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# Integrable Time-Dependent Dynamical Systems: Generalized Ermakov-Pinney and Emden-Fowler Equations

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**Abstract:** We consider the integrable time-dependent classical dynamics studied by Bartuccelli and Gentile (Phys Letts. **A307** (2003) 274–280; Appl. Math. Lett. **26** (2013) 1026–1030) and show its power to compute the first integrals of the (generalized) Ermakov-Pinney systems. A two component generalization of the Bartuccelli-Gentile equation is also given and its connection to Ermakov-Ray-Reid system and coupled Milne-Pinney equation has been illucidated. Finally, we demonstrate its application in other integrable ODEs, in particular, using the spirit of Bartuccelli-Gentile algorithm we compute the first integrals of the Emden-Fowler and describe the Lane-Emden type equations. A number of examples are given to illustrate the procedure.

**Keywords:** *time-dependent harmonic oscillator; Ermakov-Pinney equation; first integrals; Ermakov-Lewis invariant; Emden-Fowler equation.*

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## 1 Introduction

The study of nonlinear time-dependent ordinary differential equations (ODEs) has been going on for several years now. Since there are hardly any general methods for dealing with such equations one is often forced to look for interesting transformations which either enable us to simplify the equation or to map it to some linear or nonlinear equation whose features are already known. The linear harmonic oscillator has been a time honored favorite and has enhanced our understanding of several key areas of mathematics and physics. It has the added advantage of being a Hamiltonian system and serves as a first approximation for many nonlinear differential equations. In [4] Bartuccelli and Gentile made a beautiful observation regarding the equation of a linear harmonic oscillator,

$$\ddot{x} + \omega^2 x = 0. \tag{1.1}$$

Here the over dot represents differentiation with respect to the independent variable $t$. As is well known its solution is

$$x(t) = A\sin(\omega t + \phi), \tag{1.2}$$

where $A$ and $\phi$ are arbitrary constants representing the amplitude and phase respectively. They observed that if (1.1), which may also be written as

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega}\right) + \omega x = 0, \tag{1.3}$$

one assumes that $\omega$, instead of being a constant, is any arbitrary function of the independent variable $t$ so that one actually has the following equation:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)x = 0, \tag{1.4}$$

then its solution is similar in structure to (1.2) in the sense that

$$x(t) = A\sin(\int \omega(t)dt + \phi). \tag{1.5}$$

It was stressed in [4,5] that the equation in the form (1.4) is still quite interesting and can be generalized to various directions and gives new results. Our main aim is to explore all these directions in this paper.

It is obvious that (1.4) is not reducible to the equation of a time-dependent linear harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0. \tag{1.6}$$

Nevertheless the fact that the solution of (1.4) clearly reduces to that of the usual harmonic oscillator when $\omega$ is a constant is indeed remarkable. In fact the following generalization is also possible, namely we replace (1.4) by

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)F(x) = 0, \tag{1.7}$$

where $F(x)$ is some nonlinear $C^1$ function of $x$. Note that (1.7) may be written as the following system

$$\dot{x} = \omega(t)y, \quad \dot{y} = -\omega(t)\frac{dU(x)}{dx}, \tag{1.8}$$

with $F(x) = dU/dx$. In this paper equations (1.4) and (1.7) will be called the Bartuccelli-Gentile equations. In general for the linear equation

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0, \tag{1.9}$$

one can make use of Jacobi's Last Multiplier to derive a suitable Lagrangian. Indeed the last multiplier turns out to be the integrating factor of such an equation given by

$$M = \exp(\int^t P(s)ds).$$

The relationship between a last multiplier and the Lagrangian is given by

$$M = \frac{\partial^2 L}{\partial \dot{x}^2},$$

from which it follows that a Lagrangian for (2.12) is given by

$$L(t, x, \dot{x}) = e^{\int P(t)dt}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}Q(t)x^2\right). \tag{1.10}$$

By using the standard Legendre transformation it follows that the corresponding Hamiltonian is

$$H(t, x, p_x) = \frac{1}{2}\left(e^{-\int P(t)dt}p_x^2 + Q(t)e^{\int P(t)dt}x^2\right), \tag{1.11}$$

where the conjugate momentum is defined in the usual manner

$$p_x = \frac{\partial L}{\partial \dot{x}} = e^{\int P(t)dt}\dot{x}.$$

In case of (1.4) it is clear that $P(t) = -\dot{\omega(t)}/\omega(t)$ and $Q(t) = \omega^2(t)$, so that $M = \omega^{-1}(t)$, and the Hamiltonian therefore assumes the form

$$H = \frac{1}{2}\omega(t)\left(p_x^2 + x^2\right).$$

Note that (1.7) admits the following first integral

$$I(x, \dot{x}, t) = \frac{1}{2}\left(\frac{\dot{x}}{\omega(t)}\right)^2 + U(x), \tag{1.12}$$

where $U(x)$ is a primitive of $F(x)$, as is easy to verify. Clearly the level sets $I(x, \dot{x}, t) = E$ allow us to write

$$\int \frac{dx}{\sqrt{E - U(x)}} = \pm\sqrt{2}\int \omega(t)dt,$$

which in turn means that it is effectively a time-reparametrization of the usual time-independent case. The invariant (1.12) will be referred to as the Bartuccelli-Gentile invariant. The special case of $F(x) = x$ allows us to express this invariant as

$$I = \frac{1}{2}\left(p_x^2 + x^2\right),$$

where the definition of $p_x = \dot{x}/\omega$ has been used, and write

$$H(t, x, p_x) = I\omega.$$

Clearly the invariant $I$ must have the dimension of action. It can be readily seen that $\frac{dH}{dt} \neq 0$, as is to be expected of a dissipative system. It is also necessary to mention that the expressions for the Lagrangian and Hamiltonian given in (2.32) and (2.36) reduce to those of Caldirola [9] and Kanai [22] when $P(t) = \gamma(t)$ and the case of $H = I\omega$ also appeared in connection with the derivation of Hannay's angle in [42].

The celebrated Ermakov-Pinney equation (see [21] for brief introduction) was introduced in the nineteenth century by V.P. Ermakov [15] to find the first integral for the time-dependent harmonic oscillator. In 1950 E. Pinney [34] found the solution of this equation. Ermakov systems have been extensively studied in physics as they often arise in the context of Bose-Einstein condensates, cosmological models, plasma confinements etc. Lewis [28, 29] found independently an exact invariant for this system. Several methods have subsequently been devised for the derivation of the Lewis invariant, which was originally obtained in closed form through an application of the asymptotic theory of Kruskal [24]. Leach [26] has obtained the same result using a time-dependent canonical transformation. On the other hand Lutzky's [30] derivation was based on Noether's theorem. Moyo and Leach [31] used Noether symmetries to discuss the source of the Ermakov-Lewis invariant. Ray and Reid [37, 38] by resurrecting Ermakov's original technique were able to obtain the existence of a Lewis-type invariant for the case of two coupled nonlinear equations. Grammaticos and Dorizzi [20] proposed a direct method to investigate the existence of an exact invariant for $2D$ time-dependent Hamiltonian systems. The construction of Bartuccelli and Gentile didn't consider the Ermakov issue. Although it is clear from their construction that there should be an explicit link between the Ermakov-Pinney equation and the Bartuccelli-Gentile equation.

The Emden-Fowler equation was first studied in an astrophysical context by Emden [14] and subsequently by Fowler who was instrumental in laying its mathematical foundation [16]. The celebrated Emden-Fowler equation appears in many areas in physics [33]. More recently Berry and Shukla [7] presented a class of models for particles moving under *curl forces* alone. They could not find closed-form solutions for general motions, but the dynamics can be reduced to the Emden-Fowler equation, for which a particular exact solution exists for a wide class of cases. In the study of stellar structure a star is usually considered as a gaseous sphere in thermodynamic and hydrostatic equilibrium described by a certain equation of state. In particular the polytropic equation of state yields the Lane-Emden equation, given by

$$xy'' + 2y' + xy^n = 0.$$

This was originally proposed by Jonathan Lane [25] and was analysed by R. Emden [14]. Several applications of the Emden-Fowler and Lane-Emden equations of various forms arising in astrophysics [11] and nonlinear dynamics have been reported. The Lane-Emden equation also arises in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemical-reaction systems. A detailed account, though somewhat dated, can be found in the survey by Wong [43].

In recent years this equation has been generalized in many ways. For example, Goenner [17] studied a generalized class of the Lane-Emden equation

$$xy'' + k_1 y' + k_2 x^\nu y^n = 0, \quad \text{first kind},$$

$$y'' + f(x)y' + g(x)y^n = 0, \quad \text{second kind}.$$

Kara and Mahomed [23] showed that when $n = -3$ the Lane-Emden equation,

$$y'' + (k/x)y' = \sigma x^w y^n, \qquad n \neq 0, 1, \ \sigma \neq 0,$$

generates the three-dimensional algebra $sl(2, R)$ in which case general solutions are known for $w = -2k$. Ranganathan [35, 36] has obtained solutions and first integrals for some classes of the Emden-Fowler equation.

### 1.1 Motivation, result and organization

In this paper we explore two important sets of integrable ODEs, namely, the Ermakov-Pinney systems and the Emden-Fowler systems. Many papers were devoted to the construction of the first integrals of these set of equations. We demonstrate in this survey that one can give a unified method to describe the first integrals of all these equations using Bartuccelli-Gentile's method.

At first we show how the Bartuccelli-Gentile invariant can be mapped to invariants of Ermakov type systems, then we present the two-component generalization of the Bartuccelli-Gentile construction. We extend their method to compute the first integrals of the Emden-Fowler equations and second first integrals for Lane-Emden type systems. It is true that the first integrals for many of these equations have already been found by means of a variety of different methods [6, 8, 19, 27, 35, 36, 40, 41]. In this paper we give an alternative and easy method to compute the first integrals of the Emden-Fowler class of equations.

This paper is organized as follows. In Section 2 we give an intimate connection between the Bartuccelli-Gentile construction and the Ermakov-Pinney equation, and extend this connection to coupled system also. We illustrate our construction through examples. Section 3 is devoted to Emden-Fowler type equations. We show just extending slightly the method of Bartuccelli-Gentile's construction one can easily obtain the first integrals of the Lane-Emden equations.

## 2 Ermakov-Pinney Equation and Bartuccelli-Gentile Construction

We begin by considering the equation of motion of a linear harmonic oscillator with time-dependent frequency, namely,

$$\ddot{x} + \omega^2(t)x = 0. \tag{2.1}$$

The problem of the time-dependent oscillator was first solved by Ermakov [15] who obtained an invariant for (2.1) by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}. \tag{2.2}$$

Equation (2.2) is usually called the Ermakov-Pinney equation since Pinney provided the solution, some years after Ermakov's derivation of its first integral [34]. Ermakov obtained a first integral for the system of equations (2.1) and (2.2), by first of all eliminating $\omega^2(t)$ by multiplying (2.1) with $\rho$ and (2.2) with $x$ and subtracting the two and then finally by multiplying the resulting equation with the integrating factor $(\dot{x}\rho - x\dot{\rho})$. The resulting first integral is given by

$$I = \frac{1}{2}\big[(\rho\dot{x} - \dot{\rho}x)^2 + (x/\rho)^2\big], \tag{2.3}$$

and is called the Ermakov-Lewis invariant after Lewis independently *recalculated* it in 1966.

As mentioned in the previous section equation (1.7) which is explicitly given by

$$\ddot{x} - \frac{\dot{\omega}}{\omega}\dot{x} + \omega^2(t)F(x) = 0, \tag{2.4}$$

admits the first integral (1.12). Upon introducing the substitution

$$x = \frac{y}{\rho}, \tag{2.5}$$

into the first integral (1.12) the latter has the following appearance

$$I = \frac{1}{2}\left(\frac{\rho\dot{y} - y\dot{\rho}}{\omega(t)\rho^2}\right)^2 + U(y/\rho). \tag{2.6}$$

The transformation (2.5) is a particular case of a general transformation contained in Magnus and Winkler's book [32]. Moreover, under the above change of variables, (2.4) becomes

$$\frac{\rho\ddot{y} - y\ddot{\rho}}{\rho^2} - \left(\frac{\rho\dot{y} - y\dot{\rho}}{\rho^2}\right)\left(\frac{\dot{\omega}}{\omega} + 2\frac{\dot{\rho}}{\rho}\right) + \omega^2(t)F(y/\rho) = 0. \tag{2.7}$$

Setting

$$\frac{\dot{\omega}}{\omega} + 2\frac{\dot{\rho}}{\rho} = 0$$

so that

$$\omega(t)\rho^2 = c(>0) \text{ then leads to } \rho^2 = \frac{c}{\omega(t)}, \tag{2.8}$$

and causes (2.7) after partial elimination of the variable $\rho$, to reduce to the following equation (assuming $c = 1$),

$$\ddot{y} + \frac{1}{2}\left(\frac{\ddot{\omega}}{\omega} - \frac{3}{2}\left(\frac{\dot{\omega}}{\omega}\right)^2\right)y + \omega^2\rho F(y/\rho) = 0. \tag{2.9}$$

In view of (2.8) the first integral (2.6) therefore becomes

$$I = \frac{1}{2}\left(\frac{\rho\dot{y} - \dot{\rho}y}{c}\right)^2 + U(y/\rho).$$

Such a form of the first integral is suggestive of a deeper relation with the Ermakov system. Indeed if one assumes $F(x) = x$, then clearly (2.9) reduces to the time-dependent linear harmonic oscillator equation,

$$\ddot{y} + \Omega^2(t)y = 0, \tag{2.10}$$

with

$$\Omega^2(t) = \omega^2(t) + \frac{1}{2}\left(\frac{\ddot{\omega}}{\omega} - \frac{3}{2}\left(\frac{\dot{\omega}}{\omega}\right)^2\right). \tag{2.11}$$

On the other hand elimination of $y$ from (2.7) leads to

$$\ddot{\rho} + (\Omega^2(t) - w^2(t))\rho = 0,$$

which in view of (2.8) is equivalent to the equation

$$\ddot{\rho} + \Omega^2(t)\rho = \rho^{-3}. \tag{2.12}$$

We are thus led to the following proposition.

**Proposition 2.1** *Given the second-order linear time-dependent differential equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)x(t) = 0, \tag{2.13}$$

*then under the transformation $x = y/\rho$, the equation is equivalent to the coupled system*

$$\ddot{y} + \Omega^2(t)y = 0, \qquad \ddot{\rho} + \Omega^2(t)\rho = \rho^{-3}, \tag{2.14}$$

*provided $\rho^2\omega = 1$, where $\Omega^2(t)$ is defined by (2.11). The solution $x = \sin\big(\int \omega(t)\,dt\big)$ of the time-dependent equation (2.13) can also be mapped to the solution of the $(y, \rho)$ pair of equations.*

As to the proof of the latter part of the above proposition we note that the solution of the Bartuccelli-Gentile equation is $x = \sin\big(\int \omega(t)\,dt\big)$. Consequently substituting $x = y/\rho$ we obtain $y = \rho\sin\big(\int 1/\rho^2\,dt\big)$, which is a solution of

$$1 = \frac{\rho}{\sqrt{\rho^2 - y^2}}(\dot{y}\rho - \dot{\rho}y). \tag{2.15}$$

Differentiating (2.15) one can easily obtain the TDHO and the Ermakov-Pinney equations.□

### 2.1 Generalized Ermakov-Pinney equations

By an unbalanced Ermakov system [1] is meant a coupled second-order nonlinear system of the form

$$\ddot{x} + \omega_1^2(t)x = x^{-3}f(y/x), \qquad \ddot{y} + \omega_2^2(t)y = y^{-3}g(x/y), \tag{2.16}$$

where $f$ and $g$ are arbitrary functions of their arguments and where in general $\omega_1 \neq \omega_2$. When $\omega_1 = \omega_2$ the system is said to be balanced. Systems of the former type were studied by Ray and Reid [38] and as a result (2.16) is also known as the Ermakov-Ray-Reid system.

A crucial property of the balanced Ermakov system (i.e., when $\omega_1 = \omega_2 = \omega(t)$) is that it possesses an invariant, given by

$$I_{ERR} = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + \int^{y/x} [uf(u) - u^{-3}g(u)]du. \tag{2.17}$$

The invariance of $I_{ERR}$ can be directly verified by checking that $dI_{ERR}/dt = 0$ along the trajectories of the Ermakov-Ray-Reid system.

The generalized Ermakov-Pinney equation is an Ermakov system in two-dimension given by a pair of coupled nonlinear second-order differential equations of the form

$$\ddot{x} + \omega^2(t)x = \frac{1}{yx^2}f(y/x), \qquad \ddot{y} + \omega^2(t)y = \frac{1}{xy^2}g(x/y), \tag{2.18}$$

where $f$ and $g$ are once again arbitrary functions of their arguments. This coupled system possesses the Lewis-Ray-Reid invariant

$$I_{GE} = \frac{1}{2}(x\dot{y} - \dot{x}y)^2 + U(y/x), \tag{2.19}$$

where $U(y/x) = \int^{y/x} f(u)du + \int^{x/y} g(u)du$.

We can generalize this result to the time-dependent damped harmonic oscillator equation

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0, \tag{2.20}$$

in which case the invariant turns out to be

$$I_{dampedTD} = \frac{1}{2}\Big((x/\rho)^2 + (\dot{\rho}x - \rho\dot{x})^2 \exp\big(2\int_0^t P(t)\,dt\big)\Big) \tag{2.21}$$

with $\rho(t)$ satisfying the equation

$$\ddot{\rho} + P(t)\dot{\rho} + Q(t)\rho = \rho^{-3}\exp\big(-2\int_0^t P(t)\,dt\big). \tag{2.22}$$

The invariant $I_{dampedTD}$ of the damped time-dependent oscillator equation is called the Eliezer-Grey invariant.

**Proposition 2.2** *The Eliezer-Grey invariant may be mapped to that of the time-dependent harmonic oscillator (TDHO) equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)x(t) = 0$$

*by setting* $P = -\dot{w}/w$, $Q = w^2(t)$ *and* $\rho = 1$.

***Proof.*** If we expand the time-dependent equation we can easily map it to damped TDHO provided $P = -\dot{\omega}/\omega$ and $Q = \omega^2(t)$. Hence we obtain

$$\exp\big(2\int_0^t P(t)\,dt\big) = \frac{1}{\omega^2}.$$

If we put $\rho = 1$, then from the Eliezer-Grey invariant we obtain the invariant

$$I = \frac{1}{2}\left(\frac{\dot{x}^2}{\omega^2(t)} + x^2\right). \quad \square$$

## 2.2 Ermakov-Ray-Reid system and Bartuccelli-Gentile construction

Let us consider the generalized time-dependent system

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)F(x) = 0,$$

where $F(x)$ is some nonlinear $C^1$ function of $x$, such that

$$F(x(t)) = \frac{1}{x^3}g(x^{-1}) + xf(x). \tag{2.23}$$

**Proposition 2.3** *Given the second-order nonlinear time-dependent differential equation*

$$\frac{d}{dt}\left(\frac{\dot{x}}{\omega(t)}\right) + \omega(t)F(x) = 0,$$

*if $x = y/\rho$, then this equation may be transformed to the coupled system:*

$$\ddot{y} + \frac{1}{y^3}g\big(\frac{\rho}{y}\big) = 0, \qquad \ddot{\rho} + \frac{1}{\rho^3}f\big(\frac{y}{\rho}\big) = 0. \tag{2.24}$$

***Proof.*** By direct calculation. □

Moreover, if we set $f = g = 1$ then an invariant can be readily found as

$$I = c_1\big(\frac{y}{\rho}\big)^2 + c_2\big(\frac{\rho}{y}\big)^2 + (y\dot{\rho} - \dot{y}\rho)^2.$$

**Proposition 2.4** *Given the matrix second-order linear time-dependent differential equation*

$$\frac{d}{dt}\big(\Theta^{-1}\dot{X}\big) + \Theta X = 0, \tag{2.25}$$

*where $\Theta = \Theta(t)$ is a differentiable function, such that its entries are all positive functions of time. This system has a first integral of motion given by*

$$H = \frac{1}{2}\big( < \Theta^{-1}\dot{X}, \Theta^{-1}\dot{X} > + < X, X > \big) = \mathbb{E} = \textit{ constant }. \tag{2.26}$$

***Proof.*** By explicit differentiation. □

Let

$$\Theta = \begin{pmatrix} \omega_1(t) & \omega_0(t) \\ \omega_0(t) & \omega_2(t) \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the time-dependent matrix equation yields

$$\ddot{x} - \frac{1}{\Delta}\big((\dot{\omega_1}\omega_2 - \dot{\omega_0}\omega_0)\dot{x}) + (\dot{\omega_0}\omega_1 - \dot{\omega_1}\omega_0)\dot{y}\big) + (\omega_1^2 + \omega_2^2)x + \omega_0(\omega_1 + \omega_2)y = 0, \tag{2.27}$$

$$\ddot{y} - \frac{1}{\Delta}\big((\dot{\omega_2}\omega_1 - \dot{\omega_0}\omega_2)\dot{y}) + (\dot{\omega_0}\omega_2 - \dot{\omega_2}\omega_0)\dot{x}\big) + (\omega_2^2 + \omega_2^2)x + \omega_0(\omega_1 + \omega_2)x = 0, \tag{2.28}$$

where $\Delta(t) = \omega_1\omega_2 - \omega_0^2$. We consider now a special case.

Suppose $\omega_0 = 0$ and $\omega_1 \neq \omega_2$ then we obtain two decoupled equations of the form deduced earlier by Bartuccelli and Gentile, *viz*

$$\frac{d}{dt}\big(\frac{\dot{x}}{\omega_1}\big) + \omega_1 x = 0, \qquad \frac{d}{dt}\big(\frac{\dot{y}}{\omega_2}\big) + \omega_2 y = 0. \tag{2.29}$$

Finally if we define $\omega_2 = i\omega_1 \equiv \omega$ and $z = x + iy$, then equations (2.27) and (2.28) can be expressed as the following single complex differential equation

$$\frac{d}{dt}\big(\frac{\dot{z}}{\omega}\big) - i\omega z = 0. \tag{2.30}$$

**Proposition 2.5** *The complex version of the Bartuccelli-Gentile equation has a first integral of motion given by*

$$I_{complex} = \frac{1}{2}\left(\frac{\dot{z}}{\omega}\right)^2 - iz^2. \tag{2.31}$$

***Proof.*** By explicit differentiation we may obtain the desired first integral. □

### 2.3 Integrable coupled Milne-Pinney type dissipative systems

The study of coupled nonlinear ordinary differential equations of Ermakov-type originated in 1880 and in modern days the classical Ermakov-Pinney system was extended by Ray-Reid [37]. There is a class of Ermakov systems [2] given by

$$\ddot{q} + \omega^2(t)q = \frac{1}{q^3}f(q/p), \qquad \ddot{p} + \omega^2(t)p = \frac{1}{p^3}g(p/q), \tag{2.32}$$

where $\omega(t)$, $f$ and $g$ are essentially arbitrary functions of their arguments. In this case the Lewis-Ray-Reid invariant is

$$I = \frac{1}{2}(q\dot{p} - \dot{q}p)^2 - \int^{q/p} \left(u^{-3}f(u) - u\, g(u)\right)^2 du. \tag{2.33}$$

We propose to study, in this section, the following time-dependent generalization of (2.32)

$$\frac{d}{dt}\Big(\frac{\dot{q}}{\omega(t)}\Big) + \omega(t)q = \frac{\omega(t)}{q^3}f(q/p), \qquad \frac{d}{dt}\Big(\frac{\dot{p}}{\omega(t)}\Big) + \omega(t)p = \frac{\omega(t)}{p^3}g(p/q). \tag{2.34}$$

In the following proposition an invariant of this system of coupled equation is provided.

**Proposition 2.6** *The first integral of the coupled integrable Bartuccelli-Gentile equation of type (2.34) is*

$$I = \frac{1}{2}\frac{(\dot{q}p - q\dot{p})^2}{\omega(t)^2} + \int^{p/q} \left(uf(u^{-1}) - \frac{1}{u^3}g(u)\right) du, \tag{2.35}$$

*where $\omega(t)$ is a differentiable function.*

We can extend this result to a more general case. Consider the following generalized Ermakov system

$$\frac{d}{dt}\Big(\frac{\dot{q}}{\omega(t)}\Big) + \omega(t)q = \omega(t)q^m p^n f(q/p), \qquad \frac{d}{dt}\Big(\frac{\dot{p}}{\omega(t)}\Big) + \omega(t)p = \omega(t)q^n p^m g(p/q), \tag{2.36}$$

where $\omega(t)$ is a differentiable positive function.

**Proposition 2.7** *The system (2.36) has a first integral of motion given by*

$$I = \frac{1}{2}\frac{(\dot{q}p - q\dot{p})^2}{\omega(t)^2} + \int^{p/q} \left(u^{n+1}f - \frac{1}{u^{n+3}}g\right)du, \tag{2.37}$$

*where $m = -(n+3)$ and $u = p/q$.*

#### 2.3.1 Generalized Ince equation and coupled Bartuccelli-Gentile equation

Consider the class of second-order homogeneous differential equations

$$\frac{d^2p}{dt^2} + \frac{\alpha + \beta\cos 2t + \gamma\cos 4t}{(1 + a\cos 2t)^2}p = 0, \qquad \text{where } |a| < 1. \tag{2.38}$$

It is a four parameter family of Hill's equation which has been christened as the Ince equation by Magnus and Winkler [32]. A subclass of this system was studied by Athorne [3], and is given by

$$\frac{d^2p}{dt^2}+\big(1+\frac{\alpha'}{(1+a\cos 2t)^2}\big)p=0. \tag{2.39}$$

One must note that $q(t)=B(1+a\cos 2t)^{1/2}$ is a solution of the Ermakov-Pinney equation. It has been shown by Athorne that this equation can be replaced by the following coupled nonlinear equations of Ermakov type, namely

$$\ddot{p}+p=-\frac{\alpha' B^4}{q^4}p, \qquad \ddot{q}+q=\frac{\delta}{q^3}. \tag{2.40}$$

We propose to analyze a time-dependent generalization of (2.39) and consider the following generalization of the two-parameter version of the Ince equation

$$\frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big)+\big(1+\frac{\alpha'}{(1+acos\,2t)^2}\big)\omega(t)p=0. \tag{2.41}$$

This equation may also be replaced by the pair of equations:

$$\frac{d}{dt}\big(\frac{\dot{p}}{\omega(t)}\big)+\omega(t)p=-\frac{\alpha' B^4}{q^4}p, \qquad \frac{d}{dt}\big(\frac{\dot{q}}{\omega(t)}\big)+\omega(t)q=\frac{\omega(t)\delta}{q^3}, \tag{2.42}$$

and possesses a first integral which, in this case, is given by

$$I=\frac{1}{2}\Big[\frac{1}{\omega^2(t)}(q\dot{p}-\dot{q}p)^2+\big(\frac{p}{q}\big)^2\Big], \tag{2.43}$$

as may easily be checked.

## 3 A Simple Algorithmic Method to Compute First Integrals of the Emden-Fowler Family

We can apply this straight forward scheme to compute the first integrals of the Lane-Emden equation. Consider the equation

$$y''+p(x)y'=Ke^{-2F}y^n,$$

where $\int^x F\,dx=p(x)$. We can rewrite this equation as

$$(y'e^F)'=Ke^{-F}y^n,$$

from the prescription of Bartuccelli and Gentile one can immediately obtain the first integral

$$I=\frac{1}{2}(y'e^F)^2-K/(n+1)y^{n+1},$$

where $\omega(x)=e^{-F}$.

We modify the preceding scheme to incorporate the Emden-Fowler equation. This will now be described.

**Proposition 3.1** *The second-order ODE $y'' + dx^r y^s = 0$ with $d > 0$ and $s \neq 1$ admits a first integral of the form*

$$I = \frac{1}{2}(y'x - y)^2 + V(x, y),$$

*where $V(x, y) = dx^{r+2}y^{s+1}/(s + 1)$ and $r + s = -3$.*

***Proof.*** Setting $dI/dx = 0$ and using the given equation lead to

$$V_x = -dx^{r+1}y^{s+1} \text{ and } V_y = dx^{x+2}y^s,$$

respectively. The consistency of these partial derivatives then yields the condition $r+s = -3$ and $V(x, y)$ has the stated form. □

**Remark 3.1** If we compare with the Bartucelli-Gentile construction we can readily see here $\omega(x) = x^{-1}$, furthermore there is a shift to define the first integral $I$ of the Emden-Fowler equation. The nature of $\omega(x)$ is fixed for the entire family of the Emden-Fowler systems.

**Proposition 3.2** *The second-order ODE $y'' = \gamma_1^2 y + e^{-(2\gamma_1-\gamma_2)x}h(y)$ admits a first integral of the form*

$$I = \frac{1}{2}(y' - \gamma_1 y)^2 e^{2\gamma_1 x} - e^{\gamma_2 x} \int^y h(u)du$$

*provided $h(y) = y^{-(1+\gamma_2/\gamma_1)}$.*

***Proof.*** By an explicit calculation. □

**Example 3.1** We can apply this scheme to compute the first integrals of the following Lane-Emden-Fowler equation [17]

$$y'' + \frac{k_1}{x}y' = \lambda x^{k_2} y^n. \tag{3.1}$$

This equation has been the subject of study by Rosenau [39] for its solution. It is worth mentioning here that from this equation one obtains immediately a generalization of Chandrasekhar's homology theorem. We can rewrite this equation as

$$(y'x + (k_1 - 1)x)' = x^{-1}\lambda x^{k_2+2}y^n$$

and from our prescription one can immediately obtain the first integral

$$I = \frac{1}{2}(y'x + (k_1 - 1)x)^2 - \frac{\lambda}{n+1}x^{k_2+2}y^{n+1},$$

where $(n + 1)(k_1 - 1) = \lambda(k_2 + 2)$.

We present a slightly different method to compute the first integrals for the Emden-Fowler equation $y''+dx^r y^s = 0$ for other sets of values of $(r, s)$ than given in the previous section.

**Proposition 3.3** *The Emden-Fowler equation $y'' + dx^r y^s = 0$ with $d > 0$ and $r \neq 1$ admits a first integral of the form $I = y'(y'x - y) + V(x, y)$, where $V(x, y) = dx^{r+1}y^{s+1}/(r + 1)$ and $2r + s = -3$.*

***Proof.*** It is clear that $(y'x-y)' = y''x$. We can recast the equation $y''+dx^r y^s = 0$ as $(y'x - y)' + dx^{r+1}y^s = 0$. We compute $\frac{d}{dx}(y'(y'x - y))$ using the Emden-Fowler equation and equate it with the derivative of $V(x, y)$. This immediately yields the condition $2r + s = -3$. □

### 3.1 The (Generalized) Lane-Emden equation

Consider the Lane-Emden equation

$$y'' + 2\frac{y'}{x} + y^5 = 0.$$

One can rewrite this equation in either of the following two different forms, namely

$$(x^2y')' + x^2y^5 = 0, \qquad (y'x + y)' + xy^5 = 0.$$

Once again we use these two equations to compute $\big(x^2y'(y'x+y)\big)'$. Finally equating with a potential $V(x, y) = Kx^n y^m$ we obtain the first integral of the Lane-Emden equation

$$I = x^3(y')^2 + x^2yy' + \frac{1}{3}x^3y^6.$$

We can extend this scheme to more complicated systems. Let us compute the first integrals of the above Emden-Fowler equation for different values of $(r, s)$. The generalized Lane-Emden equation as proposed by Goenner (3.1) in [17, 18] can be expressed either as

$$(y'x + (k_1 - 1)x)' = \lambda x^{k_2+1}y^n \quad \text{or} \quad (y'x^{k_1})' = \lambda x^{k_2+k_1}y^n.$$

Using these two forms we obtain

$$\frac{d}{dx}\big(y'x^{k_1}(y'x + (k_1 - 1)x)\big) = 2\lambda x^{k_1+k_2+1}y^ny' + \lambda(k_1 - 1)x^{k_1+k_2}y^{n+1}.$$

If we take $V = -2\lambda/(n+1)y^{n+1}x^\beta$ we obtain $\beta = k_1 + k_2 + 1 = (k_1 - 1)(n + 1)/2$. Thus we can get the first integral for equation (3.1)

$$I = y'x^{k_1}(y'x + (k_1 - 1)x) - 2\frac{\lambda}{n+1}y^{n+1}x^{(k_1-1)(n+1)/2}, \ n \neq -1.$$

Incidentally this first integral was first derived by Crespo Da Silva [12]. In this way we can find new first integrals for the Emden-Fowler type systems.

### 3.2 First integrals for other type of equations

One can extend the scheme to compute the first integral of more complicated equation with more terms, such as

$$y'' + \frac{k_1}{x}y' + \frac{k_3}{x^2}y = \lambda x^{k_2}y^n. \tag{3.2}$$

We then use our old trick to club the first two terms and express them either as

$$(y'x + (k_1 - 1)x)' = \lambda x^{k_2+1}y^n - \frac{k_3}{x}y \quad \text{or} \quad (y'x^{k_1})' = \lambda x^{k_2+k_1}y^n - k_3x^{k_1-1}y.$$

Once again we differentiate $(y'x_1^k)(y'x + (k_1 - 1)x)$ and equate it with the derivative of $V$ and obtain the first integral of (3.2) in the form

$$I = y'x^{k_1}(y'x + (k_1 - 1)x) - 2\frac{\lambda}{n+1}y^{n+1}x^{(k_1-1)(n+1)/2} + k_3y^2x^{k_1-1}.$$

For an isothermal gaseous sphere, Emden studied also the equation

$$xy'' + 2y' + xe^{ny} = 0.$$

We can also compute the first integral from our method. It is easy to see that this equation can be rewritten either $(x^2y')' + x^2e^{ny} = 0$ or $(xy' + y)' + xe^{ny} = 0$. Again using our scheme we obtain the first integral

$$I = (x^2y'(xy' + y) + \frac{1}{3}x^3e^{ny}, \qquad \text{for} \quad n = 6.$$

Hence we have shown in this section how one can generalize the Barucelli-Gentile scheme to encompass various classes of Emden-Fowler systems.

## 4 Conclusion

In this paper we have examined the connection between a time-dependent second-order ODE due to Bartuccelli and Gentile which was derived by modifying the equation of a linear harmonic oscillator and the Ermakov-Pinney system of ODEs. It is interesting to note that though the system (1.8) can be generalized further to the following

$$\dot{x} = \omega(x, y, t)\frac{\partial G}{\partial y}, \quad \dot{y} = -\omega(x, y, t)\frac{\partial G}{\partial x},$$

with $G = G(x, y)$ and one can easily verify that $G(x, y)$ is an invariant, the solution of the above system is in general not known in closed form unlike that of (1.8) which can be obtained explicitly. This is the main reason for our interest in the Bartuccelli and Gentile construction. It is found that by a simple rational transformation of the dependent variable one can easily extract the well known Ermakov-Lewis invariant. Furthermore a matrix formulation is also considered and a decoupled version of the Bartuccelli-Gentile equation is obtained. Finally we present a simple scheme to compute the first integrals of several equations belonging to the Emden-Fowler and Lane-Emden class.

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