[\frac{a_i}{b_i}\, e^{b_i(q_{i+1}\,-\,q_i)} \;-\; a_i\,(q_{i+1} \,-\, q_i) \;-\; \frac{a_i}{b_i}\right], \quad (33)$$

where $q_o \equiv q_4 \equiv 0$ describe the boundary conditions of the fixed-fixed chain. The equations of motion of the unconstrained (uncontrolled) system can be written down in matrix form as $M\ddot{q} = Q$, or more explicitly as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} = \begin{bmatrix} a_1(e^{b_1(q_2-q_1)}-1) \;-\; a_0(e^{b_0(q_1)}-1) \\ a_2(e^{b_2(q_3-q_2)}-1) \;-\; a_1(e^{b_1(q_2-q_1)}-1) \\ a_3(e^{b_3(-q_3)}-1) \;-\; a_2(e^{b_2(q_3-q_2)}-1) \end{bmatrix}. \quad (34)$$

We take, for example, the initial conditions of this Toda chain to be

$$\begin{aligned} &q_1(0) = 1,\; q_2(0) = 2,\; q_3(0) = 1, \\ &\dot{q}_1(0) = 2,\; \dot{q}_2(0) = 0,\; \dot{q}_3(0) = 2. \end{aligned} \quad (35)$$

Figure 2 shows the parameter values of the masses ($m_i,\ i = 1,\ 2,\ 3$) used as well as the parameter values $a_i,\ b_i,\ i = 0,\ 1,\ 2,\ 3$ that characterize the four different Toda springs. Using these parameter values and the initial conditions given in (35), the unconstrained equations of motion given in (34) can be numerically integrated. We note that for all the simulations presented in this section, the equations of motion have been integrated using the 'ode45' scheme in the Matlab environment with a relative integration error tolerance of 10^{-10} and an absolute error tolerance of 10^{-13}. Figure 5 (top) shows a plot of the displacements of the three masses from $t = 0$ to $t = 10$ time units for the unconstrained (uncontrolled) system.

The unconstrained Toda chain is a conservative system and the energy, being an integral of motion, remains constant throughout the duration of the simulation (see

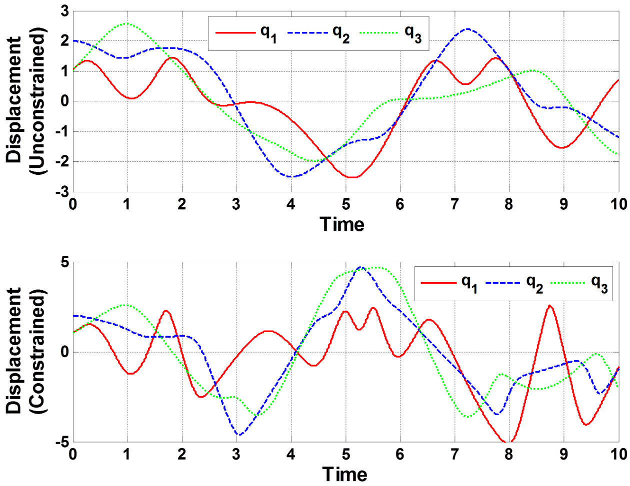

Figure 5: Time history of displacements for the unconstrained system (top) and constrained system (bottom).

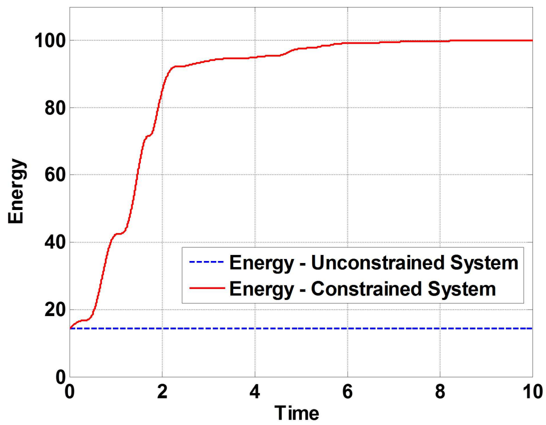

Figure 6: Time history of energy of the 3-DOF Toda chain.

dotted line in Figure 6). For the parameter values chosen, the energy level of the chain is $H_0 = 14.22$ units. Our aim is to increase the energy of the chain to a new and different value.

(iii) Constraints: We shall assume that we want the nonlinear Toda chain described by equation (34) (with the parameter values as shown in Figure 2) to have an energy level $H^* = 100$ units by controlling only mass m_1. In order to achieve this control objective, we impose the following two types of constraints on the unconstrained system. The first deals with our objective to change the energy of the system to its desired value, H^*; the second deals with the fact that we want to achieve this by actuating just a single mass from amongst the three masses in the chain, namely, only mass m_1 (see Figure 2).

1. *Energy Control Constraint*: The energy control constraint is given by

$$\frac{d}{dt}\left(H(q,\dot{q})-H^*\right)+\beta(H(q,\dot{q})-H^*) \;=\; 0, \tag{36}$$

where $\beta > 0$. The solution to this differential equation shows that as $t \to \infty$, $H(q,\dot{q}) \to H^*$. Notice that this constraint allows the 3-DOF Toda chain to be started from any arbitrary initial energy state H_0 (see equation (22)) so that it reaches its desired energy state, H^*, as $t \to \infty$.

2. *No-Control Constraints*: Since no control force is to be applied to masses m_2 and m_3 of the Toda chain, the second and third equations in the equation set (34) must remain unchanged in the controlled system. Therefore, the unconstrained equations of motion of masses m_2 and m_3 are themselves the constraints and guarantee that no control is applied to either of these two masses! Thus, in addition to the energy constraint given by (36), the unconstrained system (equation (34)) is also subjected to the following two constraints.

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} \;=\; \begin{bmatrix} a_2(e^{b_2(q_3-q_2)}-1) \;-\; a_1(e^{b_1(q_2-q_1)}-1) \\ a_3(e^{b_3(-q_3)}-1) \;-\; a_2(e^{b_2(q_3-q_2)}-1) \end{bmatrix}. \tag{37}$$

When this set of constraints (equations (36) and (37)) are expressed in the general constraint matrix form of equation (12), we obtain $A\ddot{q} = b$, or more explicitly

$$\begin{bmatrix} m_1\dot{q}_1 & m_2\dot{q}_2 & m_3\dot{q}_3 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} \;=\; \begin{bmatrix} \dot{q}^T Q - \beta(H-H^*) \\ a_2(e^{b_2(q_3-q_2)}-1) \;-\; a_1(e^{b_1(q_2-q_1)}-1) \\ a_3(e^{b_3(-q_3)}-1) \;-\; a_2(e^{b_2(q_3-q_2)}-1) \end{bmatrix}. \tag{38}$$

(iv) Explicit Control Force: With the matrices M, Q, A, b at our disposal, the control force Q^C can be calculated using (14) and is given by

$$Q^C(q,\ \dot{q}) \;=\; \begin{bmatrix} -\xi_o\ (H-H^*)\ m_1\ \dot{q}_1 \\ 0 \\ 0 \end{bmatrix}, \tag{39}$$

where the value of $\beta = \xi_o\ m_1\ \dot{q}_1^2$ has been chosen to avoid any singularities in the control force, which might arise when the actuated mass m_1 has zero velocity. In the present example, for illustration, the positive constant ξ_o has been chosen to be 0.03. The control force (equation (39)) obtained is optimal and it minimizes the control cost given by (20) at each instant of time, with $N = M^{-1}$. Notice from equation (39) that the control force acting on the first mass appears to make it move like a self-excited oscillator!

(v) Dynamics of Constrained System: The equations of motion of the constrained (controlled) Toda chain can now be written down using equations (14) and (15), where M and Q are given by (34), and Q^C is given in (39). A plot of the displacements of the three masses of the controlled system (using the parameters shown in Figure 2), is shown in Figure 5 (bottom) from $t = 0$ to $t = 10$ time units. A plot of the time history of the energy is depicted in Figure 6 for the constrained system. The solid line in the figure shows that the application of the control force has resulted in an increase of the energy of the 3-DOF Toda chain from an initial energy level of $H_0 = 14.22$ units to the desired energy level of $H^* = 100$ units. Figure 7 shows a plot of the time history of the nonlinear control force acting on the first mass to achieve the desired transition. Once the desired energy level is attained, the control force automatically becomes zero and we make use of the conservative nature of the chain to remain at the desired energy level for all future time.

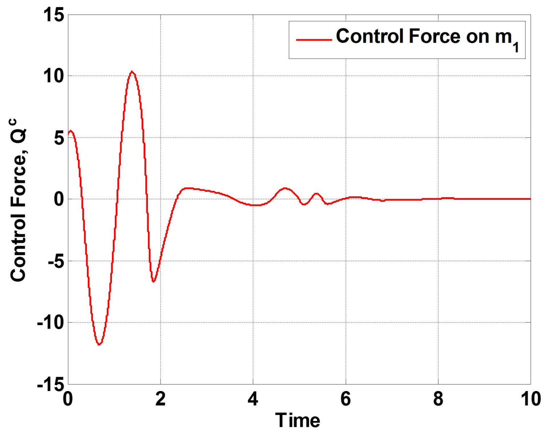

Figure 7: Time history of control forces acting on the 3-DOF Toda chain

It can be shown with some effort that the nonhomogeneous Toda chain that we have considered is controllable using control on just mass m_1 in the sense that the system can be "moved" from any arbitrary energy state $H_0 \neq 0$ to any other energy state $H^* \neq 0$ using the control described in (39). We don't prove that here, since it will take us too far afield from the central theme of this paper.

5 Conclusions and Open Problems

In this paper, we have established a connection between the problem of constrained motion and the problem of control of nonlinear mechanical systems. An example illustrating the development of exact, closed-form energy control of a highly nonlinear multi-degree of freedom system that utilizes this connection has been demonstrated. The developments outlined herein form just the beginnings of a new path to our understanding of the synthesis of analytical dynamics and control. Numerous open questions remain unanswered, such as, robustness of control, extensions to multi-body dynamics and the dynamics of continua, and applications to robotics, space systems, and fluid mechanical systems.

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