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Stability Conditions for a Class of Nonlinear Time Delay Systems

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Abstract: In this paper, stability analysis for a class of nonlinear time delay system is done. A state space representation of the class of system under consideration is used and a transformation is carried out to represent the system by an arrow form matrix. Taking advantage of this representation and applying the Kotelyanski lemma in combination with properties of M-matrices, some new sufficient stability conditions are determined. An illustrative example is presented to show the effectiveness of the proposed approach.

Keywords: *nonlinear time delay systems; arrow matrix; stability analysis.*

Mathematics Subject Classification (2010): 34K20.

1 Introduction

Time delay exists in many practical systems. This includes chemical processes, teleoperators, mechanical systems, network control systems etc. see [2, 3, 8, 11]. The delay can be an inherent part of the dynamics of the system or can be a result of actuators and sensors used and the time needed for transmission of control signals. Presence of delay complicates the analysis of such systems and can even cause instability [6,10,11]. In many situations industrial models have to represent nonlinear phenomena for the delay or the system itself. This is justified by the insufficiency of the first order linear approximations to explain the typically nonlinear problem of instability linked to excessive initial conditions or perturbations. Difficulties are greater when delays appear in nonlinear systems, see [1, 3–5] for an excellent exposition of nonlinear delay equations. For all these reasons, there has been an extensive literature on stability of time delay systems [7, 19, 21]. In this

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paper, we determine sufficient stability conditions for nonlinear systems with constant delay.

There are mainly two main approaches in determining stability conditions for time delay systems, namely, delay independent conditions and delay dependent conditions. To this extent most of the existing results are delay-independent [6, 9, 12, 20] and few are delay-dependent, see [13, 18, 22] and the references therein. Even fewer give practical results which can be applied to nonlinear systems. In this paper, we determine sufficient delay dependent stability conditions for nonlinear systems with a constant delay.

The paper is organized as follows. In Section 2, the main result is given. Delay dependent sufficient conditions for stability of the nonlinear system with delay are derived. Section 3 is devoted to the application of the obtained result to delayed Lurie systems. An illustrative example is given in Section 4. We finish this paper by some concluding remarks in Section 5.

2 Sufficient Stability Conditions

Our work consists of determining stability conditions for systems described by the following equation:

$$\tilde{S}:\begin{cases} y^{(n)}(t)+\sum_{i=0}^{n-1}\tilde{f}_i(t,x_t,\wp)y^{(i)}(t)+\sum_{i=0}^{n-1}\tilde{g}_i(t,x_t,\wp)y^{(i)}(t-\tau)=0, \\ y^{(i)}(t)=\phi_i(t), \forall t\in[-\tau\ \ 0],\, i=0,\dots,n-1, \end{cases} \tag{1}$$

where τ is a constant delay and $\tilde{f}_i$, $\tilde{g}_i$ $i=0,\dots,n-1$ are nonlinear functions. Let us fix the notation used. Let $C_n = C([-\tau\ 0]\ ,R^n)$ be the Banach space of continuous functions mapping the interval $[-\tau\ 0]$ into R^n with the topology of uniform convergence. Let $x_t\in C_n$ be defined by $x_t(\theta)=x(\theta)$, $\theta\in[-\tau\ 0]$. For a given $\phi\in C_n$, we define $\|\phi\|=\sup_{-\tau\le\theta\le 0}\|\phi(\theta)\|$, $\phi(\theta)\in R^n$. Let $x_t\in C_n$ be defined by $x_t(\theta)=x(\theta)$, $\theta\in[-\tau\ 0]$. The functions $\tilde{f}_i$, $\tilde{g}_i$, $i=0,1,..,n-1$ are completely continuous mapping the set $J_a\times C_n^H\times S_\wp$ into R, where $C_n^H=\{\phi\in C_n, \|\phi\|<H\}$, $H>0$, $J_a=[a\ +\infty)$, $a\in R$ and $S_\wp=\{\wp\in R, \underline{\wp}\le\wp\le\overline{\wp}$ where $\underline{\wp}\le\overline{\wp}\in R\}$. Finally we say that the function g satisfies the finite sector condition if $g\in E([k_1\ ,\ k_2])=\{g\mid g(0)=0, k_1\sigma^2<\sigma g(\sigma)<k_2\sigma^2, \sigma\neq 0$ and $k_1<k_2\}$. In the sequel, we denote $(t,x_t,\wp)=(.)$. We start by making the following changes:

$$x_{i+1}(t)=y^{(i)}(t),\, i=0,\dots,n-1$$

which implies that

$$\dot{x}_i(t)=x_{i+1}(t),\, i=0,\dots,n-1,$$

therefore,

$$\dot{x}_n(t)=-\sum_{i=1}^{n}\tilde{f}_{i-1}(.)x_i(t)-\sum_{i=1}^{n}\tilde{g}_{i-1}(.)x_i(t-\tau).$$

The studied system is described by the following state space representation:

$$\begin{cases} \dot{x}(t)=\tilde{F}(.)x(t)+\tilde{G}(.)x(t-\tau), \\ x(t)=\phi(t), \forall t\in[-\tau\ \ 0], \end{cases} \tag{2}$$

where

$$x(t)=\begin{pmatrix} x_1(t) & x_2(t) & \dots & x_{n-1}(t) & x_n(t)\end{pmatrix}^{'},$$

$$\phi(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_{n-1}(t) & \phi_n(t) \end{pmatrix}'.$$

The matrices $\tilde{F}(.)$ and $\tilde{G}(.)$ are given by

$$\tilde{F}(.) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -\tilde{f}_0(.) & -\tilde{f}_1(.) & \cdots & -\tilde{f}_{n-2}(.) & -\tilde{f}_{n-1}(.) \end{pmatrix} \tag{3}$$

and

$$\tilde{G}(.) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ -\tilde{g}_0(.) & -\tilde{g}_1(.) & \dots & -\tilde{g}_{n-2}(.) & -\tilde{g}_{n-1}(.) \end{pmatrix}. \tag{4}$$

Applying the following transformation:

$$x = Pz, \tag{5}$$

where

$$P = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots & \alpha_{n-1}^{n-2} & 0 \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_{n-1}^{n-1} & 1 \end{pmatrix} \quad \alpha_i \neq \alpha_j \quad \forall i, j \tag{6}$$

leads to the following state representation

$$S : \dot{z}(t) = F(.)z(t) + \Delta(.)z(t - \tau) \tag{7}$$

which describes the dynamics of system (1) by using the new state vector z. The matrix $F(.)$ is given by

$$F(.) = P^{-1}\tilde{F}(.)P = \begin{pmatrix} \alpha_1 & & & & \beta_1 \\ & \alpha_2 & & & \beta_2 \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & \beta_{n-1} \\ \gamma_1(.) & \gamma_2(.) & \cdots & \gamma_{n-1}(.) & \gamma_n(.) \end{pmatrix}. \tag{8}$$

Elements of the matrix $F(.)$ are defined in [15] by

$$\gamma_i(.) = -D(\alpha_i, .) \quad i = 1...n - 1, \tag{9}$$

where

$$D(s, .) = s^n + \sum_{i=0}^{n-1} \tilde{f}_i(.)s^i \tag{10}$$

and

$$\gamma_n(.) = -\tilde{f}_{n-1}(.) - \sum_{i=1}^{n-1} \alpha_i, \tag{11}$$

$$\beta_i = \left.\frac{\alpha_i - \lambda}{Q(\lambda)}\right|_{\lambda=\alpha_i} \quad i = 1...n-1, \tag{12}$$

where

$$Q(\lambda) = \prod_{j=1}^{n-1} (\lambda - \alpha_j). \tag{13}$$

The matrix $\Delta(.)$ is given by

$$\Delta(.) = P^{-1}\tilde{G}(.)P = \begin{pmatrix} O_{n-1,n-1} & O_{n-1,1} \\ \delta_1(.) \cdots \delta_{n-1}(.) & \delta_n(.) \end{pmatrix} \tag{14}$$

with

$$\delta_i(.) = -N(\alpha_i, .),\ i = 1, \ldots, n-1, \tag{15}$$

where

$$N(s, .) = \sum_{i=0}^{n-1} \tilde{g}_i(.) s^i \tag{16}$$

and

$$\delta_n(.) = -\tilde{g}_{n-1}(.). \tag{17}$$

Based on this transformation and the arbitrary choice of parameters α_i, $i = 1, \ldots, n-1$ which play an important role in simplifying the use of aggregate techniques, we give now the main result. Let us start by writing our system in another form. By using the Newton-Leibniz formula

$$z(t-\tau) = z(t) - \int_{t-\tau}^{t} \dot{z}(\theta) d\theta, \tag{18}$$

equation (7) becomes

$$\dot{z}(t) = (F(.) + \Delta(.))z(t) - \Delta(.) \int_{t-\tau}^{t} \dot{z}(\theta) d\theta. \tag{19}$$

Let Ω be a domain of R^n, containing a neighborhood of the origin, and $\sup\limits_{J_\tau,\, \Omega, S_\wp}$ the suprema calculated for $t \in J_\tau$ (i.e $t \geq \tau$), for functions x with values in Ω, and for $\wp$ in $S_\wp$. Next, using the special form of system (1) and applying the notation $\sup_{[.]} = \sup\limits_{J_\tau,\, \Omega, S_\wp}$, we can announce the following theorem.

Theorem 2.1 *The system (1) is asymptotically stable, if there exist distinct parameters $\alpha_i < 0$ $i = 1, ..., n-1$, such that the matrix $T(.)$ is the opposite of an M-matrix, where $T(.)$ is given by*

$$T(.) = \begin{pmatrix} \alpha_1 & & & & |\beta_1| \\ & \alpha_2 & & & |\beta_2| \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & |\beta_{n-1}| \\ t_1(.) & t_2(.) & \cdots & t_{n-1}(.) & t_n(.) \end{pmatrix} \tag{20}$$

and the elements $t_i(.)$, $i = 1, ..., n$ are given by

$$t_i(.) = \frac{|\gamma_i(.) + \delta_i(.)| + \tau|\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \tag{21}$$

and

$$t_n(.) = \gamma_n(.) + \delta_n(.) + \frac{\tau \sup_{[.]} |\delta_n(.)||\gamma_n(.) + \delta_n(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau \sum_{i=1}^{n-1} |\beta_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|}. \tag{22}$$

Proof. We use the following vector norm

$$p(z) = \begin{pmatrix} p_1(z) & p_2(z) & p_3(z) & ... & p_n(z) \end{pmatrix}^{'}, \tag{23}$$

where $p_i(z) = |z_i|$, $i = 1, \ldots, n-1$ and $p_n(z)$ is given by

$$p_n(z) = |z_n| + \frac{\sum_{i=1}^{n} \sup_{[.]} |\delta_i(.)|}{1 - \tau \left(\sup_{[.]} |\delta_n(.)| \right)} \int_{-\tau}^{0} \int_{t+\theta}^{t} |\dot{z}_i(\vartheta)| d\vartheta d\theta \tag{24}$$

with the condition

$$\tau \left(\sup_{[.]} |\delta_n(.)| \right) < 1. \tag{25}$$

Let $V(t)$ be a radially unbounded Lyapunov function given by (26).

$$V(t) = \left\langle (p(z(t)))^{'}, w \right\rangle = \sum_{i=1}^{n} w_i p_i(z(t)), \tag{26}$$

where $w \in R_+^n$, $w_i > 0$, $i = 1, \ldots, n$. First, note that

$$V(t_0) \leq \sum_{i=1}^{n-1} w_i |z_i(t_0)| + w_n (|z_n(t_0)| + \frac{\sup_{[.]}(|\delta_n(.)|)}{1 - \tau \sup_{[.]}(|\delta_n(.)|)} \sup_{[-\tau,0]} |\dot{\phi}_n| \frac{\tau^2}{2}) := r < +\infty \tag{27}$$

and

$$V(t) \geq \sum_{i=1}^{n} w_i |z_i(t)|. \tag{28}$$

The right Dini derivative of $V(t)$, along the solution of (19), gives

$$D^+V(t) = \sum_{i=1}^{n} w_i \frac{d^+ p_i(z(t))}{dt^+}. \tag{29}$$

For clarification reasons, each element of $\frac{d^+ p_i(z(t))}{dt^+}$, $i = 1, ..., n$ is calculated separately. Let us begin with the first $(n-1)$ elements. Because $|z_i| = z_i sign(z_i)$, we can write, for

$i = 1, ..., n-1,$

$$\begin{aligned} \frac{d^+ p_i(z(t))}{dt^+} &= \frac{d^+ |z_i(t)|}{dt^+} \\ &= \frac{d^+ z_i(t)}{dt^+} sign(z_i(t)) \\ &= (\alpha_i z_i(t) + \beta_i z_n(t)) sign(z_i(t)) \\ &\leq \alpha_i |z_i(t)| + |\beta_i| |z_n(t)| \end{aligned} \tag{30}$$

and $\frac{d^+ p_n(z)}{dt^+}$ is given by

$$\frac{d^+ p_n(z)}{dt^+} = \frac{d^+ |z_n|}{dt^+} + \frac{\sum\limits_{i=1}^{n} \sup\limits_{[.]} |\delta_i(.)|}{1 - \tau \sup\limits_{[.]} |\delta_n(.)|} \frac{d^+}{dt^+} \left[\int_{-\tau}^{0} \int_{t+\theta}^{t} |\dot{z}_i(\vartheta)| d\vartheta d\theta \right]. \tag{31}$$

Finally, it is easy to see that equation (31) can be overvalued by the following one

$$\frac{d^+ p_n(z)}{dt^+} \leq \sum_{i=1}^{n} t_i(.) |z_i|, \tag{32}$$

where elements $t_i(.)$, $i = 1, ..., n$ are given by

$$\begin{aligned} t_i(.) &= |\gamma_i(.) + \delta_i(.)| + \frac{\tau \sup_{[.]} |\delta_n(.)| |\gamma_i(.) + \delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau |\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \\ &= \frac{|\gamma_i(.) + \delta_i(.)| + \tau |\alpha_i| \sup_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} \end{aligned} \tag{33}$$

and

$$t_n(.) = \gamma_n(.) + \delta_n(.) + \frac{\tau \sup_{[.]} |\delta_n(.)| |\gamma_n(.) + \delta_n(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|} + \frac{\tau \sum\limits_{i=1}^{n-1} |\beta_i| \sup\limits_{[.]} |\delta_i(.)|}{1 - \tau \sup_{[.]} |\delta_n(.)|}. \tag{34}$$

Then the inequality (29) becomes

$$D^+ V(t) < \left\langle T'(.) w, |z| \right\rangle, \tag{35}$$

where $T(.)$ is given by (36)

$$T(.) = \begin{pmatrix} \alpha_1 & & & & |\beta_1| \\ & \alpha_2 & & & |\beta_2| \\ & & \ddots & & \vdots \\ & & & \alpha_{n-1} & |\beta_{n-1}| \\ t_1(.) & t_2(.) & \cdots & t_{n-1}(.) & t_n(.) \end{pmatrix}. \tag{36}$$

Because the nonlinear elements of $T(.)$ are isolated in the last row, the eigenvector $v(t, x_t, \wp)$ relative to the eigenvalue λ_m is constant [17], where λ_m is such that $Re(\lambda_m) = \max\{Re(\lambda), \lambda \in \lambda(T(.))\}$. Then, in order to have $D^+ V(t) < 0$, it is sufficient to have $T(.)$ as the opposite of an M$-$matrix. Indeed, according to properties of

M$-$matrices, we have $\forall\ \sigma \in R_{+}^{*n}$, $\exists\ w \in R_{+}^{*n}$ such that $-\left(T^{'}(.)\right)^{-1}\sigma = w$. This enables us to write the following equation

$$\left\langle T^{'}(.)w, |z(t)|\right\rangle = \langle -\sigma, |z(t)|\rangle = -\sum_{i=1}^{n}\sigma_i|z_i(t)| \tag{37}$$

which yields

$$D^{+}V(t) \leq -\sum_{i=1}^{n}\sigma_i|z_i(t)|. \tag{38}$$

This completes the proof of theorem.

Remark 2.1 If the couple $(D(s,.) + N(s,.), Q(s))$ forms a positive pair, then there exist distinct negative parameters α_i, $i = 1, ..., n-1$, verifying the condition $(\gamma_i(.) + \delta_i(.))\,\beta_i > 0$ for $i = 1, ..., n-1$.

Using Theorem 2.1 and Remark 2.1, the obtained supremum is a function of α_i values, $i = 1, ..., n-1$. As a result, a sufficient condition for asymptotic stability of our system is when values of the time delay are less than this supremum.

Corollary 2.1 *If the couple $(D(s,.) + N(s,.), Q(s))$ forms a positive pair and there exist distinct negative parameters α_i, $i = 1, ..., n-1$, such that:*

$$2\tau\left((\gamma_n(.) + \delta_n(.))\sup_{[.]}|\delta_n(.)| - \nu(.)\right) + \frac{D(0,.) + N(0,.)}{Q(0)} > 0, \tag{39}$$

then the system (1) is asymptotically stable.

Proof. According to Remark 2.1, we find that

$$\begin{aligned}\gamma_n(.) + \delta_n(.) - \sum_{j=1}^{n-1}\frac{|\gamma_j(.) + \delta_j(.)||\beta_j|}{\alpha_j} &= \gamma_n(.) + \delta_n(.) - \sum_{j=1}^{n-1}\frac{(\gamma_j(.) + \delta_j(.))\beta_j}{\alpha_j} \\ &= -\frac{D(0,.) + N(0,.)}{Q(0)}.\end{aligned}$$

The result of Theorem 2.1 becomes

$$-2\tau(\gamma_n(.) + \delta_n(.))\sup_{[.]}|\delta_n(.)| + 2\tau\nu(.) - \frac{D(0,.) + N(0,.)}{Q(0)} < 0$$

which is equivalent to

$$2\tau\left((\gamma_n(.) + \delta_n(.))\sup_{[.]}|\delta_n(.)| - \nu(.)\right) + \frac{D(0,.) + N(0,.)}{Q(0)} > 0.$$

This completes the proof of corollary.

Remark 2.2

- Theorem 2.1 depends on the new basis change, where parameters α_i of the matrix P are arbitrary chosen such that matrix $T(.)$ is the opposite of an M-matrix. The appropriate choice of the set of free parameters α_i makes the given stability conditions satisfied.

- The theorem takes into account the fact that delayed terms may stabilize our system [22]. Theorem 2.1 can hold even if $D(s,.)$ is unstable. This is another advantage as the majority of previously published results assume that $D(s)$ is linear and stable.
- The theorem can easily be extended to the study of systems with multiple time-delays and can generalize the work of [14] in the case of fuzzy TS systems with time-delay and the work of [16] in the case of discrete time delay system.

3 Application to Delayed Nonlinear n-th Order All Pole Plant

Consider the complex system S given in Figure 1.

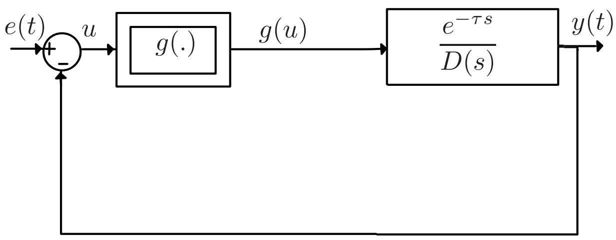

Figure 1: Block representation of the studied system.

$D(s)$ is defined by (10) and $N(s) = 1$, respectively. In this case $\tilde{f}_i(.)$ are constants and g is a function satisfying the finite sector condition. Let $\hat{g}$ be a function defined as follows

$$\hat{g}(e(\theta), y(\theta)) = \frac{g(e(\theta) - y(\theta))}{e(\theta) - y(\theta)}, e(\theta) \neq y(\theta) \quad \forall\theta \in [-\tau \ \ +\infty[, \tag{40}$$

$$\sup_{[.]} |\hat{g}(e(t), y(t))| = \bar{g} \in R_+^*.$$

The presence of delay in the system of Figure 1 makes stability study difficult. The following steps show how to represent this system in the form of system (1). Then we can write

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = -\hat{g}(e(t-\tau), y(t-\tau))y(t-\tau) + \hat{g}(e(t-\tau), y(t-\tau))e(t-\tau). \tag{41}$$

We use the following notation

$$\hat{g}(.) = \hat{g}(e(t-\tau), bx(t-\tau)),$$

therefore,

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \hat{g}(.)y(t-\tau) = \hat{g}(.)e(t-\tau). \tag{42}$$

It is clear that system (42) is equivalent to system (1) in the special cases $e(\theta) = 0$ and $e(\theta) = -Kx(\theta)$, $x(t) = \left(y(t), \dot{y}(t), ..., y^{(n)}(t)\right)'$, $\forall\ \theta \in [-\tau \ \ +\infty[$. We will now consider each case separately.

3.1 Case $e(t) = 0$

In the case $e(t) = 0 \ \ \forall t \in [-\tau \ +\infty[$, the description of the system becomes

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) + \hat{g}(.)y(t-\tau) = 0. \tag{43}$$

This is a special representation of system (1) where $\tilde{f}_i(.) = a_i$,$\tilde{g}_1(.) = \hat{g}(.)$ $\tilde{g}_i(.) = 0$ $\forall\ i = 2, ..., n-1,\ D(s,.) = D(s),\ N(s,.) = \hat{g}(.),\ \gamma_n(.) = \gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i$ and $\delta_n(.) = 0$. A sufficient stability condition for this system is given in the following proposition.

Proposition 1 *If there exist distinct $\alpha_i < 0$ $i = 1, ..., n-1$, such that the following conditions*

$$\begin{cases} \gamma_n < 0, \\ \mu_1(.) + 2\tau\nu_1(.) - \xi_1(.) < 0, \end{cases} \tag{44}$$

where

$$\begin{cases} \mu_1(.) = \gamma_n, \\ \nu_1(.) = \bar{g}, \\ \xi_1(.) = \frac{|D(\alpha_1)+\hat{g}(.)||\beta_1|}{\alpha_1} + \sum_{i=2}^{n-1} \frac{|D(\alpha_i)||\beta_i|}{\alpha_i}, \end{cases} \tag{45}$$

are satisfied. Then the system S is asymptotically stable.

Suppose that $D(s)$ admits n distinct real roots p_i, $\ i = 1, ..., n$ among which there are $n-1$ negative ones. We use the fact that $a_{n-1} = -\sum_{i=1}^{n} p_i$, then the choice $\alpha_i = p_i$, $\forall i = 1, .., n-2$ and $\alpha_{n-1} = p_{n-1} + \varepsilon$ permits us to write $\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} p_i = p_n - \varepsilon$. In this case the last proposition becomes

Proposition 2 *If $D(s)$ admits $n-1$ distinct real negative roots such that the following conditions*

$$\begin{cases} p_n - \varepsilon < 0, \\ \mu_2(.) + 2\tau\nu_2(.) - \xi_2(.) < 0, \end{cases} \tag{46}$$

are satisfied, where

$$\begin{cases} \mu_2(.) = p_n - \varepsilon, \\ \nu_2(.) = \bar{g}, \\ \xi_2(.) = \frac{|\hat{g}(.)||\beta_1|}{\alpha_1} + \frac{|D(\alpha_{n-1})||\beta_{n-1}|}{\alpha_{n-1}}, \end{cases} \tag{47}$$

then the system S is asymptotically stable.

3.2 Case $e(t) = -Kx(t)$

In this case, take $e(t) = -Kx(t)$ with $K = (k_0, k_1, \ldots, k_{n-1})$, then the obtained system has the same form as (1), with $\hat{g}_1^K(.) = \hat{g}^K(.)\,(k_0+1)$ and $\hat{g}_i^K(.) = \hat{g}^K(.)k_{i-1}$, $i = 2, \ldots, n$. The stabilizing values of K can be obtained by making the following changes: $\gamma_n = -a_{n-1} - \sum_{i=1}^{n-1} \alpha_i$, $\delta_n^K(.) = -\hat{g}^K(.)k_{n-1}$, $\nu_1^K(.) = \bar{g}^K \sum\limits_{i=1}^{n-1} \left|\tilde{N}(\alpha_i)\right|$ where $\bar{g}^K = \sup_{[.]} |\hat{g}^K(.)|$ and $\tilde{N}(\alpha) = (1+k_0) + \sum\limits_{i=1}^{n-1} (b_i + k_i)\alpha^i$.

Proposition 3 *If there exist distinct $\alpha_i < 0$, $i = 1, ..., n-1$, such that the following conditions*

$$\begin{cases} \gamma_n - \hat{g}^K(.)k_{n-1} < 0, \\ \tau < \dfrac{1}{2\bar{g}^K|k_{n-1}|}, \\ \mu_1^K(.) + 2\tau\nu_1^K(.) - \xi_1^k(.) < 0, \end{cases} \tag{48}$$

where

$$\begin{cases} \mu_1^K(.) = (1 - 2\bar{g}^K\tau|k_{n-1}|)(\gamma_n + \delta_n^K(.)), \\ \nu_1^K(.) = \bar{g}^K \sum_{i=1}^{n-1} |\beta_i||\tilde{N}(\alpha_i)|, \\ \xi_1^K(.) = \sum_{i=1}^{n-1} \frac{|D(\alpha_i)+\hat{g}^K(.)\tilde{N}(\alpha_i)||\beta_i|}{\alpha_i}, \end{cases} \tag{49}$$

are satisfied. Then the system S is asymptotically stable.

By a special choice of K the result of Proposition 3 can be simplified. In fact, if the conditions of this proposition are verified we can choose the vector K such that $D(p_i) = \tilde{N}(p_i)$. In this case we obtain $D(p_i) = \tilde{N}(p_i) = 0$, $\forall$, $i = 1, ..., n-1$ and $\nu_1(.) = \xi_1(.) = 0$ which yields the following new proposition.

Proposition 4 *If $D(s)$ admits $n-1$ distinct real negative roots p_i such that the following conditions*

$$\begin{cases} p_n - \bar{g}^K(.)k_{n-1} < 0, \quad p_n \textit{ is the n-th root of } D(s), \\ \tau < \dfrac{1}{2\bar{g}^K|k_{n-1}|}, \end{cases} \tag{50}$$

are satisfied. Then the system S is asymptotically stable.

4 Illustrative Example

Let us consider the block diagram in Figure 2 which describes the dynamics of a time-delayed DC motor speed control system with nonlinear gain, where:

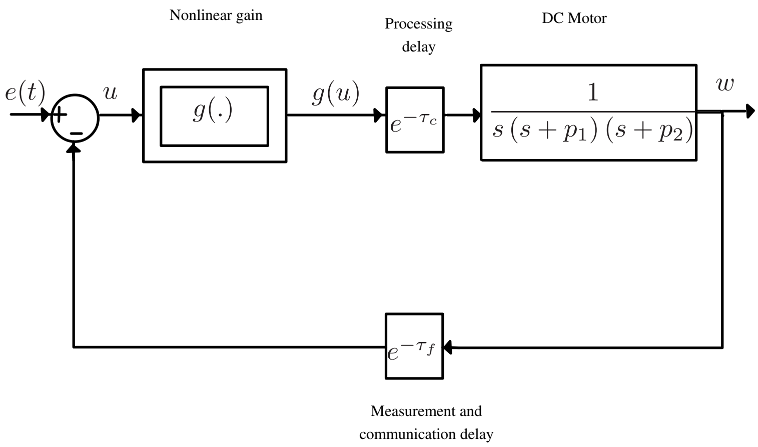

Figure 2: Block diagram of time-delayed DC motor speed control system with nonlinear gain.

- $p_1 = \frac{1}{T_e}$ and $p_2 = \frac{1}{T_m}$ where T_e and T_m are respectively electrical constant and mechanical constant.

- τ_f present the feedback delay between the output and the controller. This delay represents the measurement and communication delays (sensor-to-controller delay).

- τ_c the controller processing and communication delay (controller-to-actuator delay) is placed in the feedforward part between the controller and the DC motor.

- $g(.) : R \to R$ is a function that represents a nonlinear gain.

The process of Figure 2 can also be modeled by Figure 3, where $\tau = \tau_f + \tau_c$.

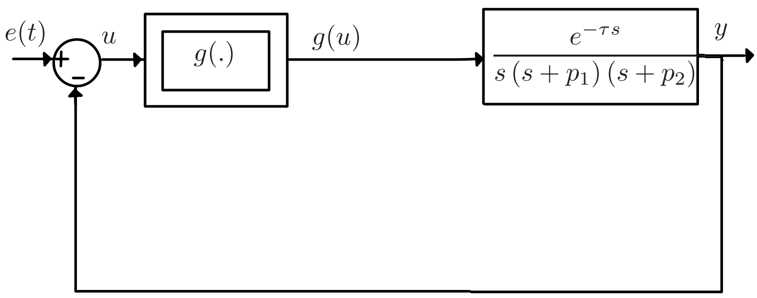

Figure 3: Delayed nonlinear model of DC motor speed control.

It is clear that model of Figure 3 is a particular form of delayed Lurie system in the case where $D(s) = s(s+p_1)(s+p_2) = s^3 + (p_1+p_2)s^2 + p_1p_2s$ and $N(s) = 1$. Thereafter, applying the result of Theorem 2.1, a stability condition of the system is that the matrix $T(.)$ given by

$$T(.) = \begin{pmatrix} \alpha_1 & 0 & |\,(\alpha_1-\alpha_2)^{-1}\,| \\ 0 & \alpha_2 & |\,(\alpha_2-\alpha_1)^{-1}\,| \\ t_1(.) & t_2(.) & t_3(.) \end{pmatrix}, \tag{51}$$

where

$$t_1(.) = |\gamma_1 + \hat{g}(.)| + \tau|\alpha_1|\bar{g}, \quad t_2(.) = |\gamma_2|, \quad t_3(.) = \gamma_3 + \tau|\beta_1|\bar{g},$$

must be the opposite of an M-matrix. By choosing α_i, $i = 1, 2$, negative real and distinct, we get the following stability condition:

$$\gamma_3 + 2\tau|\beta_1|\bar{g} - \frac{|\beta_1||\gamma_1 + \hat{g}(.)|}{\alpha_1} - \frac{|\beta_2||\gamma_2|}{\alpha_2} < 0. \tag{52}$$

For the particular choice of $\alpha_1 = -p_1$ and $\alpha_2 = -p_2 + \varepsilon$, $\varepsilon > 0$ yields $|\beta_1| = |\beta_2| = |(\varepsilon + p_1 - p_2)^{-1}|$ and we obtain the new stability condition:

$$2\tau\bar{g} + |p_1|^{-1}|\hat{g}(.)| + |\alpha_2|^{-1}|D(\alpha_2)| < \varepsilon|\varepsilon + p_1 - p_2|. \tag{53}$$

Assume that we have this inequality:

$$\bar{g} < |D(\alpha_2)|.$$

We can find from (53) the stabilizing delay given by the following condition:

$$\tau < \frac{1}{2}\left(\frac{\varepsilon|\varepsilon + p_1 - p_2|}{|D(\alpha_2)|} - |p_1|^{-1}| - |\alpha_2|^{-1}\right).$$

By applying the control $e(t) = -Kx(t)$ with $K = (k_0, k_1, k_2)$, we can determine the stabilizing values of K that can be obtained by making the following changes: $\gamma_3 = -(p_1 + p_2) - \sum_{i=1}^{2}\alpha_i$, $\delta_1^K(.) = -\hat{g}^K(.)\,(k_0 + 1)\,, \delta_i^K(.) = -\hat{g}^K(.)k_{i-1},\ \ i = 2, 3.$ $\nu_1^K(.) = \bar{g}^K \sum_{i=1}^{2}|\beta_i||\tilde{N}(\alpha_i)|$ where $\bar{g}^K = \sup_{[.]}|\hat{g}^K(.)|$ and $\tilde{N}(\alpha) = 1 + k_0 + \sum\limits_{i=1}^{2} k_i\alpha^i$.

If we choose $\alpha_i < 0$, $i = 1, 2$, such that the following conditions $D(\alpha_i) = \tilde{N}(\alpha_i) = 0$, $\forall$, $i = 1, 2$ hold, we get $\dfrac{1 + k_0}{k_2} = p_1 + p_2$, $\dfrac{k_1}{k_2} = p_1p_2$ and from Proposition 3 the stabilizing gain values satisfying the following relations:

$$\begin{cases} 0 - \bar{g}^K(.)k_2 < 0, \\ |k_2| < \dfrac{1}{2\tau\bar{g}^K}. \end{cases}$$

Finally we find the domain of stabilizing $k_0,\ k_1,\ k_2$ as follows

$$\begin{cases} 0 < k_2 < \dfrac{1}{2\tau\bar{g}^K}, \\ k_1 = p_1p_2k_2, \\ k_0 = (p_1 + p_2)\,k_2 - 1. \end{cases}$$

5 Conclusion

In this paper, new sufficient stability conditions for a class of time delay systems are derived. The proposed method is based on a specific choice of a Lyapunov function. The obtained conditions are explicit and easy to apply. Indeed, the proposed approach is successfully applied to nonlinear n-th order all pole plant that is a particular form of delayed Lurie Postnikov systems. In addition, the simplicity of the application of these criteria is demonstrated on model of time-delayed DC motor speed control.

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