Nonlinear Dynamics and Systems Theory, 14 (3) (2014) 313–322

On the Convergence of Solutions of Some Nonlinear Differential Equations of Fourth Order

E. Korkmaz [1] and C. Tunc [2*]

[1] *Department of Mathematics, Faculty of Arts and Sciences, Mus Alparslan University, 49100, Muş – Turkey*
[2] *Department of Mathematics, Faculty of Sciences, Yüzüncü Yil University, 65080, Van – Turkey*

Received: January 11, 2013; Revised: July 4, 2014

Abstract: In this paper, we consider a nonlinear differential equation of fourth order. By the Lyapunov function approach, we discuss the convergence of the solutions of the equation considered. Our findings generalize some well known results in the literature.

Keywords: *convergence of solutions; nonlinear fourth order equation; Routh-Hurwitz interval; Lyapunov functions.*

Mathematics Subject Classification (2010): 34D20, 34C11.

1 Introduction

As we know the qualitative theory refers to the investigation of the behaviors of solutions of differential equations such as the stability, instability, boundedness,convergence of solutions etc. without determining explicit formulas for the solutions.The relative works can be summarized as follows:

In [1, 15, 16], the authors investigated the asymptotic behaviour of the solutions of certain fourth-order differential equations. In [11, 13, 19–25], the authors considered the stability, instability and boundedness properties of the solutions of some nonlinear third, fourth and fifth-order differential equations (see, also, [10, 14]). In [7], Afuwape studied the existence of a limiting regime in the sense of Demidovic for a certain fourth-order nonlinear differential equations. These studies were done using the Lyapunov's second method. In [2, 5, 8, 9], the authors created conditions for the existence of periodic, almost periodic, exponential stability and dissipative solutions by using the frequency domain

* Corresponding author: mailto:cemtunc@yahoo.com

method. In [3, 4, 6, 12], the authors discussed the convergence of solutions. In [17], Tejumola studied periodic solutions of boundary value problems for some fifth, fourth and third order ordinary differential equations. In [18], Tiryaki and Tunc created Lyapunov functions for certain fourth-order autonomous differential equations.

This paper is concerned with differential equations of the form

$$x^{(iv)} + f_1(x, x', x'', x''') + f_2(x, x', x'') + f_3(x, x') + f_4(x) = p(t, x, x', x'', x'''), \qquad (1)$$

where the functions f_1, f_2, f_3, f_4 and p are real valued and continuous in their respective arguments such that the uniqueness theorem is valid, the solutions are continuously dependent on the initial conditions. The function $p(t, x, x', x'', x''')$ is assumed to have the form

$$p(t, x, x', x'', x''') = q(t) + r(t, x, x', x'', x''')$$

with the functions q and r depending explicitly on the arguments displayed and being continuous in their respective arguments. Furthermore, it is assumed that $r(t, 0, 0, 0, 0) = 0$ for all t.

Definition 1.1 Any two solutions $x_1(t)$, $x_2(t)$ of Eq.(1) are said to converge (to each other) if $x_1 - x_2 \to 0$, $x_1' - x_2' \to 0$, $x_1'' - x_2'' \to 0$, $x_1''' - x_2''' \to 0$ as $t \to \infty$.

Our results assert the existence of convergence of solutions with the functions f_1, f_2, f_3 and f_4 not necessarily differentiable. Here, the functions f_4 are only required to satisfy the increment ratio

$$\frac{f_4(\xi + \eta) - f_4(\xi)}{\eta} \in I_0,$$

where I_0 is closed sub-interval of the Routh -Hurwitz interval defined by

$$I_0 = \left[\Delta_0, K_0 \left[\frac{(ab - c)\, c}{a^2}\right]\right],$$

for some positive constants $a, b, c, d, D, \Delta_0, K_0$, and $(ab - c)\, c - a^2 d > 0, ab - c > 0$.

2 Main Results

Theorem 2.1 *In addition to the basic assumptions imposed on the functions* f_1, f_2, f_3 *and* f_4, *we assume that* $f_1(x, y, z, 0) = f_2(x, y, 0) = f_3(x, 0) = f_4(0) = 0$ *and that:*

(i) there are positive constants δ, δ_0, γ, γ_0, β *and* β_0 *such that*

$$\begin{aligned} &\delta \le \frac{f_1(x_2, y_2, z_2, u_2) - f_1(x_1, y_1, z_1, u_1)}{u_2 - u_1} \le \delta_0, && (u_2 \ne u_1), \\ &\gamma \le \frac{f_2(x_2, y_2, z_2) - f_2(x_1, y_1, z_1)}{z_2 - z_1} \le \gamma_0, && (z_2 \ne z_1), \\ &\beta \le \frac{f_3(x_2, y_2) - f_3(x_1, y_1)}{y_2 - y_1} \le \beta_0, && (y_2 \ne y_1), \end{aligned} \qquad (2)$$

(ii) for any $\xi, \eta \quad (\eta \ne 0)$, *the increment ratios for* f_4 *satisfy*

$$\frac{f_4(\xi + \eta) - f_4(\xi)}{\eta} \in I_0,$$

(iii) there is a continuous function $\phi(t)$ such that

$$\begin{aligned} & |r(t, x_2, y_2, z_2, u_2) - r(t, x_1, y_1, z_1, u_1)| \\ \leq \quad & \phi(t) \left\{ |x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |u_2 - u_1| \right\} \end{aligned} \tag{3}$$

holds for arbitrary $t, x_1, y_1, z_1, u_1, x_2, y_2, z_2, u_2$.

Then,there exists a constant D_1 such that if

$$\int_0^t \phi^v(\tau) d\tau \leq D_1 t \tag{4}$$

for some v in the range $1 \leq v \leq 2$, then all solutions of Eq.(1) converge.

Theorem 2.2 *Let $x_1(t), x_2(t)$ be any two solutions of Eq.(1). Suppose that all the conditions of Theorem 2.1 hold. Then, for each fixed v in the range $1 \leq v \leq 2$, there exist constants $D_2, D_3,$ and D_4 such that*

$$S(t_2) \leq D_2 S(t_1) \exp \left\{ -D_3 (t_2 - t_1) + D_4 \int_{t_1}^{t_2} \phi^v(\tau) d\tau \right\} \quad \text{for } t_2 \geq t_1, \tag{5}$$

where

$$S(t) \quad = \quad [x_2(t) - x_1(t)]^2 + [x_2'(t) - x_1'(t)]^2 + [x_2''(t) - x_1''(t)]^2 + [x_2'''(t) - x_1'''(t)]^2 .$$

We have the following corollaries when $x_1(t) = 0$ and $t_1 = 0$.

Corollary 2.1 *Suppose that $p = 0$ in Eq.(1) and assumptions (i) and (ii) of Theorem 2.1 hold for arbitrary $\eta \neq 0$. Then the trivial solution of Eq.(1) is exponentially stable.*

Corollary 2.2 *If $p \neq 0$ and assumptions (i) and (ii) of Theorem 2.1 hold for arbitrary $\eta \neq 0$ and $\xi = 0$, then there exists a constant $D_5 > 0$ such that every solution $x(t)$ of Eq.(1) satisfies*

$$|x(t)| \leq D_5, \quad |x'(t)| \leq D_5, \quad |x''(t)| \leq D_5, \quad |x'''(t)| \leq D_5.$$

Proof of Theorem 2.2 Writing Eq.(1) as a system of first order equations, we obtain

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= u, \\ u' &= -f_1(x, y, z, u) - f_2(x, y, z) - f_3(x, y) - f_4(x) + r(t, x, y, z, u) + q(t). \end{aligned} \tag{6}$$

Let $(x_i(t), y_i(t), z_i(t), u_i(t))$, $(i = 1, 2)$, be two solutions of (1), such that

$$\Delta_0 \leq \frac{f_4(x_2) - f_4(x_1)}{x_2 - x_1} \leq K_0 \left[\frac{(ab - c)\, c}{a^2} \right]$$

hold. For the proof of the convergence theorem, we define a function

$$\begin{aligned} 2V \quad = \quad & [\beta(1-\epsilon)x + \gamma y + \delta z + u]^2 + [(1-\epsilon)D - 1] (\delta z + u)^2 \\ & + \beta\delta [\epsilon + (1-\epsilon)D - 1] y^2 + \gamma (D - 1) z^2 + \epsilon D u^2 \\ & + \beta^2 \epsilon (1-\epsilon) x^2 + 2\gamma\delta \left[(1-\epsilon)^2 D - 1 \right] yz, \end{aligned} \tag{7}$$

where $0 < \epsilon < \frac{1}{2}$, $\frac{\gamma\delta}{\beta} > (1-\epsilon)$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta - \beta(1-\epsilon)]}{\gamma\delta - \beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always. Indeed, we can rearrange the terms in (7) to obtain

$$2V = 2V_1 + 2V_2, \tag{8}$$

where

$$\begin{aligned} 2V_1 &= [\beta(1-\epsilon)x + \gamma y + \delta z + u]^2 + [(1-\epsilon)D - 1](\delta z + u)^2 \\ &\quad + \epsilon D u^2 + \beta^2 \epsilon (1-\epsilon) x^2 + \epsilon\beta\delta y^2, \\ 2V_2 &= \beta\delta[(1-\epsilon)D - 1] y^2 + 2\gamma\delta\left[(1-\epsilon)^2 D - 1\right] yz + \gamma(D-1) z^2. \end{aligned}$$

We note that V_1 is obviously positive definite. This follows from the condition above. Also V_2 can be regarded as quadratic form in y and z, and is always positive.

Let us recall that a real 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is positive definite $\Leftrightarrow a_1 > 0$, $a_4 > 0$ and $a_1 a_4 - a_2 a_3 > 0$. Thus we can rearrange the terms in V_2 as

$$(y, z)\begin{pmatrix} \beta\delta[(1-\epsilon)D - 1] & \gamma\delta\left[(1-\epsilon)^2 D - 1\right] \\ \gamma\delta\left[(1-\epsilon)^2 D - 1\right] & \gamma(D-1) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

Hence V is positive definite. We can therefore find a constant $D_6 > 0$, such that

$$D_6(x^2 + y^2 + z^2 + u^2) \le V. \tag{9}$$

Furthermore, by using the Schwartz inequality $|y|\,|z| \le \frac{1}{2}(y^2 + z^2)$, we obtain the following estimate:

$$2|V_2| \le D^*(y^2 + z^2), \qquad D^* = D^*(\beta, \gamma, \delta, D, \epsilon) > 0.$$

Thus there exists a constant $D_7 > 0$ such that

$$V \le D_7(x^2 + y^2 + z^2 + u^2), \tag{10}$$

Using inequalities (9) and (10), we obtain

$$D_6(x^2 + y^2 + z^2 + u^2) \le V \le D_7(x^2 + y^2 + z^2 + u^2). \tag{11}$$

The following lemma can be easily verified for $W \equiv V$. □

Lemma 2.1 *Let the function $W(t) = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, u_2 - u_1)$ be defined by*

$$\begin{aligned} 2W &= [\beta(1-\epsilon)(x_2 - x_1) + \gamma(y_2 - y_1) + \delta(z_2 - z_1) + (u_2 - u_1)]^2 \\ &\quad + [(1-\epsilon)D - 1](\delta(z_2 - z_1) + (u_2 - u_1))^2 \\ &\quad + \beta\delta[\epsilon + (1-\epsilon)D - 1](y_2 - y_1)^2 + \gamma(D-1)(z_2 - z_1)^2 \\ &\quad + \epsilon D(u_2 - u_1)^2 + \beta^2\epsilon(1-\epsilon)(x_2 - x_1)^2 \\ &\quad + 2\gamma\delta\left[(1-\epsilon)^2 D - 1\right](y_2 - y_1)(z_2 - z_1), \end{aligned}$$

where $0 < \epsilon < \frac{1}{2}$, $\frac{\gamma\delta}{\beta} > (1-\epsilon)$, β, γ, δ are positive real numbers and $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta-\beta(1-\epsilon)]}{\gamma\delta-\beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always.

i) $W(0,0,0,0) = 0$.

ii) There exist finite positive constants D_6, D_7 such that

$$\begin{aligned} W &\geq D_6 \left\{ (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (u_2-u_1)^2 \right\}, \\ W &\leq D_7 \left\{ (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 + (u_2-u_1)^2 \right\}. \end{aligned} \tag{12}$$

The solutions (x_i, y_i, z_i, u_i), $(i = 1, 2)$ satisfy the system (6). Then $S(t)$ as defined in (6) becomes

$$S(t) = [x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2 + [z_2(t) - z_1(t)]^2 + [u_2(t) - u_1(t)]^2 .$$

Next we prove a result on the derivative of $W(t)$ with respect to t.

Lemma 2.2 *Assume that conditions (i) and (ii) of Theorem 2.1 hold. Then there exist positive constants D_8 and D_9 such that*

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{\frac{1}{2}} |\theta| , \tag{13}$$

where $\theta = r(t, x_2, y_2, z_2, u_2) - r(t, x_1, y_1, z_1, u_1)$.

Proof of Lemma 2.2 Using the system (6), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\dot{W} = \frac{dW}{dt} = -W_1 + W_2, \tag{14}$$

where

$$\begin{aligned} W_1 = {} & \beta(1-\epsilon)F_4(x_2-x_1)^2 + \gamma\left[F_3 - \beta(1-\epsilon)\right](y_2-y_1)^2 \\ & + (1-\epsilon)D\delta\left[F_2 - \gamma(1-\epsilon)\right](z_2-z_1)^2 + D\left[F_1 - \delta(1-\epsilon)\right](u_2-u_1)^2 \\ & + \{\beta(1-\epsilon)[F_3-\beta] + \gamma F_4\}(x_2-x_1)(y_2-y_1) \\ & + \{\beta(1-\epsilon)[F_2-\gamma] + (1-\epsilon)D\delta F_4\}(x_2-x_1)(z_2-z_1) \\ & + \{\beta(1-\epsilon)[F_1-\delta] + DF_4\}(x_2-x_1)(u_2-u_1) \\ & + \{\gamma[F_2-\gamma] + (1-\epsilon)D\delta[F_3-\beta]\}(y_2-y_1)(z_2-z_1) \\ & + \Big\{\gamma[F_1-\delta] + D[F_3-\beta] + \gamma\delta + D\beta - \beta(1-\epsilon) \\ & -\gamma\delta(1-\epsilon)^2 D\Big\}(y_2-y_1)(u_2-u_1) \\ & + \{D[F_2-\gamma] + (1-\epsilon)D\delta[F_1-\delta]\}(z_2-z_1)(u_2-u_1), \end{aligned}$$

$$\begin{aligned} W_2 = {} & \theta(t)\{\beta(1-\epsilon)(x_2-x_1) + \gamma(y_2-y_1) + (1-\epsilon)D\delta(z_2-z_1) \\ & + D(u_2-u_1)\}, \end{aligned} \tag{15}$$

$$F_1 = \frac{f_1(x_2,y_2,z_2,u_2) - f_1(x_1,y_1,z_1,u_1)}{u_2 - u_1}, \qquad (u_2 \neq u_1),$$
$$F_2 = \frac{f_2(x_2,y_2,z_2) - f_2(x_1,y_1,z_1)}{z_2 - z_1}, \qquad (z_2 \neq z_1),$$
$$F_3 = \frac{f_3(x_2,y_2) - f_3(x_1,y_1)}{y_2 - y_1}, \qquad (y_2 \neq y_1),$$
$$F_4 = \frac{f_4(x_2) - f_4(x_1)}{x_2 - x_1}, \qquad (x_2 \neq x_1),$$

and $\lambda_i, \mu_i, \tau_i, \sigma_i$ are strictly positive constants such that

$$\sum_{i=1}^{7}\lambda_i = 1, \quad \sum_{i=1}^{8}\mu_i = 1, \qquad \sum_{i=1}^{7}\tau_i = 1, \sum_{i=1}^{8}\sigma_i = 1.$$

Then W_1 can be rearranged as

$$\begin{aligned} W_1 \;=\;& W_{11} + W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18} + W_{19} \\ & + W_{20} + W_{21} + W_{22} + W_{23} + W_{24}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} W_{11} \;=\;& \lambda_1\beta(1-\epsilon)F_4(x_2-x_1)^2 + \{\gamma\,[F_3-\beta] + \mu_1\gamma\beta\epsilon\}\,(y_2-y_1)^2 \\ & + \{(1-\epsilon)\,D\delta\,[F_2-\gamma] + \tau_1\,(1-\epsilon)\,D\delta\gamma\epsilon\}\,(z_2-z_1)^2 \\ & + \{D\,[F_1-\delta] + \sigma_1 D\delta\epsilon\}\,(u_2-u_1)^2, \end{aligned}$$

$$W_{12} = \lambda_2\beta(1-\epsilon)F_4(x_2-x_1)^2 + \beta(1-\epsilon)\,[F_3-\beta]\,(x_2-x_1)\,(y_2-y_1) + \mu_2\gamma\beta\epsilon\,(y_2-y_1)^2,$$
$$W_{13} = \lambda_3\beta(1-\epsilon)F_4(x_2-x_1)^2 + \gamma F_4\,(x_2-x_1)\,(y_2-y_1) + \mu_3\gamma\beta\epsilon\,(y_2-y_1)^2,$$

$$\begin{aligned} W_{14} \;=\;& \lambda_4\beta(1-\epsilon)F_4(x_2-x_1)^2 + \beta(1-\epsilon)\,[F_2-\gamma]\,(x_2-x_1)\,(z_2-z_1) \\ & + \tau_2\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2, \end{aligned}$$

$$\begin{aligned} W_{15} \;=\;& \lambda_5\beta(1-\epsilon)F_4(x_2-x_1)^2 + (1-\epsilon)\,D\delta F_4\,(x_2-x_1)\,(z_2-z_1) \\ & + \tau_3\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2, \end{aligned}$$

$$W_{16} = \lambda_6\beta(1-\epsilon)F_4(x_2-x_1)^2 + \beta(1-\epsilon)\,[F_1-\delta]\,(x_2-x_1)\,(u_2-u_1) + \sigma_2 D\delta\epsilon\,(u_2-u_1)^2,$$
$$W_{17} = \lambda_7\beta(1-\epsilon)F_4(x_2-x_1)^2 + DF_4\,(x_2-x_1)\,(u_2-u_1) + \sigma_3 D\delta\epsilon\,(u_2-u_1)^2,$$
$$W_{18} = \mu_4\gamma\beta\epsilon\,(y_2-y_1)^2 + \gamma\,[F_2-\gamma]\,(y_2-y_1)\,(z_2-z_1) + \tau_4\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2,$$

$$\begin{aligned} W_{19} \;=\;& \mu_5\gamma\beta\epsilon\,(y_2-y_1)^2 + (1-\epsilon)\,D\delta\,[F_3-\beta]\,(y_2-y_1)\,(z_2-z_1) \\ & + \tau_5\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2, \end{aligned}$$

$$W_{20} = \mu_6\gamma\beta\epsilon\,(y_2-y_1)^2 + \gamma\,[F_1-\delta]\,(y_2-y_1)\,(u_2-u_1) + \sigma_4 D\delta\epsilon\,(u_2-u_1)^2,$$
$$W_{21} = \mu_7\gamma\beta\epsilon\,(y_2-y_1)^2 + D\,[F_3-\beta]\,(y_2-y_1)\,(u_2-u_1) + \sigma_5 D\delta\epsilon\,(u_2-u_1)^2,$$

$$\begin{aligned} W_{22} &= \mu_8\gamma\beta\epsilon\,(y_2-y_1)^2+\left\{\gamma\delta+D\beta-\beta(1-\epsilon)-\gamma\delta\,(1-\epsilon)^2 D\right\}(y_2-y_1)\,(u_2-u_1)\\ &\quad +\sigma_6 D\delta\epsilon\,(u_2-u_1)^2, \end{aligned}$$

$$W_{23}=\tau_6\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2+D\,[F_2-\gamma]\,(z_2-z_1)\,(u_2-u_1)+\sigma_7 D\delta\epsilon\,(u_2-u_1)^2,$$

$$\begin{aligned} W_{24} &= \tau_7\,(1-\epsilon)\,D\delta\gamma\epsilon\,(z_2-z_1)^2+(1-\epsilon)\,D\delta\,[F_1-\delta]\,(z_2-z_1)\,(u_2-u_1)\\ &\quad +\sigma_8 D\delta\epsilon\,(u_2-u_1)^2. \end{aligned}$$

It is clear that $W_{11}\geq 0$. Since each $W_{12}, W_{13}, ..., W_{23}, W_{24}$ are quadratic forms in their respective variables, then from the fact that any quadratic of the form $Ap^2+Bpq+Cq^2$ is non negative if $4AC-B^2\geq 0$, it follows that

$$W_{12}\geq 0 \;\text{ if }\; [F_3-\beta]^2\leq 16\lambda_3\mu_3\lambda_2\mu_2(\epsilon\beta)^2,$$

$$W_{13}\geq 0 \;\text{ if }\; F_4\leq \frac{4\lambda_3\mu_3\epsilon\beta^2\,(1-\epsilon)}{\gamma},$$

$$W_{14}\geq 0 \;\text{ if }\; [F_2-\gamma]^2\leq 16\lambda_4\lambda_5\tau_2\tau_3\,(\gamma\epsilon)^2,$$

$$W_{15}\geq 0 \;\text{ if } F_4\leq \frac{4\lambda_5\tau_3\epsilon\beta\gamma}{D\delta},$$

$$W_{16}\geq 0 \;\text{ if }\; [F_1-\delta]^2\leq 16\lambda_6\lambda_7\sigma_2\sigma_3\,(\delta\epsilon)^2,$$

$$W_{17}\geq 0 \;\text{ if }\; F_4\leq \frac{4\lambda_7\sigma_3\beta\,(1-\epsilon)\,\delta\epsilon}{D},$$

$$W_{18}\geq 0 \;\text{ if }\; [F_2-\gamma]^2\leq 4\mu_4\tau_4\beta D\delta\epsilon^2\,(1-\epsilon),$$

$$W_{19}\geq 0 \;\text{ if }\; [F_3-\beta]^2\leq \frac{4\mu_5\tau_5\beta\,(\gamma\epsilon)^2}{(1-\epsilon)\,D\delta},$$

$$W_{20}\geq 0 \;\text{ if }\; [F_1-\delta]^2\leq \frac{4\mu_6\sigma_4\beta\epsilon^2 D\delta}{\gamma},$$

$$W_{21}\geq 0 \;\text{ if }\; [F_3-\beta]^2\leq \frac{4\mu_7\sigma_5\beta\epsilon^2\gamma\delta}{D},$$

$$W_{22}\geq 0 \;\text{ if } 4\mu_8\gamma\beta\epsilon\sigma_6 D\delta\epsilon\geq \left\{\gamma\delta+D\beta-\beta(1-\epsilon)-\gamma\delta\,(1-\epsilon)^2 D\right\}^2,$$

$$W_{23}\geq 0 \;\text{ if }\; [F_2-\gamma]^2\leq 4\tau_6\sigma_7\gamma\,(1-\epsilon)\,(\delta\epsilon)^2,$$

$$W_{24}\geq 0 \;\text{ if }\; [F_1-\delta]^2\leq \frac{4\tau_7\sigma_8\gamma\epsilon^2}{(1-\epsilon)}.$$

That is,

$$[F_1-\delta]^2\leq \min\left\{\frac{4\tau_7\sigma_8\gamma\epsilon^2}{(1-\epsilon)},\; \frac{4\mu_6\sigma_4\beta\epsilon^2 D\delta}{\gamma},\; 16\lambda_6\lambda_7\sigma_2\sigma_3\,(\delta\epsilon)^2\right\},$$

$$[F_2-\gamma]^2\leq \min\left\{16\lambda_4\lambda_5\tau_2\tau_3\,(\gamma\epsilon)^2,\, 4\mu_4\tau_4\beta D\delta\epsilon^2\,(1-\epsilon),\; 4\tau_6\sigma_7\gamma\,(1-\epsilon)\,(\delta\epsilon)^2\right\},$$

$$[F_3-\beta]^2 \le \min\left\{16\lambda_3\mu_3\lambda_2\mu_2(\epsilon\beta)^2, \frac{4\mu_5\tau_5\beta\,(\gamma\epsilon)^2}{(1-\epsilon)\,D\delta}, \frac{4\mu_7\sigma_5\beta\epsilon^2\gamma\delta}{D}\right\},$$

$$F_4 \le \min\left\{\frac{4\lambda_3\mu_3\epsilon\beta^2\,(1-\epsilon)}{\gamma}, \ \frac{4\lambda_5\tau_3\epsilon\beta\gamma}{D\delta}, \ \frac{4\lambda_7\sigma_3\beta\,(1-\epsilon)\,\delta\epsilon}{D}\right\},$$

Because of $W_{12}\ge 0$, $W_{13}\ge 0,\dots,W_{24}\ge 0$, we obtain $W_1 \ge W_{11}$. Then we find a constant D_8 such that

$$W_1 \ge W_{11} \ge 2D_8 S(t), \tag{17}$$

where

$$2D_8 = \min\left\{\beta(1-\epsilon)\Delta_0, \gamma\beta\epsilon, (1-\epsilon)\,D\delta\gamma\epsilon, D\delta\epsilon\right\}.$$

Similarly, we can find from the value of W_2, a constant $D_9 > 0$ small enough such that

$$W_2 \le D_9 S^{\frac{1}{2}}\,|\theta|, \tag{18}$$

where $D_9 = \max\left\{\beta(1-\epsilon), \gamma, (1-\epsilon)\,D\delta, D\right\}.$

Writing (17) and (18) in (14), we get

$$\frac{dW}{dt} \le -2D_8 S + D_9 S^{\frac{1}{2}}\,|\theta|.$$

Let v be any constant in the range $1 \le v \le 2$ and $2\mu = 2-v$, so that $0 \le \mu \le 1/2$. One can arrange the estimate in (13) as

$$\frac{dW}{dt} + D_8 S \le -D_8 S + D_9 S^{1/2}\,|\theta| = D_{10} S^{\mu} W^*,$$

where

$$W^* = \left(|\theta| - D_{11} S^{1/2}\right) S^{1/2-\mu}, \tag{19}$$

with $D_{11} = D_8 D_{10}^{-1}$. We consider the following two cases:
a) $|\theta| < D_{11} S^{1/2}$, b) $|\theta| \ge D_{11} S^{1/2}$.

If $|\theta| < D_{11} S^{1/2}$, then $W^* < 0$. On the other hand, if $|\theta| \ge D_{11} S^{1/2}$, then the definition of W* in (19) gives at least

$$W^* \le S^{1/2-\mu}\,|\theta|,$$

and also $S^{1/2} \le |\theta|/D_{11}$. The foregoing inequality leads to

$$S^{1/2(1-2\mu)} \le \left[\frac{|\theta|}{D_{11}}\right]^{(1-2\mu)},$$

so that

$$S^{1/2(1-2\mu)}\,|\theta| \le \left[\frac{|\theta|}{D_{11}}\right]^{(1-2\mu)} |\theta|.$$

The above estimate implies

$$W^* \le D_{12}\,|\theta|^{2(1-\mu)},$$

where $D_{12} = D_{11}^{(2\mu-1)}$. Hence, it is clear that

$$\frac{dW}{dt} + D_8 S \leq D_{10} D_{12} S^{\mu} |\theta|^{2(1-\mu)} \leq D_{13} S^{\mu} \phi^{2(1-\mu)} S^{(1-\mu)},$$

where $D_{13} = S^{1-\mu} D_{10} D_{12}$ which follows from

$$|\theta| = |r(t, x_2, y_2, z_2, u_2) - r(t, x_1, y_1, z_1, u_1)| \\ \leq \phi(t) \{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |u_2 - u_1|\}.$$

Using the estimate $v = 2(1-\mu)$, we obtain

$$\frac{dW}{dt} \leq -D_8 S + D_{13} \phi^v S.$$

By the inequality (12), we find

$$\frac{dW}{dt} + (D_{14} - D_{15} \phi^v(t)) W \leq 0 \tag{20}$$

for some positive constants D_{14} and D_{15}. Integrating (20) from t_1 to t_2 $(t_2 \geq t_1)$, we have

$$W(t_2) \leq W(t_1) \exp\left\{-D_{14}(t_2 - t_1) + D_{15} \int_{t_1}^{t_2} \phi^v(\tau) d\tau\right\}.$$

Again, using Lemma 2.1, we obtain (5) with $D_2 = D_7 D_6^{-1}$, $D_3 = D_{14}$, and $D_4 = D_{15}$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1 Choose $D_1 = D_3 D_4^{-1}$ in (4). From the estimate (5), if

$$\int_{t_1}^{t_2} \phi^v(\tau) d\tau \leq D_3 D_4^{-1} (t_2 - t_1),$$

then the exponential index remains negative for all $t_2 - t_1 \geq 0$. Then, as $t = t_2 - t_1 \to \infty$, we have $S(t) \to 0$, and this gives

$$x_2 - x_1 \to 0, \quad y_2 - y_1 \to 0, \quad z_2 - z_1 \to 0, \quad u_2 - u_1 \to 0$$

as $t \to \infty$. This completes the proof of Theorem 2.1. □

References

[1] Abou-El-Ela, A. M. A. and Sadek, A. I. On the asymptotic behaviour of solutions of some differential equations of the fourth-order. *Ann. Differential Equations.* **8** (1) (1992) 1–12.

[2] Adesina, O. A. On the qualitative behaviour of solutions of some fourth-order non-linear differential equations. In: *Ordinary differential equations.* Abuja, 2000, 87–93.

[3] Adesina, O. A. and Ogundare, B. S. New results on the convergence of solutions for a certain fourth order nonlinear differential equation. *Journal of Computational Analysis and Appl.* **14** (1) (2012) 10–18.

[4] Afuwape, A. U. On the convergence of solutions of certain fourth order differential equations. *An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat.* (N.S.) **27** (1) (1981) 133–138.

[5] Afuwape, A. U. On some properties of solutions for certain fourth-order nonlinear differential equations. *Analysis.* **5** (2) (1985) 175–183.

[6] Afuwape, A. U. Convergence of the solutions for the equation $x^{(4)} + ax''' + bx'' + g(x') + h(x) = p(t, x, x', x'', x''')$. *Internat. J. Math. Math. Sci.* **11** (4) (1988) 727–733.

[7] Afuwape, A. U. On the existence of a limiting regime in the sense of Demidovic for a certain fourth-order nonlinear differential equation. *J. Math. Anal. Appl.* **129** (2) (1988) 389–393.

[8] Afuwape, A. U. Uniform dissipativity of some fourth-order nonlinear differential equations. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* **35** (1) (1989) 9–16.

[9] Afuwape, A. U. Uniform dissipative solutions for some fourth order nonlinear differential equations. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat.* 37 (4) (1991) 431–445.

[10] Ezeilo, J. O. C. New properties of the equation $x''' + ax'' + bx' + h(x) = p(t, x, x', x'')$ for certain special values of the incrementary ratio $\mathrm{y}^{-1}\{h(x+y) - h(x)\}$. In: *Équations différentielles et fonctionnelles non linéaires (Actes Conférence Internat. "Equa-Diff 73", Brussels/Louvain-la-Neuve*, 1973, 447–462.

[11] Ogundare, B. S. Boundedness of solutions to fourth order differential equations with oscillatory restoring and forcing terms. Electron. *J. Differential Equations.* (6) (2006) 6 pp.

[12] Ogundare, B. S. and Okecha, G. E. Convergence of solutions of certain fourth-order nonlinear differential equations. *Int. J. Math. Math. Sci.* (2007), Art. ID 12536, 13 pp.

[13] Ogundare, B. S. and Okecha, G. E. Boundedness and stability properties of solution to certain fourth order non-linear differential equation. *Nonlinear Stud.* **15** (1) (2008) 61–70.

[14] Reissig, R.; Sansone, G. and Conti, R. Non-linear Differential Equations of Higher Order. *Translated from the German.* Noordhoff International Publishing, Leyden, 1974. xiii+669 pp.

[15] Sadek, A. I. On the asymptotic behaviour of solutions of certain fourth-order ordinary differential equations. *Math. Japon.* **45** (3) (1997) 527–540.

[16] Sadek, A. I. and AL-Elaiw, A. S. Asymptotic behaviour of the solutions of a certain fourth-order differential equation. *Ann. Differential Equations* **20** (3) (2004) 221–234.

[17] Tejumola, H. O. Periodic boundary value problems for some fifth, fourth and third order ordinary differential equations. *J. Nigerian Math. Soc.* **25** (2006) 37–46.

[18] Tiryaki, A. and Tunç, C. Constructing Lyapunov functions for certain fourth-order autonomous differential equations. *Indian J. Pure Appl. Math.* **26** (3) (1995) 225–232.

[19] Tiryaki, A. and Tunç, C. Boundedness and stability properties of solutions of certain fourth order differential equations via the intrinsic method. *Analysis.* **16** (4) (1996) 325–334.

[20] Tunc, C. A note on the stability and boundedness results of solutions of certain fourth order differential equations. *Appl. Math. Comput.* **155** (3) (2004) 837–843.

[21] Tunc, C. Some stability and boundedness results for the solutions of certain fourth order differential equations. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **44** (2005) 161–171.

[22] Tunc, C. About stability and boundedness of solutions of certain fourth order differential equations. *Nonlinear Phenom. Complex Syst.* **9** (4) (2006) 380–387.

[23] Tunc, C. The boundedness of solutions to nonlinear third order differential equations. *Nonlinear Dynamics and systems Theory* **10** (1) (2010) 97–102.

[24] Tunc, C. Instability for nonlinear differential equations of fifth order subject to delay. *Nonlinear Dynamics and systems Theory* **12** (2) (2012) 207–214.

[25] Wu, X. and Xiong, K. Correspondence remarks on stability results for the solutions of certain fourth-order autonomous differential equations. *Internat. J. Control* **69** (2) (1998) 353–360.