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Using Dynamic Vibration Absorber for Stabilization of a Double Pendulum Oscillations

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Abstract: In this paper a stability problem for double pendulum is discussed. A damping device of passive type is used to stabilize small free oscillations of perturbed system. The simplified approach is suggested to prove the asymptotic stability of equilibrium.

Keywords: *double pendulum; dynamic vibration absorber; asymptotic stability; Lagrange function.*

Mathematics Subject Classification (2010): 34C46, 34D20, 70E50, 70E55, 70K20.

1 Introduction

The double pendulum may be considered as a simplified model of the coupled rigid bodies and finds wide use in engineering and technology. Both mathematical and physical interest to this model arises from the phenomena of its motion. Although this motion is described by rather simple ODE system, the pendulum exhibits the dynamical behavior which may be complex and unpredictable [1,2]. In particular, the motion of the double pendulum has the ability of ***beats*** and is strongly sensitive to the initial perturbations. These perturbations may provoke an increased amplitude of the second limb oscillations and, as a result, the switch from regular regime to chaotic one [3,4].

The problem of elimination or reduction of the undesired vibration in various technical systems has a long history and great achievements [5], mostly during the last century. For this purpose the damping devices are used, which may be divided into active and passive dampers. The classical example of passive damper is a dynamic vibration absorber (DVA) [6,7] or vibration neutralizer. It represents the mechanical appendage comprising inertia, stiffness, and damping elements and is connected to a given structure, named herein the primary [5] or original [8] system, with the aim to absorb the excessive vibratory energy. A DVA may be used both in cases of free oscillations and vibrations caused by harmonic excitations. For the case of a simple pendulum, DVA was used in papers [8,9].

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2 Description of the Model

Consider the double pendulum with distributed mass (Fig. 1) which has a fixed point O and is in a gravitational field. Assume that the mass center of the first limb is located at C_1. At the point O_1 located on the axis OC_1 a second limb is pivotally attached. The point C_2 is mass center of the second link. The first limb (configuration A) is attached with a dynamic absorber with stiffness k and damping coefficient h. The absorber oscillates along the axis O_2x', which is orthogonal to the line OO_1 and intersects it at the point O_2. Hinges at the points $O,\ O_1$ are supposed frictionless.

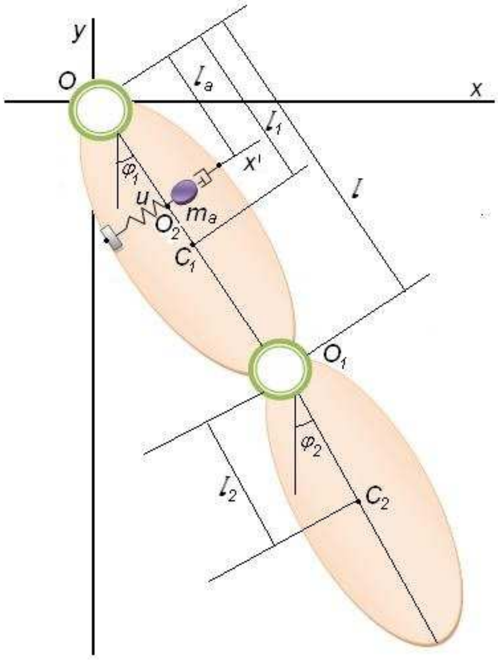

Figure 1: Double pendulum with dynamic vibration absorber in first limb.

Let us write the Lagrange function for the described mechanical system. One can get the kinetic energy K of the system in the form

$$K = K_p + K_a,$$

where K_p, K_a are the kinetic energies of the primary system (pendulum without absorber) and vibration absorber, respectively, calculated by the formulas

$$K_p = \frac{1}{2}[J_1\dot{\varphi}_1^2 + J_2\dot{\varphi}_2^2 + m_2 l^2\dot{\varphi}_1^2 + 2m_2 l l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)],$$

$$K_a = \frac{1}{2}m_a[\dot{\varphi}_1^2(l_a^2 + u^2) + 2l_a\dot{\varphi}_1\dot{u} + \dot{u}^2].$$

Here J_1, J_2 are the moments of inertia of the first and second limbs of pendulum with respect to poles $O,\ O_1$ respectively, m_1, m_2, m_a are the masses of the first and second links, and absorber respectively, u is the extension of the spring, φ_1, φ_2 are the angles of deflection of the pendulum limbs about a vertical axis, l is the length of the first limb, l_1, l_2 are the distances from the suspension points of each of the links to its mass center, l_a is the distance OO_2.

The potential energy can be written as

$$\Pi = -g\cos\varphi_1(m_a l_a + m_1 l_1 + m_2 l) - m_2 l_2 g\cos\varphi_2 + m_a g u\sin\varphi_1 + \frac{1}{2}ku^2.$$

The equations of motion can be written in the form of Lagrange

$$\begin{gathered}(J_1 + m_2 l^2 + m_a l_a^2)\ddot{\varphi}_1 + m_2 l l_2 \ddot{\varphi}_2\cos(\varphi 1 - \varphi 2) + m_2 l l_2 \dot{\varphi}_2(\dot{\varphi}_1 - \dot{\varphi}_2)\sin(\varphi_1 - \varphi_2) + \\ + m_a l_a \ddot{u} + g\sin\varphi_1(m_1 l_1 + m_2 l + m_a l_a) + m_a g u\cos\varphi_1 = 0, \qquad (2.1) \\ J_2\ddot{\varphi}_2 + m_2 l l_2 \ddot{\varphi}_1\cos(\varphi_1 - \varphi_2) - m_2 l l_2 \dot{\varphi}_1(\dot{\varphi}_1 + \dot{\varphi}_2)\sin(\varphi_1 - \varphi_2) + m_2 g l_2\sin\varphi_2 = 0, \\ m_a l_a \ddot{\varphi}_1 + m_a \ddot{u} + m_a g\sin\varphi_1 + ku = -h\dot{u}.\end{gathered}$$

Let us define the conditions of stability of motion of the system (2.1) when the pendulum is in the lower position of equilibrium, i.e. solution

$$\varphi_1 = 0, \varphi_2 = 0, u = 0, \dot{\varphi}_1 = 0, \dot{\varphi}_2 = 0, \dot{u} = 0. \qquad (2.2)$$

3 Stabilization Conditions

Firstly, we write the linear approximation of the system (2.1)

$$\begin{gathered}(J_1 + m_2 l^2 + m_a l_a^2)\ddot{\varphi}_1 + m_2 l l_2 \ddot{\varphi}_2 + m_a l_a \ddot{u} + g(m_1 l_1 + m_2 l + m_a l_a)\varphi_1 + m_a g u = 0, \\ J_2\ddot{\varphi}_2 + m_2 l l_2 \ddot{\varphi}_1 + m_2 g l_2 \varphi_2 = 0, \qquad (3.1) \\ m_a l_a \ddot{\varphi}_1 + m_a \ddot{u} + m_a g \varphi_1 + ku = -h\dot{u}.\end{gathered}$$

We introduce the dimensionless parameters by the formulas

$$\begin{gathered}\widetilde{m}_a = \frac{m_a}{m_1},\ \widetilde{m}_2 = \frac{m_2}{m_1},\ \widetilde{l}_a = \frac{l_a}{l_1},\ \widetilde{l} = \frac{l}{l_1},\ \widetilde{l}_2 = \frac{l_2}{l_1},\ \tau = \sqrt{\frac{g}{l_1}}t, \\ \widetilde{k} = \frac{kl_1}{m_1 g},\ \widetilde{h} = \frac{h}{m_1}, \widetilde{u} = \frac{u}{l_1}. \qquad (3.2)\end{gathered}$$

The system (3.1) can be rewritten as

$$\begin{gathered}(J_1 + \widetilde{m}_2\widetilde{l}^2 + \widetilde{m}_a\widetilde{l}_a^2)\widetilde{\varphi}_1'' + \widetilde{m}_2\widetilde{l}\widetilde{l}_2\widetilde{\varphi}_2'' + \widetilde{m}_a\widetilde{l}_a\widetilde{u}'' + (1 + \widetilde{m}_2\widetilde{l} + \widetilde{m}_a\widetilde{l}_a)\widetilde{\varphi}_1 + \widetilde{m}_a\widetilde{u} = 0, \\ \widetilde{J}_2\widetilde{\varphi}_2'' + \widetilde{m}_2\widetilde{l}\widetilde{l}_2\widetilde{\varphi}_1'' + \widetilde{m}_2\widetilde{l}_2\widetilde{\varphi}_2 = 0, \qquad (3.3) \\ \widetilde{m}_a\widetilde{l}_a\widetilde{\varphi}_1'' + \widetilde{m}_a\widetilde{u}'' + \widetilde{m}_a\widetilde{\varphi}_1 + \widetilde{k}\widetilde{u} = -\widetilde{h}\widetilde{u}'.\end{gathered}$$

For simplicity, we omit the symbol $\sim$ in what follows.

To investigate the problem of the stability of motion (2.2) we will use the results from [10] below.

Suppose that the motion equations of a mechanical system are described by the following system of differential equations

$$\boldsymbol{A}\ddot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{C}\boldsymbol{q} = \boldsymbol{F}(t, \dot{\boldsymbol{q}}, \boldsymbol{q})\dot{\boldsymbol{q}}_1 + \boldsymbol{N}(t, \dot{\boldsymbol{q}}, \boldsymbol{q}), \qquad (3.4)$$

where square matrices $\boldsymbol{A}$, $\boldsymbol{C}$ of order $m+n$, and $\boldsymbol{F}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of order m are symmetric, square matrix $\boldsymbol{B}$ is skew-symmetric, $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2)^T$, i.e. vector $\boldsymbol{q}$ is divided into subvectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ with orders m, n respectively. Denotation "T" means transposition, vector $N(t, \dot{\boldsymbol{q}}, \boldsymbol{q})$ represents a set of arbitrary nonlinear terms. Dependence on t is periodic or quasi-periodic.

We assume that the system provides steady motion:

$$\boldsymbol{q} = 0, \ \dot{\boldsymbol{q}} = 0. \tag{3.5}$$

It is supposed that the matrix $\boldsymbol{F}_0 = \boldsymbol{F}(t, 0, 0)$ is positive definite for $t \geq 0$. Denote by $\boldsymbol{d}, \boldsymbol{d}_{22}$ the linear differential operators

$$\boldsymbol{d} = \boldsymbol{A}\frac{d^2}{dt^2} + (\boldsymbol{B} + \boldsymbol{F}_0)\frac{d}{dt} + \boldsymbol{C}, \ \boldsymbol{d}_{22} = \boldsymbol{A}_{22}\frac{d^2}{dt^2} + \boldsymbol{B}_{22}\frac{d}{dt} + \boldsymbol{C}_{22},$$

and $\boldsymbol{D}(\lambda), \boldsymbol{D}_{22}(\lambda)$ are the corresponding λ-matrices:

$$\boldsymbol{D}(\lambda) = \boldsymbol{A}\lambda^2 + (\boldsymbol{B} + \boldsymbol{F}_0)\lambda + \boldsymbol{C}, \ \boldsymbol{D}_{22}(\lambda) = \boldsymbol{A}_{22}\lambda^2 + \boldsymbol{B}_{22}\lambda + \boldsymbol{C}_{22}.$$

Let λ_0 be an eigenvalue of $\boldsymbol{d}_{22}$, and $\boldsymbol{\gamma}_{20}$ be the corresponding eigenvector. Introduce the equality

$$\boldsymbol{D}_{12}(\lambda_0)\boldsymbol{\gamma}_{20} = 0. \tag{3.6}$$

Theorem 3.1 *Let us consider a mechanical system whose motion equations are discribed by (3.4) and suppose that none of the eigenvectors of operator $\boldsymbol{d}_{22}$ satisfies condition (3.6). Then adding to system an arbitrary dissipative force, which provides full dissipation (by linear terms) on $\dot{\boldsymbol{q}}_1$ leads to the following results:*

I) If all eigenvalues of matrix $\boldsymbol{C}$ are positive, then equilibrium (3.5) becomes asymptotically stable. Stability is exponential and uniform.

II) If matrix $\boldsymbol{C}$ has some negative eigenvalues, then equilibrium (3.5) is unstable, even if it was stabilized before by gyroscopic forces. Among particular solutions of the system at least one has negative Liapunov characteristic number.

According to the above statements, matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ ($\boldsymbol{B} = \boldsymbol{0}$) for the system (3.3) take the following form

$$\boldsymbol{A} = \begin{pmatrix} J_1 + m_2 l^2 + m_a l_a^2 & m_2 l l_2 & m_a l_a \\ m_2 l l_2 & J_2 & 0 \\ m_a l_a & 0 & m_a \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} 1 + m_2 l + m_a l_a & 0 & m_a \\ 0 & m_2 l_2 & 0 \\ m_a & 0 & k \end{pmatrix}.$$

To verify condition (3.6) one may investigate the compatibility of the following system

$$\begin{gathered} [\lambda^2(J_1 + m_2 l^2 + m_a l_a^2) + 1 + m_2 l + m_a l_a]\gamma_1 + \lambda^2 m_2 l l_2 \gamma_2 = 0, \\ \lambda^2 m_2 l l_2 \gamma_1 + (\lambda^2 J_2 + m_2 l_2)\gamma_2 = 0, \\ (\lambda^2 l_a + 1)\gamma_1 = 0. \end{gathered} \tag{3.7}$$

The third equation of (3.7) implies that $\lambda^2 = -1/l_a$. Then the condition of compatibility of the system (3.7) takes the form

$$\delta_1 = (m_2 l_2 + m_2^2 l l_2) l_a^2 - (J_1 m_2 l_2 + m_2^2 l^2 l_2 + m_2 l J_2 + J_2) l_a + \\ + J_1 J_2 + m_2 l^2 J_2 - m_2^2 l^2 l_2^2 = 0. \tag{3.8}$$

Choosing an arbitrary l_a ($l_a \le l$), excluding the value which transforms (3.8) into true equality, we obtain an inconsistent system (3.7). Consequently, the conditions of the theorem are satisfied and we have asymptotic stability of the studied solution.

For more clarity let us compare the results obtained with the standart procedure based on the Routh–Hurwitz criterion [11].

Characteristic equation of system (3.3) is written in the form

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0,$$

where the coefficients are given by the formulas

$$a_0 = m_a[J_1 J_2 + m_2 l^2 (J_2 - m_2 l_2^2)],\ a_1 = h[m_2 l^2 (J_2 - m_2 l_2^2) + J_2 (J_1 + l_a^2 m_a)],$$

$$a_2 = m_2 l^2 k (J_2 - m_2 l_2^2) + J_1 J_2 k + [m_2 l_2 (J_1 + m_2 l^2) + J_2 (1 + m_2 l)] m_a +$$

$$+ J_2 m_a l_a (k l_a - m_a),\ a_3 = h[J_2 + J_1 m_2 l_2 + m_2 l (J_2 + m_2 l l_2) + m_a l_a (J_2 + l_a m_2 l_2)],$$

$$a_4 = k[m_2 l_2 (J_1 + m_2 l^2) + J_2 (1 + m_2 l)] + (1 + m_2 l) m_a m_2 l_2 - J_2 m_a^2 +$$

$$+ (J_2 k - m_a m_2 l_2) l_a m_a + m_2 l_2 k m_a l_a^2,\ a_5 = m_2 l_2 h (1 + m_2 l + m_a l_a),$$

$$a_6 = m_2 l_2 k (1 + m_2 l) + m_a m_2 l_2 (k l_a - m_a).$$

The solution of the system will be asymptotically stable if and only if the following conditions hold

$$a_0 > 0,\ a_3 > 0,\ a_5 > 0,\ a_6 > 0,\ \Delta_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0,$$

$$\Delta_5 = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & a_6 & a_5 \end{vmatrix} > 0. \tag{3.9}$$

It is not hard to see that $a_0,\ a_3,\ a_5,\ a_6$ are positive.

$$\Delta_3 = h^2 m_a^2 \Delta_{30} = h^2 m_a^2 (p_0 - 2p_1 l_a + p_2 l_a^2 + m_a l_a^4 m_2^4 l_2^4 l^2),$$

where

$$p_0 = J_2 [l^2 m_2 (J_2 - m_2 l_2^2) + J_1 J_2]^2,$$

$$p_1 = (m_2^3 l_2^3 l^2 + m_2 l J_2^2 + J_2^2)[l^2 m_2 (J_2 - m_2 l_2^2) + J_1 J_2],$$

$$p_2 = J_2 (1 + m_2 l)[J_2^2 (1 + m_2 l) + 2 m_2^3 l_2^3 l^2] + m_2^4 l^2 l_2^4 (m_2 l^2 + J_1).$$

Let us transform the expression for Δ_{30} to the following form

$$\Delta_{30} = p_0 (l_a - \frac{p_1}{p_2})^2 + m_2^4 l_2^4 l^2 (J_1 J_2 + m_2 l^2 J_2 - m_2^2 l^2 l_2^2)^3 + m_a m_2^4 l_2^4 l^2 l_a^4.$$

So, it is obviously positive, because of $p_0 > 0$, $J_2 \geq m_2 l_2^2$.

The determinant Δ_5 can be represented as $\Delta_5 = m_2^4 l^2 l_2^4 h^3 m_a^4 \delta_1^2$.

Obviously, the conditions of criterion Routh–Hurwitz for system (3.3) are always satisfied, except for $\delta_1 = 0$.

Therefore, $\delta_1 \neq 0$ is a necessary and sufficient condition for asymptotic stability of motion of the system (2.1). That is, selecting a value of parameter l_a that does not satisfy (3.7), we can achieve the exponential stability of a double pendulum motion with additionally introduced mass.

Consider the case where the vibration absorber is located in the second link of the pendulum (Fig. 2).

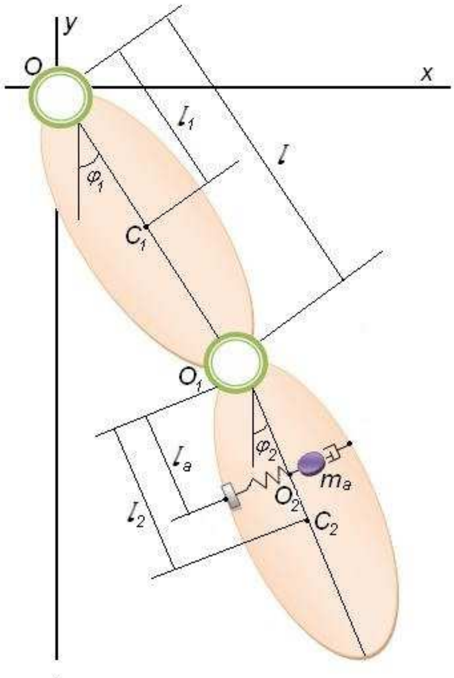

Figure 2: Double pendulum with dynamic vibration absorber in second limb.

In this situation, the choice of dimensionless parameters should be replaced by m_1 to m_2 and l_1 to l_2. Then the matrices take the form

$$\boldsymbol{A} = \begin{pmatrix} J_1 + l^2 + m_a l^2 & l + m_a l l_a & m_a l \\ l + m_a l l_a & J_2 + m_a l_a^2 & m_a l_a \\ m_a l & m_a l_a & m_a \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} m_1 l_1 + l + m_a l & 0 & 0 \\ 0 & 1 + m_a l_a & m_a \\ 0 & m_a & k \end{pmatrix}.$$

Obtain a system of conditions

$$[\lambda^2 (J_1 + l^2 + m_a l^2) + m_1 l_1 + l + m_a l]\gamma_1 + \lambda^2 (l + m_a l l_a)\gamma_2 = 0,$$

$$\lambda^2(l + m_a l l_a)\gamma_1 + [\lambda^2(J_2 + m_a l_a^2) + 1 + m_a l_a]\gamma_2 = 0, \tag{3.10}$$
$$\lambda^2 l\gamma_1 + \gamma_2\lambda^2 l_a + \gamma_2 = 0.$$

To check the consistency of the system express from the second equation (3.10)

$$\gamma_2 = -\frac{\lambda^2 l(1 + m_a l_a)\gamma_1}{\lambda^2 J_2 + \lambda^2 m_a l_a^2 + 1 + m_a l_a}.$$

Upon substituting this expression into the third equation (3.10) we obtain

$$\frac{\lambda^2 l\gamma_1(\lambda^2 J_2 + 2 + 2m_a l_a - \lambda^2 l_a)}{\lambda^2 J_2 + \lambda^2 m_a l_a^2 + 1 + m_a l_a} = 0,$$

whence $\lambda^2 = -2(1 + m_a l_a)/(J_2 - l_a)$.

The condition of compatibility of the system of (3.10) can be represented in the form

$$\delta_2 = l_a^3 m_a[(4l^2 + 4J_1 + 2l)m_a + 2l + 2m_1 l_1] - l_a^2[(6l^2 + 2lJ_2)m_a^2 - (l - 2lJ_2 - 2m_1 l_1 J_2 +$$
$$+6J_1 + 6l^2)m_a - l - m_1 l_1] + l_a[(2l^2 m_a^2 + 2J_1 m_a + 2m_a l^2)J_2 - 10m_a l^2 + 2J_1 +$$
$$+2l^2] - (m_a l + m_1 l_1 + l)J_2^2 + (2J_1 + 2m_a l^2 + 2l^2)J_2 - 4l^2 = 0. \tag{3.11}$$

Similarly to the first case by selecting l_a that does not satisfy equality (3.11) asymptotic stability of the studied solutions can be obtained.

It is possible to verify that the conditions of asymptotic stability obtained by using the Routh–Hurwitz criterion, are also satisfied for $\delta_2 \neq 0$.

Remark 3.1 In the case when equality (3.7) or (3.11) holds, this fact does not prevent the asymptotic stability of equilibrium. The linear approximation has a pair of pure imaginary roots, and we get the critical case in Liapunov sense. To prove the asymptotic stability, the Liapunov function may be constructed [8]. This function is a sum of positively defined quadratic form and form of fourth order and has negative derivative on time. Basically, this procedure is not difficult, but it leads to extremely huge analytical expressions for coefficients of the function (and its derivative) and cannot be given here.

Remark 3.2 The approach employed to prove the asymptotic stability of the motion is relatively simple and much more easier than the use of determinants or innors technique. However, it does not provide the estimation of the damping rate for perturbed oscillations of primary system. For this purpose our approach can be modified, or added by some special evaluating procedure. Obviously, in exchange for this gain, it (approach) will lose a part of simplicity.

We don't discuss now the problem about choice of absorbers parameters with the aim to optimize the decaying rate. For arbitrary set of the pendulum parameters this problem leads to extrema problem for function of high order and, probably, has no explicit finite solution. However, if the pendulum mass distribution is given, numerical calculations may help. Our simulations witness, that configuration B with small distance l_a is a bet, and values k, h strongly depend on primary system parameters. For example, with $m_1 = m_2 = m$, $l_1 = l_2 = l, J_1 = J_2 = ml^2$, $\widetilde{m}_a = 2m/5$, for configuration A one gets $\widetilde{l}_a = 0.552$, $\widetilde{k} = 0.45, \widetilde{h} = 0.463$, and $\sigma = max\{Re\lambda_j\} \approx -0.0140$. For configuration B corresponding values are $\widetilde{l}_a = 0.05, \widetilde{k} = 0.486, \widetilde{h} = 0.234$, and $\sigma = max\{Re\lambda_j\} \approx \approx -0.0943$.

4 Conclusion

In the paper we prove that attaching a DVA to double pendulum stabilizes its equilibrium i.e. provides the exponential stability. Special simple procedure to verify the conditions of stabilization is applied. Some aspects of the optimal absorber's configuration are discussed.

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