Nonlinear Dynamics and Systems Theory, **14** (4) (2014) 383–394

# Pullback Attractors of Nonautonomous Boundary Cauchy Problems

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Received: April 15, 2014; Revised: October 14, 2014

**Abstract:** In this work, we establish the existence of pullback attractors for nonautonomous nonlinear boundary Cauchy problems. We apply our result to a reaction-diffusion equation.

**Keywords:** *nonautonomous boundary Cauchy problem; pullback attractors; reaction-diffusion equation.*

**Mathematics Subject Classification (2010):** 47J35, 45Exx, 34D45, 35K57.

## 1 Introduction

Consider the nonlinear boundary Cauchy problem for arbitrary $s \in \mathbb{R}$

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \in [s, \infty), \\ L(t)u(t) = f(t, u(t)), \quad t \in [s, \infty), \\ u(s) = x, \end{cases} \tag{1}$$

where $A_{\max}(t)$ is a closed operator on a Banach space $X$ endowed with a maximal domain $D(A_{\max}(t))$, and $L(t) : D(A_{\max}(t)) \to \partial X$, with a 'boundary space' $\partial X$ and a function $f : \mathbb{R} \times X \to \partial X$, the solution $u : [s, \infty) \to X$ takes the initial value $x \in X$ at time $s$. Moreover, the restriction $A(t) := A_{\max}(t)|_{ker(L(t))}$ is assumed to generate an evolution family $(U(t, s))_{t \geq s}$, on the state space $X$. That is $U(t, s)x$ is a solution of the corresponding linear boundary Cauchy problem of (1) given by

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \in [s, \infty), \\ L(t)u(t) = 0, \quad t \in [s, \infty), \\ u(s) = x. \end{cases} \tag{2}$$

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This type of equations has recently been suggested and investigated as a model class with various applications like population equations, retarded differential (difference) equations, dynamical population equations and boundary control problems (see e.g. [2, 3, 7] and the references therein).

A crucial question concerning nonautonomous boundary equations is the existence of solutions. Recently, in [3, 9], the existence and uniqueness of classical solutions for (1) in the case that $f(t, x(t)) \equiv f(t)$ was proved. Moreover, it was shown that these solutions are given by a variation of constants formula which can be easily extended, using the contraction fixed point theorem, to the following variation of constants formula solution of (1):

$$x(t,s) = U(t,s)x_0 + \lim_{\lambda\to+\infty} \int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, x(\sigma))\, d\sigma, \quad t \geq s. \tag{3}$$

Here $L_{\lambda,t}$ is the inverse of $L(t)|_{ker(\lambda - A_{\max}(t))}$.

The study of the regularity properties and the long-time behavior of infinite dimensional dynamical systems is one of the most important problems of modern mathematical physics. In this direction, some studies have been done for the problem (1), we cite for example the compactness of solutions [3], the study of controllability [2], the almost periodicity and automorphicity of solutions [1].

Another important question concerning the long-time behavior is the existence of invariant manifolds. This question was recently studied in [7].

The long-time behavior of the above systems can be also expressed by the term of attractors. To the best of our knowledge, the existence of attractors for nonautonomous dynamical systems is not as well developed as for the autonomous case. There exist several non equivalent definitions for nonautonomous attractors, e.g. forward and pullback attractors describing, respectively, the future and the past of nonautonomous equations (see e.g. [6, 15] and the references therein).

Recently, in [5], the authors showed the existence of pullback attractors for evolution processes. Inspired by the ideas in [5], we are concerned in the present work with the study of the existence of pullback attractors for the boundary evolution equation (1), our main tool is the variation of constants formula (3).

Roughly speaking, our goal is to establish sufficient conditions for guaranteeing the existence of a pullback attractor which is a family of compact invariant subsets pullback attracting bounded subsets. More precisely, by assuming some regularity conditions on $(U(t,s))_{t\geq s}$, we will prove that the solution $x$ given in (3) is both pullback strongly bounded dissipative and pullback asymptotically compact.

Finally, to illustrate our general assumptions we give an application to the following reaction diffusion equation:

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) - \beta(t)v(t,x), \quad t \geq 0, x \in [0,1], \\ \dfrac{\partial}{\partial x} v(t,0) = g_1(t,v);\ \dfrac{\partial}{\partial x} v(t,1) = g_2(t,v), \quad t \geq 0, \\ v(0,x) = v_0(x), \quad x \in [0,1]. \end{cases} \tag{4}$$

The structure of the paper is as follows. In Section 2 we list natural assumptions for well-posedness of equation (1) and the concepts of mild solution. Section 3 is devoted to a pullback attractors theorem for (1) which yields sufficient conditions for the existence of pullback attractors. Section 4 is devoted to an application of the reaction diffusion equation (4).

## 2 Preliminaries

In this section we recall some definitions and results and formulate assumptions.

### 2.1 Linear nonautonomous boundary Cauchy problems

A family of linear (unbounded) operators $(A(t))_{t\geq 0}$ defined on a Banach space $X$ is called a *stable family* if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A(t))$ for all $t \geq 0$ and

$$\Big\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \Big\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and any finite sequence $0 \leq t_1 \leq \cdots \leq t_k$, where

$$\rho(A(t)) := \{\lambda \in \mathbb{C} \mid \lambda \, \mathrm{id}_X - A(t) : D(A(t)) \to X \text{ is bijective}\}$$

denotes the *resolvent set* of $A(t)$. For $\lambda \in \rho(A(t))$, the inverse $R(\lambda, A(t)) := (\lambda \, \mathrm{id}_X - A(t))^{-1}$ is called the *resolvent* of $A(t)$.

**Remark 2.1** If there exists a constant $\omega \in \mathbb{R}$ such that

$$\|R(\lambda, A(t))\| \leq \frac{1}{\lambda - \omega},$$

for all $\lambda > \omega$ and $t \geq 0$, then $(A(t))_{t\geq 0}$ is a stable family.

**Definition 2.1** A family of linear bounded operators $(U(t,s))_{t\geq s\in J}$, $J := \mathbb{R}_+$ or $\mathbb{R}$, on a Banach space $X$ is called *evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = \mathrm{id}_X$ for all $t \geq r \geq s \in J$,

(2) the mapping $\{(t,s) \in J \times J : t \geq s\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is strongly continuous.

The *growth bound* of $(U(t,s))_{t\geq s}$ is defined by

$$\omega(U) := inf\left\{\omega \in \mathbb{R} : \exists\, M_\omega \geq 1 \text{ with } \|U(t,s)\| \leq M_\omega e^{\omega(t-s)} \;\forall\, t \geq s \in J\right\}.$$

The evolution family $(U(t,s))_{t\geq s}$ is called *exponentially bounded* provided that $\omega(U) < \infty$ and *exponentially stable* provided that $\omega(U) < 0$.

Let $X, D, \partial X$ be Banach spaces such that $D$ is dense and continuously embedded in $X$. On these spaces, the operators $A_{\max}(t) \in \mathcal{L}(D,X), L(t) \in \mathcal{L}(D, \partial X)$, for $t \in \mathbb{R}$, are supposed to satisfy the following hypotheses:

(H1) There are positive constants $C_1$, $C_2$ such that

$$C_1\|x\|_D \leq \|x\| + \|A_{\max}(t)x\| \leq C_2\|x\|_D$$

for all $x \in D$ and $t \in \mathbb{R}$;

(H2) for each $x \in D$ the mapping $\mathbb{R} \ni t \mapsto A_{\max}(t)x \in X$ is continuously differentiable;

(H3) the operators $L(t) : D \to \partial X, t \in \mathbb{R}$, are surjective;

(H4) for each $x \in D$ the mapping $\mathbb{R} \ni t \mapsto L(t)x \in \partial X$ is continuously differentiable;

(H5) there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_{\partial X} \geq \gamma^{-1}(\lambda - \omega)||x||_X,$$

for $x \in \ker(\lambda\, \mathrm{id}_X - A_{\max}(t)), \lambda > \omega$ and $t \in \mathbb{R}$;

(H6) the family of operators $(A(t))_{t\in\mathbb{R}}$, $A(t) := A_{\max}(t)|_{\ker L(t)}$, is stable.

In the following lemma, we cite consequences of the above assumptions from [10, Lemma 1.2] which will be needed below.

**Lemma 2.1** *The restriction* $L(t)|_{\ker(\lambda\, \mathrm{id}_X - A_{\max}(t))}$ *is an isomorphism from* $\ker(\lambda\, \mathrm{id}_X - A_{\max}(t))$ *into* $\partial X$ *and its inverse* $L_{\lambda,t} := [L(t)|_{\ker(\lambda\, \mathrm{id}_X - A_{\max}(t))}]^{-1} : \partial X \to \ker(\lambda\, \mathrm{id}_X - A_{\max}(t))$ *satisfies*

$$\|L_{\lambda,t}\| \leq \gamma(\lambda - \omega)^{-1} \quad \text{for } \lambda > \omega,\ t \in \mathbb{R}.$$

Under assumptions (H1)-(H6), it was shown that the linear boundary Cauchy problem (2) is well-posed. More precisely, there exists an evolution family $(U(t,s))_{t\geq s}$ generated by the family of operators $(A(t))_{t\in\mathbb{R}}$. See [12, 13].

### 2.2 Nonlinear boundary Cauchy problems

In case $f \equiv 0$ the boundary Cauchy problem (1) reduces to the linear boundary Cauchy problem (2) which was studied in the last subsection under the assumptions (H1)-(H6). In particular, let $(U(t,s))_{t\geq s}$ denote the evolution family solution to the problem (2). We want to study nonlinear perturbations (1) of (2) and therefore assume that the nonlinearity $f$ satisfies:

(H7) The nonlinear part $f : \mathbb{R} \times X \to \partial X$ is assumed to be continuous and there exists a positive constant $\ell$ such that one has the global Lipschitz estimate

$$\|f(t,x) - f(t,\bar{x})\| \leq \ell\|x - \bar{x}\| \quad \text{for all } x, \bar{x} \in X, t \in \mathbb{R}.$$

Under the assumptions (H1)-(H7) the semilinear boundary Cauchy problem (1) admits a unique mild solution. For $s \in \mathbb{R}$, $x \in X$, a function $u = u(\cdot, s, x) : [s, \infty) \to X$ is called *mild solution of* (1) if it satisfies the integral equation

$$u(t,s,x) = U(t,s)x + \lim_{\lambda\to\infty} \int_s^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, u(\sigma,s,x))\, d\sigma, \quad t \geq s. \tag{5}$$

The unique existence follows with the usual contraction arguments (see e.g. [2,11,14]) and uses the *variation of constants formula* from [3] for solutions $v : [s,\infty) \to X$ of inhomogeneous boundary Cauchy problems, i.e. systems (1) with $f(t,u(t)) \equiv g(t)$ independent of $u(t)$

$$v(t) = U(t,s)x + \lim_{\lambda\to\infty} \int_s^\infty U(t,\sigma)\lambda L_{\lambda,\sigma} g(\sigma)\, d\sigma, \quad t \geq s.$$

Let us define on $X$ the family of operators:

$$V(t,s)x := u(t,s,x) \text{ for } x \in X \text{ and } t \geq s. \tag{6}$$

Our goal in the next section is to study the existence of pullback attractors for the family of operators $(V(t,s))_{t\geq s}$.

## 3 Pullback Attractors of Nonlinear Boundary Cauchy Problems

In this section, we consider the following system

$$\begin{cases} \frac{d}{dt}u(t) = A_{\max}(t)u(t), & t \in \mathbb{R}, \\ L(t)u(t) = f(t,u(t)), & t \in \mathbb{R}, \end{cases} \tag{7}$$

where $A_{\max}(t), L(t), f(t,x)$ are assumed to satisfy assumptions (H1)-(H7). We want to study the existence of pullback attractors of the nonlinear problem (7), therefore the evolution family $(U(t,s))_{t\geq s}$ associated with the linear problem (2) is assumed to satisfy the following:

(H8) $(U(t,s))_{t\geq s}$ is exponentially stable, that is, there exist constants $\alpha > 0$ and $M_1 \geq 1$ such that

$$\|U(t,s)\| \leq M_1 e^{-\alpha(t-s)}, \ t \geq s;$$

(H9) for all $t > s$, $U(t,s)$ is a compact operator on $X$.

To get our aim, we will use the following sufficient condition result shown in [5, Theorem 2.3].

**Theorem 3.1** *If $(V(t,s))_{t\geq s}$ is pullback strongly bounded dissipative and pullback asymptotically compact, then it has a pullback attractor $(\mathcal{A}(t))_{t\in\mathbb{R}}$ with the property that $\bigcup_{s\leq t} \mathcal{A}(s)$ is bounded for each $t \in \mathbb{R}$.*

The concepts of pullback strongly bounded dissipative and pullback asymptotically compact are given in the following definitions.

**Definition 3.1** We say that $(V(t,s))_{t\geq s}$ is *pullback strongly bounded dissipative* if, for each $t \in \mathbb{R}$, there is a bounded subset $B(t)$ of $X$ which pullback attracts bounded subsets of $X$ at time $t$, that is, given a bounded subset $B \subset X$ and $t \in \mathbb{R}$, there exists $s(t,B) \leq t$ such that $V(t,s)B \subset B(t)$ for all $s \leq s(t,B)$.

**Definition 3.2** We say that $(V(t,s))_{t\geq s}$ is *pullback asymptotically compact* if, for each $t \in \mathbb{R}$, sequence $(s_k)_{k\in\mathbb{N}}$ in $(-\infty,t]$ and bounded sequence $(x_k)_{k\in\mathbb{N}}$ in $X$ such that $s_k \longrightarrow -\infty$ as $k \to +\infty$ and $\{V(t,s_k)x_k \ : \ k \in \mathbb{N}\}$ is bounded, the sequence $(V(t,s_k)x_k)_{k\in\mathbb{N}}$ has a convergent subsequence.

We first show the following lemma.

**Lemma 3.1** *The family of operators $(V(t,s))_{t\geq s}$ is pullback strongly bounded dissipative provided that $M_1\gamma\ell - \alpha < 0$.*

***Proof.*** Let $x \in X$ and $t \geq s$. From (H7) we obtain

$$\|f(t,x)\| \leq \|f(t,0)\| + \|f(t,x) - f(t,0)\| \leq \|f(t,0)\| + \ell\|x\|.$$

We put $C := \sup_{t \in \mathbb{R}} \|f(t,0)\|$. Using (H8) and Lemma 2.1 we obtain

$$\begin{aligned}
\|V(t,s)x\| &\leq M_1 e^{-\alpha(t-s)}\|x\| + \lim_{\lambda\to\infty}\int_s^t M_1 e^{-\alpha(t-\sigma)}\frac{\lambda\gamma}{\lambda-\omega}\|f(\sigma,V(\sigma,s)x)\|d\sigma \\
&\leq M_1 e^{-\alpha(t-s)}\|x\| + M_1\gamma\int_s^t e^{-\alpha(t-\sigma)}(C + \ell\|V(\sigma,s)x\|)d\sigma \\
&\leq M_1 e^{-\alpha(t-s)}\|x\| + M_1\gamma C\int_s^t e^{-\alpha(t-\sigma)}\, d\sigma \\
&\quad + M_1\gamma\ell\int_s^t e^{-\alpha(t-\sigma)}\|V(\sigma,s)x\|\, d\sigma
\end{aligned}$$

then we get

$$\begin{aligned}
e^{\alpha t}\|V(t,s)x\| &\leq M_1 e^{\alpha s}\|x\| + M_1\gamma C\int_s^t e^{\alpha\sigma}\, d\sigma + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|V(\sigma,s)x\|\, d\sigma \\
&= M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha t} - e^{\alpha s}) + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|V(\sigma,s)x\|\, d\sigma
\end{aligned}$$

Using the generalized Gronwall's lemma we obtain

$$\begin{aligned}
e^{\alpha t}\|V(t,s)x\| &\leq M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha t} - e^{\alpha s}) \\
&\quad + \int_s^t \left[M_1 e^{\alpha s}\|x\| + \frac{M_1\gamma C}{\alpha}(e^{\alpha\sigma} - e^{\alpha s})\right] M_1\gamma\ell e^{\int_\sigma^t M_1\gamma\ell\, du}\, d\sigma \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} + M_1 e^{\alpha s}\|x\| e^{M_1\gamma\ell(t-s)} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)}e^{\alpha t} \\
&\quad - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)}e^{M_1\gamma\ell t}e^{(\alpha - M_1\gamma\ell)s} - \frac{M_1\gamma C}{\alpha}e^{\alpha s}e^{M_1\gamma\ell(t-s)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|V(t,s)x\| &\leq \frac{M_1\gamma C}{\alpha} + M_1 e^{-\alpha(t-s)}\|x\| e^{M_1\gamma\ell(t-s)} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} \\
&\quad - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)}e^{M_1\gamma\ell(t-s)}e^{(-\alpha(t-s)} - \frac{M_1\gamma C}{\alpha}e^{-\alpha(t-s)}e^{M_1\gamma\ell(t-s)} \\
&= \frac{M_1\gamma C}{\alpha} + \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} \\
&\quad + e^{(M_1\gamma\ell - \alpha)(t-s)}\left[M_1\|x\| - \frac{M_1\gamma\ell M_1\gamma C}{\alpha(\alpha - M_1\gamma\ell)} - \frac{M_1\gamma C}{\alpha}\right] \\
&= \frac{M_1\gamma C}{\alpha - M_1\gamma\ell} + \left(M_1\|x\| - \frac{M_1\gamma C}{\alpha - M_1\gamma\ell}\right)e^{(M_1\gamma\ell - \alpha)(t-s)}.
\end{aligned}$$

We have then

$$\|V(t,s)x\| \leq K + (M_1\|x\| - K)e^{-\beta s}e^{\beta t}$$

with $K := \dfrac{M_1\gamma C}{\alpha - M_1\gamma\ell}$ and $\beta := M_1\gamma\ell - \alpha$. By hypothesis, $\beta < 0$.
Since $(M_1\|x\| - K)e^{-\beta s} \longrightarrow 0$ as $s \longrightarrow -\infty$. Then for fixed $t \in \mathbb{R}$ and $x \in B$ bounded, there exists $s_0(t, B)$ such that $(M_1\|x\| - K)e^{-\beta s} < 1$ for all $s \le s_0(t, B)$. This implies

$$\|V(t,s)x\| \le K + e^{\beta t}.$$

We take $B(t) = B(0, K + e^{\beta t})$ the ball with center 0 and radius $K + e^{\beta t}$. Then the dissipativity of the family $(V(t,s))_{t\ge s}$ holds.

To get the main result, it remains to show that $V(t,s)$, $t \ge s$, is pullback asymptotically compact. To do that, from [5, Theorem 2.4], it is sufficient to prove the following lemma.

**Lemma 3.2** *There exist $(T(t,s))_{t\ge s}$ and $(R(t,s))_{t\ge s}$ such that $V(t,s) = T(t,s) + R(t,s)$, where*

*(i) $R(t,s)$, $t > s$, is compact,*

*(ii) there exists a non-increasing function $k : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ with $k(\sigma, r) \longrightarrow 0$ when $\sigma \to \infty$, and for all $s \le t$ and $x \in X$ with $\|x\| \le r$, $\|T(t,s)\| \le k(t-s, r)$.*

***Proof.*** Define the families of operators $R(t,s) := U(t,s)$ and

$$T(t,s) := \lim_{\lambda\to+\infty} \int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma, V(\sigma,s)\cdot))\, d\sigma. \tag{8}$$

The assertion (i) is satisfied by hypothesis (H9).

To prove (ii), we assume that $M_1\gamma l < \alpha$ and we will show that

$$\begin{aligned}\|T(t,s)x\| \ \le\ & M_1\|x\|e^{(M_1\gamma l-\alpha)(t-s)} - \frac{M_1\gamma C}{\alpha - M_1\gamma l}e^{(M_1 Ml-\alpha)(t-s)} \\ & -M_1\|x\|e^{-\alpha(t-s)} + \frac{M_1\gamma C}{\alpha - M_1 Ml}.\end{aligned}$$

In fact, we have

$$\begin{aligned}\|T(t,s)x\| \ \le\ & \lim_{\lambda\to\infty}\int_s^t M_1 e^{-\alpha(t-\sigma)}\frac{\lambda\gamma}{\lambda-\omega}\|f(\sigma, V(\sigma,s)x)\| d\sigma \\ \le\ & M_1\gamma \int_s^t e^{-\alpha(t-\sigma)}(C + \ell\|V(\sigma,s)x\|)d\sigma \\ \le\ & M_1\gamma C\int_s^t e^{-\alpha(t-\sigma)}\, d\sigma \\ & + M_1\gamma\ell\int_s^t e^{-\alpha(t-\sigma)}(\|U(\sigma,s)x\| + \|T(\sigma,s)x\|)\, d\sigma.\end{aligned}$$

Then we get

$$\begin{aligned} e^{\alpha t}\|T(t,s)x\| &\le M_1\gamma C\int_s^t e^{\alpha\sigma}\,d\sigma + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|U(\sigma,s)x\|\,d\sigma \\ &\quad + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma \\ &\le M_1\gamma C\int_s^t e^{\alpha\sigma}\,d\sigma + M_1 M_1\gamma\ell\int_s^t e^{\alpha s}\|x\|\,d\sigma \\ &\quad + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma \\ &= \frac{M_1\gamma C}{\alpha}(e^{\alpha t}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\ &\quad + M_1\gamma\ell\int_s^t e^{\alpha\sigma}\|T(\sigma,s)x\|\,d\sigma. \end{aligned}$$

Using the generalized Gronwall's lemma we obtain

$$\begin{aligned} e^{\alpha t}\|T(t,s)x\| &\le \frac{M_1\gamma C}{\alpha}(e^{\alpha t}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\ &\quad + \int_s^t \left[\frac{M_1\gamma C}{\alpha}(e^{\alpha\sigma}-e^{\alpha s}) + M_1M_1\gamma\ell e^{\alpha s}(\sigma-s)\|x\|\right] \\ &\qquad\qquad \times M_1\gamma\ell e^{\int_\sigma^t M_1\gamma\ell\,du}\,d\sigma \\ &= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\ &\quad + \int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha\sigma}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\ &\quad - \int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha s}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\ &\quad + \int_s^t M_1M_1\gamma_1\gamma\ell e^{\alpha s}(\sigma-s)\|x\|M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\ &= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\ &\quad + \int_s^t \frac{M_1\gamma C}{\alpha}M_1\gamma\ell e^{M_1\gamma\ell t}e^{(\alpha-M_1\gamma\ell)\sigma}\,d\sigma \\ &\quad - \int_s^t \frac{M_1\gamma C}{\alpha}e^{\alpha s}M_1\gamma\ell e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\ &\quad + M_1M_1\gamma\ell\|x\|M_1\gamma\ell e^{\alpha s}\int_s^t (\sigma-s)e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \end{aligned}$$

$$\begin{aligned}
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad + \frac{M_1\gamma C}{\alpha}M_1\gamma\ell e^{M_1\gamma\ell t}\frac{1}{\alpha - M_1\gamma\ell}\left(e^{(\alpha-M_1\gamma\ell)t} - e^{(\alpha-M_1\gamma\ell)s}\right) \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{\alpha s}\left(-1 + e^{M_1\gamma\ell(t-s)}\right) \\
&\quad + M_1M_1\gamma\ell\|x\|M_1\gamma\ell e^{\alpha s}\int_s^t(\sigma - s)e^{M_1\gamma\ell(t-\sigma)}\,d\sigma \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}e^{\alpha s} + M_1M_1\gamma\ell e^{\alpha s}(t-s)\|x\| \\
&\quad + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{M_1\gamma\ell t}\left(e^{(\alpha-M_1\gamma\ell)t} - e^{(\alpha-M_1\gamma\ell)s}\right) \\
&\quad + \frac{M_1\gamma C}{\alpha}e^{\alpha s} - \frac{M_1\gamma C}{\alpha}e^{\alpha s}e^{M_1\gamma\ell(t-s)} \\
&\quad + M_1M_1\gamma\ell\|x\|M_1\gamma\ell\left[-\frac{(t-s)}{M_1\gamma\ell} - \frac{1}{(M_1\gamma\ell)^2}(1 - e^{M_1\gamma\ell(t-s)})\right] \\
&= \frac{M_1\gamma C}{\alpha}e^{\alpha t} + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{\alpha t} - \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{\alpha s}e^{M_1\gamma\ell(t-s)} \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{\alpha s}e^{M_1\gamma\ell(t-s)} - \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{\alpha s}\|x\| + \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{\alpha s}\|x\|e^{M_1\gamma\ell(t-s)}.
\end{aligned}$$

Multiplying both sides by $e^{-\alpha t}$, we get

$$\begin{aligned}
\|T(t,s)x\| &\leq \frac{M_1\gamma C}{\alpha} + \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell} - \frac{M_1\gamma C}{\alpha}\frac{M_1\gamma\ell}{\alpha - M_1\gamma\ell}e^{(M_1\gamma\ell-\alpha)(t-s)} \\
&\quad - \frac{M_1\gamma C}{\alpha}e^{(M_1\gamma\ell-\alpha)(t-s)} - \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}e^{-\alpha(t-s)}\|x\| \\
&\quad + \frac{M_1M_1\gamma\ell}{M_1\gamma\ell}\|x\|e^{(M_1\gamma\ell-\alpha)(t-s)} \\
&= M_1\|x\|e^{(M_1\gamma l-\alpha)(t-s)} - \frac{M_1\gamma C}{\alpha - M_1\gamma l}e^{(M_1\gamma l-\alpha)(t-s)} \\
&\quad - M_1\|x\|e^{-\alpha(t-s)} + \frac{M_1\gamma C}{\alpha - M_1\gamma l}.
\end{aligned}$$

To end the proof, we take the function $k(\cdot,\cdot)$ as follows

$$k(\sigma, r) = M_1re^{\beta\sigma} + \frac{M_1\gamma C}{\beta}e^{\beta\sigma} - M_1re^{-\alpha\sigma} - \frac{M_1\gamma C}{\beta}$$

with $\beta := M_1\gamma l - \alpha$. Since, by hypothesis, $\beta < 0$, it is clear that $k(t,s)$ satisfies assertion (ii). Then the proof is achieved.

From the previous lemmas, we are now ready to state our main result.

**Theorem 3.2** *Assume that* (7) *satisfies the assumptions (H1)-(H9) with* $M_1\gamma l < \alpha$. *Then the family of operators* $(V(t,s))_{t\geq s}$ *has a pullback attractor* $(\mathcal{A}(t))_{t\in\mathbb{R}}$ *with the property that* $\bigcup_{s\leq t}\mathcal{A}(s)$ *is bounded for each* $t \in \mathbb{R}$.

## 4 Application

consider the following reaction diffusion equation

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) - \beta(t)v(t,x), t \geq 0, x \in [0,1], \\ \dfrac{\partial}{\partial x}v(t,0) = g_1(t,v);\ \dfrac{\partial}{\partial x}v(t,1) = g_2(t,v) \quad t \geq 0, \\ v(0,x) = v_0(x), \quad x \in [0,1]. \end{cases} \tag{9}$$

Here $\beta(\cdot)$ is a continuously differentiable positive function. Moreover, we assume that

(i) There exist positive constants $\overline{\beta}$ and $\underline{\beta}$ such that $\underline{\beta} \leq \beta(t) \leq \overline{\beta}$ for all $t \geq 0$.

(ii) $g_1 : \mathbb{R}^+ \times L^1[0,1] \longrightarrow \mathbb{R}$ and $g_2 : \mathbb{R}^+ \times L^1[0,1] \longrightarrow \mathbb{R}$ are continuous functions and globally Lipschitz with respect to the second variable uniformly to the first one.

Our aim is to write equation (9) as a boundary Cauchy problem of the form (7) satisfying the assumptions (H1)-(H9). For this purpose, we define the Banach spaces

$$\partial X := \mathbb{R}^2,\ X := L^1[0,1] \text{ and } D := W^{2,1}[0,1],$$

with

$$W^{2,1}[0,1] = \Big\{ u \in L^1[0,1] \mid u^{'}, u^{''} \in L^1[0,1] \Big\}$$

endowed with the norm

$$\|u\|_D := \|u\|_1 + \|u'\|_1 + \|u''\|_1 \text{ for } u \in W^{2,1}[0,1].$$

Here $\|u\|_1$ denotes the norm of $L^1[0,1]$.

$(D, \|\cdot\|_D)$ is a Banach space dense and continuously embedded in $X$.

For each $t \geq 0$ the operator $A_{\max}(t) : X \to X$ is defined by $D(A_{\max}(t)) = D$ and

$$(A_{\max}(t)\varphi)(a) = \varphi^{''} - \beta(t)\varphi \tag{10}$$

for all $\varphi \in D$.

For each $t \geq 0$, we define $L(t) : D \longrightarrow \partial X$ by

$$L(t)\varphi = (\varphi^{'}(0), \varphi^{'}(1))^T \quad \text{for all } \varphi \in D. \tag{11}$$

We show now that the hypotheses (H1)–(H9) are satisfied.

Verification of (H1): since, from [4, Remarque 11], the norms $\|\varphi\|_D$ and $\|\varphi\|_1 + \|\varphi''\|_1$ are equivalent in $D$, then (H1) holds.

Verification of (H2): holds from assumptions on $t \to \beta(t)$.

Verification of (H3): to show the surjectivity of $L(t)$, let $(a,b) \in \mathbb{R}^2$ be arbitrary. Define

$$u(x) = bx + a(1-x) \text{ for all } x \in [0,1].$$

We have $u \in D$ and $L(t)u = (a,b)$. Therefore $L(t)$ is surjective.

Verification of (H4): is obvious since $L(t)$ is independent of $t$.

Verification of (H5): let $u \in ker(\lambda - A_{\max}(t))$ for $\lambda > \overline{\beta}$, then there exists $(a,b) \in \mathbb{R}^2$ such that $u(x) = ae^{\alpha(t)} + be^{-\alpha(t)}$ for $x \in [0,1]$ with $\alpha(t) := \sqrt{\lambda - \beta(t)}$. We have

$$|u(x)| = |ae^{\alpha(t)x} + be^{-\alpha(t)x}| \leq |a|e^{\alpha(t)x} + |b|e^{-\alpha(t)x} = \frac{1}{\alpha(t)}\left[|a|\frac{d}{dx}e^{\alpha(t)x} - |b|\frac{d}{dx}e^{-\alpha(t)x}\right].$$

Integrating both sides on $x$, one can have

$$\begin{aligned}\int_0^1 |u(x)|\,dx &\leq \frac{1}{\alpha(t)}\int_0^1 |a|\frac{d}{dx}e^{\alpha(t)x} - |b|\frac{d}{dx}e^{-\alpha(t)x}\,dx \\ &= \frac{1}{\alpha(t)}\left[|a|e^{\alpha(t)} - |b|e^{-\alpha(t)} - |a| + |b|\right] \\ &\leq \frac{1}{\alpha(t)}\left[|ae^{\alpha(t)} - be^{-\alpha(t)}| + |a-b|\right] \\ &= \frac{1}{\alpha^2(t)}(|u^{'}(0)| + |u^{'}(1)|).\end{aligned}$$

We obtain then $\|Lu\|_{\mathbb{R}^2} \geq (\lambda - \overline{\beta})\|u\|_1$. This shows (H5) with $\gamma = 1$ and $\omega = \overline{\beta}$.

Verification of (H6): Define the operator

$$Au := \Delta u, \quad D(A) = \left\{u \in W^{2,1}[0,1] \mid u^{'}(0) = u^{'}(1) = 0\right\}.$$

It is known that $A$ generates an immediately compact analytic semigroup $(T(t))_{t\geq 0}$ of contraction on the Banach space $L^1[0,1]$, that is $T(t)$ is compact for all $t > 0$ and

$$\|T(t)u\| \leq 1 \text{ for } t \geq 0 \text{ and } u \in L^1[0,1]. \tag{12}$$

See, for example, [8]. Then from Hille-Yosida theorem (see [8, Theorem II.3.8]), $\forall\lambda > 0$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}.$$

Then, for every $\lambda + \underline{\beta} > 0$, we have $\lambda \in \rho(A(t))$. Moreover,

$$R(\lambda, A(t)) = R(\lambda + \beta(t), A).$$

Therefore,

$$\|R(\lambda, A(t))\| \leq \frac{1}{\lambda + \underline{\beta}} = \frac{1}{\lambda - (-\underline{\beta})},$$

by Remark 2.1, it follows that

$$\Big\|\prod_{i=1}^{n} R(\lambda, A(t_i))\Big\| \leq \frac{1}{(\lambda - (-\underline{\beta}))^n}$$

for $\lambda > -\underline{\beta}$ and any finite sequence $0 \leq t_1 \leq \cdots \leq t_n$. Hence (H6) is satisfied.

Verification of (H7): Follows from assumptions on the functions $g_1$ and $g_2$.

Verification of (H8): We note that the evolution family $(U(t,s))_{t\geq s}$ generated by $(A(t))_{t\geq 0}$ is given by

$$U(t,s) = exp\left(\int_s^t -\beta(\sigma)\,d\sigma\right)T(t-s),\ t \geq s \geq 0.$$

Then, from (12) one can see that $\|U(t,s)\| \leq e^{-\underline{\beta}(t-s)},\ t \geq s$. Hence $(U(t,s))_{t\geq s}$ is exponentially stable and (H8) holds.

Verification of (H9): Is obvious from the fact that the semigroup $T(t)$ is compact for all $t > 0$.

### Acknowledgment

The author thanks Doan Thai Son for a helpful discussion concerning this work.

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