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Perturbed Partial Fractional Order Functional Differential Equations with Infinite Delay in Fréchet Spaces

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Abstract: In this paper we investigate the existence of solutions of perturbed partial hyperbolic differential equations of fractional order with infinite delay and Caputo's fractional derivative by using a nonlinear alternative of Avramescu on Fréchet spaces.

Keywords: *partial functional differential equation; fractional order; solution; left-sided mixed Riemann-Liouville integral; Caputo fractional-order derivative; infinite delay; Fréchet space; fixed point.*

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1 Introduction

In this paper we are concerned with the existence of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) = f(t,x,u_{(t,x)}) + g(t,x,u_{(t,x)}),\ \text{if } (t,x)\in J, \tag{1}$$

$$u(t,x) = \phi(t,x),\ \text{if } (t,x)\in \tilde{J}, \tag{2}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ (t,x)\in J, \tag{3}$$

where $\varphi(0) = \psi(0)$, $J := [0,\infty)\times[0,\infty)$, $\tilde{J} := (-\infty,+\infty)\times(-\infty,+\infty)\backslash[0,\infty)\times[0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in$

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$(0,1] \times (0,1], f, g : J \times \mathcal{B} \Rightarrow \mathbb{R}^n$ are given functions, $\phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function with $\phi(t,0) = \varphi(t),\ \phi(0,x) = \psi(x)$ for each $(t,x) \in J, \varphi : [0,\infty) \to \mathbb{R}^n$, $\psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous functions and $\mathcal{B}$ is called a phase space that will be specified in Section 3.

We denote by $u_{(t,x)}$ the element of $\mathcal{B}$ defined by

$$u_{(t,x)}(s,\tau) = u(t+s, x+\tau);\ (s,\tau) \in (-\infty,0] \times (-\infty,0],$$

here $u_{(t,x)}(.,.)$ represents the history of the state u.

There has been a significant development in ordinary and partial fractional differential equations in recent years. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [1–5]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs [6–8], and the papers [9–15] and the references therein.

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books [16–20], and the papers [21, 22].

In this paper, we present existence result for the problem (1)-(3). Our main result for this problem is based on a nonlinear alternative for the sum of a completely continuous operator and a contraction one in Fréchet spaces due to Avramescu [23]. To our knowledge, there are very few papers devoted to fractional differential equations with delay on Fréchet spaces. This paper can be considered as a contribution in this setting case.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $p \in \mathbb{N}$ and $J_0 = [0,p] \times [0,p]$. By $C(J_0, \mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(t,x)\in J_0} \|w(t,x)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J_0, \mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0, \mathbb{R})$ is the space of Lebesgue-integrable functions $w : J_0 \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^p \int_0^p \|w(t,x)\| dt dx.$$

Definition 2.1 [24] Let $r = (r_1, r_2) \in (0,\infty) \times (0,\infty), \theta = (0,0)$ and $u \in L^1(J_0, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau) d\tau ds;\ \text{for almost all } (t,x) \in J_0,$$

where $\sigma = (1,1)$. For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)\times(0,\infty)$, when $u \in L^1(J_0,\mathbb{R}^n)$. Note also that when $u \in C(J_0,\mathbb{R}^n)$, then $(I_\theta^r u) \in C(J_0,\mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t,x \in J_0.$$

Example 2.1 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},\ \text{for almost all } (t,x) \in J_0.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1)\times[0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.2 [24] Let $r \in (0,1]\times(0,1]$ and $u \in L^1(J_0,\mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^cD_0^r u)(t,x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(t,x) = ({}^cD_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x),\ \text{for almost all } (t,x) \in J_0.$$

Example 2.2 Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1]\times(0,1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2},\ \text{for almost all } (t,x) \in J_0.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.1 *[25] Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(t,x) \le \omega(t,x) + c\int_0^t \int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c\int_0^t \int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

for every $(t,x) \in J$.

3 The Phase Space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms, which was introduced by Hale and Kato (see [22]). For further applications see for instance the books [16, 17, 19] and their references.

For any $(t, x) \in J$ denote $E_{(t,x)} := [0, t] \times \{0\} \cup \{0\} \times [0, x]$, furthermore in case $t = a,\ x = b$ we write simply E. Consider the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $y : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ continuous on J and $y_{(t,x)} \in \mathcal{B}$, for all $(t, x) \in E$, then there are constants $H, K, M > 0$ such that for any $(t, x) \in J$ the following conditions hold:

(i) $y_{(t,x)}$ is in $\mathcal{B}$;

(ii) $\|y(t, x)\| \le H\|y_{(t,x)}\|_{\mathcal{B}}$,

(iii) $\|y_{(t,x)}\|_{\mathcal{B}} \le K \sup_{(s,\tau)\in[0,t]\times[0,x]} \|y(s, \tau)\| + M \sup_{(s,\tau)\in E_{(t,x)}} \|y_{(s,\tau)}\|_{\mathcal{B}}$,

(A_2) For the function $y(.,.)$ in (A_1), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [26, 27].

Example 3.1 Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0],\ \alpha, \beta \ge 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 3.2 Let $\gamma \in \mathbb{R}$ and let C_γ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,\tau)\|\to\infty} e^{\gamma(s+\tau)}\phi(s, \tau)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s, \tau)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-\gamma(a+b)}, 1\}$.

Example 3.3 Let $\alpha, \beta, \gamma \ge 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s, \tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s, \tau)\| d\tau ds$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1, \; K = \int_{-\alpha}^{0} \int_{-\beta}^{0} e^{\gamma(s+\tau)} d\tau ds, \; M = 2.$$

4 Some Properties in Fréchet Spaces

Let X be a Fréchet space with a family of semi-norms $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$. We assume that the family of semi-norms $\{\| \cdot \|_n\}$ verifies :

$$\|u\|_1 \leq \|u\|_2 \leq \; \|u\|_3 \leq \ldots \quad \textit{for every } u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \textit{for all } y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote by $X^n = (X|_{\sim_n}, \| \cdot \|_n)$ the quotient space, the completion of X^n with respect to $\| \cdot \|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $u \in X$, we denote by $[u]_n$ the equivalence class of u of subset X^n and we define $Y^n = \{[u]_n : u \in Y\}$. We denote by $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\| \cdot \|_n$ in X^n. For more information about this subject see [28].

Definition 4.1 Let X be a Fréchet space. A function $N : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in [0, 1)$ such that

$$\|N(u) - N(v)\|_n \leq k_n \|u - v\|_n \; \text{ for all } u, v \in X.$$

Theorem 4.1 *(Nonlinear Alternative of Avramescu) [23] Let $(X, |.|_n)$ be a Fréchet space and let $A, B : X \to X$ be two operators. Suppose that the following hypotheses are fulfilled:*

(i) A is a compact operator;

(ii) B is a contraction operator with respect to a family of seminorms $||.||_n$ equivalent to the family $|.|_n$;

(iii) the set $\mathcal{E} = \{u \in X : u = \lambda A(u) + \lambda B(\frac{u}{\lambda})$ for some $\lambda \in (0, 1)\}$ is bounded.

Then there is $u \in X$ such that $u = Au + Bu$.

5 Existence of Solutions

In this section, we give our main existence result for problem (1)-(3). Before starting and proving this result, we give what we mean by a solution of this problem. Let the space

$$\Omega := \{u : \mathbb{R}^2 \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B} \text{ for } (t, x) \in E \text{ and } u|_J \in C(J, \mathbb{R}^n)\}.$$

Definition 5.1 A function $u \in \Omega$ is said to be a solution of (1)-(3) if u satisfies equations (1) and (3) on J and the condition (2) on $\tilde{J}$.

For the existence of solutions for the problem (1)–(3), we need the following lemma:

Lemma 5.1 *A function $u \in \Omega$ is a solution of problem (1)-(3) if and only if u satisfies the equation*

$$\begin{aligned} u(t,x) &= z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)}) d\tau ds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)}) d\tau ds, \end{aligned}$$

for all $(t,x) \in J$ and the condition (2) on $\tilde{J}$.

For each $p \in \mathbb{N}$ we consider following sets,

$$C_p = \{u : (-\infty,p] \times (-\infty,p] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B},\ \ u_{(t,x)} = 0 \text{ for } (t,x) \in E \text{ and } u|_{J_0} \in C(J_0,\mathbb{R}^n)\},$$

and $C_0 = \{u \in \Omega : u_{(t,x)} = 0 \text{ for } (t,x) \in E\}$.

On C_0 we define the semi-norms:

$$\|u\|_p = \sup_{(t,x)\in E} \|u_{(t,x)}\| + \sup_{(t,x)\in J_0} \|u(t,x)\| = \sup_{(t,x)\in J_0} \|u(t,x)\|, \quad u \in C_p.$$

Then C_0 is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Theorem 5.1 *Assume:*

$(H1)$ *The functions $f, g : J \times \mathcal{B} \to \mathbb{R}^n$ are continuous.*

$(H2)$ *For each $p \in \mathbb{N}$, there exist constants $\ell_p(t,x) \in C(J_0,\mathbb{R}^n)$ such that*

$$\|g(t,x,u) - g(t,x,v)\| \le \ell_p(t,x)\|u-v\|_{\mathcal{B}}, \text{ for any } u,v \in \mathcal{B} \text{ and } (t,x) \in J_0.$$

$(H3)$ *For each $p \in \mathbb{N}$, there exist $p, q \in C(J,\mathbb{R}_+)$ such that*

$$\|f(t,x,u)\| \le p(t,x) + q(t,x)\|u\|_{\mathcal{B}}, \text{ for } (t,x) \in J_0 \text{ and each } u \in \mathcal{B}.$$

If

$$\frac{K\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{4}$$

where $\ell_p^* = \sup_{(t,x)\in J_0} \ell_p(t,x)$, then there exists a unique solution for IVP (1)-(3) on $(-\infty,+\infty) \times (-\infty,+\infty)$.

Proof. Transform the problem (1)-(3) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by,

$$(Nu)(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) \\ \quad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)}) d\tau ds \\ \quad + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)}) d\tau ds, & (t,x) \in J. \end{cases} \tag{5}$$

Let $v(.,.) : \mathbb{R}^n \to \mathbb{R}^n$ be a function defined by,

$$v(t,x) = \begin{cases} z(t,x), & (t,x) \in J. \\ \phi(t,x), & (t,x) \in \tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x) \in E$.

For each $w \in C(J,\mathbb{R}^n)$ with $w(t,x) = 0$ for each $(t,x) \in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x), & (t,x) \in J, \\ 0, & (t,x) \in \tilde{J}. \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$\begin{aligned} u(t,x) \;&=\; z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,u_{(s,\tau)}) d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,u_{(s,\tau)}) d\tau ds, \end{aligned}$$

we can decompose $u(.,.)$ as $u(t,x) = \overline{w}(t,x) + v(t,x);\ (t,x) \in J$, which implies $u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}$, for every $(t,x) \in J$, and the function $w(.,.)$ satisfies

$$\begin{aligned} w(t,x) \;&=\; \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)}) d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)}) d\tau ds. \end{aligned}$$

Let the operators $A, B :\ C_0 \to C_0$ be defined by

$$(Aw)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)}) d\tau ds$$

and

$$(Bw)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\overline{w}_{(t,x)} + v_{(t,x)}) d\tau ds.$$

Obviously, the operator N has a fixed point which is equivalent to finding the fixed point of the operator equation $(Aw)(t,x) + (Bw)(t,x) = w(t,x), (t,x) \in J$. We shall show that the operators A and B satisfy all the conditions of Theorem 4.1.

For better readability, we break the proof into a sequence of steps.

Step 1: A is continuous.

Let $\{w_n\}$ be a sequence such that $w_n \to w$ in C_0. Then

$$\begin{aligned} \|(Aw_n)(t,x) - (Aw)(t,x)\| \;&\leq\; \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\times\; \|f(s,\tau,\overline{w}_{n(s,\tau)} + v_{n(s,\tau)}) - f(s,\tau,\overline{w}_{(s,\tau)} + v_{(s,\tau)})\| d\tau ds. \end{aligned}$$

Since f is a continuous function, we have

$$\|(Aw_n) - (Aw)\|_p \leq \frac{p^{r_1+r_2} \|f(.,.,\overline{w_n}_{(.,.)} + v_{n(.,.)}) - f(.,.,\overline{w}_{(.,.)} + v_{(.,.)})\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} \to 0 \text{ as } n \to \infty.$$

Thus A is continuous.

Step 2: A maps bounded sets into bounded sets in C_0. Indeed, it is enough to show that, for any $\eta > 0$, there exists a positive constant $\widetilde{\ell}$ such that, for each $w \in B_\eta = \{w \in C_0 : \|w\|_p \leq \eta\}$, we have $\|A(w)\|_p \leq \widetilde{\ell}$.

Let $w \in B_\eta$. By $(H3)$ we have for each $(t,x) \in J_0$,

$$\begin{aligned}\|(Aw)(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})\|d\tau ds\\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}p(s,\tau)\\ &\quad +q(s,\tau)\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds\\ &\leq \frac{\|p\|_p}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\ &\quad +\frac{\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\ &\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}p^{r_1+r_2} := \ell^*,\end{aligned}$$

where

$$\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} \leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}} \leq K_p\eta+K_p\|\phi(0,0)\|+M_p\|\phi\|_{\mathcal{B}} := \eta^*.$$

Hence $\|(Aw)\|_p \leq \ell^*$.

Step 3: A maps bounded sets into equicontinuous sets in C_0.
Let $(t_1,x_1),(t_2,x_2) \in J_0$, $t_1 < t_2$, $x_1 < x_2$, B_η be a bounded set as in Step 2, and let $w \in B_\eta$. Then

$$\begin{aligned}&\|(Aw)(t_2,x_2)-(Aw)(t_1,x_1)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}\\ &-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\ +&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\ +&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\\ +&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})||d\tau ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}d\tau ds\\
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[x_2^{r_2}(t_2-t_1)^{r_1}+t_2^{r_1}(x_2-x_1)^{r_2}\\
&\quad -(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}]\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[t_2^{r_1}-(t_2-t_1)^{r_1}](x_2-x_1)^{r_2}\\
&+ \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}(t_2-t_1)^{r_1}[x_2^{r_2}-(x_2-x_1)^{r_2-1}\\
&\leq \frac{\|p\|_p+\|q\|_p\eta^*}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1}+2t_2^{r_1}(x_2-x_1)^{r_2}\\
&\quad +t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

The right-hand side of the above inequality tends to zero as $t_1 \to t_2,\ x_1 \to x_2$. The equicontinuity for the cases $t_1 < t_2 < 0,\ x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2,\ x_1 \leq 0 \leq x_2$ is obvious.

As a consequence of steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude that $A : C_0 \to C_0$ is a compact operator.

Step 4: B is a contraction.
Let $w, w^* \in C_0$. Then we have for each $(t,x) \in J_0$

$$
\begin{aligned}
\|(Bw)(t,x) &- (Bw^*)(t,x)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\\
&\times \|g(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})-g(s,\tau,\overline{w^*}_{(s,\tau)}+v_{(s,\tau)})\|d\tau ds\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)\|\overline{w}_{(s,\tau)}-\overline{w^*}_{(s,\tau)}\|_{\mathcal{B}}\\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}K\ell_p(s,\tau)\\
&\times \sup_{(s,\tau)\in[0,t]\times[0,x]}\|\overline{w}(s,\tau)-\overline{w^*}(s,\tau)\|d\tau ds\\
&\leq \frac{K\ell_p^*(s,\tau)}{\Gamma(r_1)\Gamma(r_2)}\int_0^p\int_0^p(t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\|\overline{w}-\overline{w^*}\|_p.
\end{aligned}
$$

Therefore,

$$\|(Bw)-(Bw^*)\|_p \leq \frac{K\ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|\overline{w}-\overline{w^*}\|_p,$$

since by (4), B is a contraction.

Step 5: (A priori bounds)

Now it remains to show that the set

$$\mathcal{E}=\{w \in C(J,\mathbb{R}) : w = \lambda A(w) + \lambda B(\tfrac{w}{\lambda}) \text{ for some } \lambda \in (0,1)\}$$

is bounded. Let $w \in \mathcal{E}$, then $w = \lambda A(w) + \lambda B(\frac{w}{\lambda})$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$, we have

$$\begin{aligned} w(t,x) &= \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau,\overline{w}_{(s,\tau)}+v_{(s,\tau)})d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} g(s,\tau,\frac{\overline{w}_{(s,\tau)}+v_{(s,\tau)}}{\lambda})d\tau ds. \end{aligned}$$

This implies by $(H2)$ and $(H3)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned} \|w(t,x)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}[p(s,\tau) \\ &\quad +q(s,\tau)\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}]d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \left| g(s,\tau,\frac{\overline{w}_{(s,\tau)}+v_{(s,\tau)}}{\lambda}) - g(s,\tau,0)\right| d\tau ds \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|g(s,\tau,0)|d\tau ds \\ &\leq \frac{p^{r_1+r_2}\|p\|_p}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{p^{r_1+r_2}g^*}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{||q||_p}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_p(s,\tau)||\overline{w}_{(s,\tau)}+v_{(s,\tau)}||_{\mathcal{B}}d\tau ds \\ &\leq \frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{(||q||_p+\ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}d\tau ds, \end{aligned}$$

where $g^* = \sup_{(s,\tau)\in J_0} |g(s,\tau,0)|$ and

$$\begin{aligned} \|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}} + \|v_{(s,\tau)}\|_{\mathcal{B}} \\ &\leq K\sup\{w(\tilde{s},\tilde{\tau}) : (\tilde{s},\tilde{\tau}) \in [0,s]\times[0,\tau]\} \\ &\quad + M\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \end{aligned} \tag{6}$$

If we name $y(s,\tau)$ the right hand side of (6), then we have

$$\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}}\leq y(t,x),$$

and therefore, for each $(t,x)\in J_0$ we obtain

$$\|w(t,x)\|\leq\frac{p^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$+\frac{||q||_p+\ell_p^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,\tau)d\tau ds. \quad (7)$$

Using the above inequality and the definition of y for each $(t,x)\in J_0$ we have

$$\begin{aligned}y(t,x) &\leq M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+\frac{Kp^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}\\ &\quad+\frac{K(\|q\|_p+\ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}y(s,t)d\tau ds.\end{aligned}$$

Then by Lemma 2.1, there exists $\delta=\delta(r_1,r_2)$ such that we have

$$\|y(t,x)\| \quad\leq\quad R+\delta\frac{K(\|q\|_p+\ell_p^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}Rd\tau ds,$$

where

$$R=M\|\phi\|_{\mathcal{B}}+K\|\phi(0,0)\|+\frac{Kp^{r_1+r_2}(\|p\|_p+g^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|y\|_p\leq R+\frac{R\delta Kp^{r_1+r_2}(\|q\|_p+\ell_p^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}:=\widetilde{R}.$$

Then, (7) implies that

$$\|w\|_p\leq\frac{p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}[\|p\|_p+g^*+\widetilde{R}(\|q\|_p+\ell_p^*)]:=R_p^*.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 4.1 we deduce that $A+B$ has a fixed point which is a solution of problem (1)-(3). □

6 An Example

As an application of our results we consider the following partial perturbed hyperbolic functional differential equations of the form

$$({}^cD_0^ru)(t,x)=\frac{2+e^{t+x}(|u(t-2,x-3)|+3)}{c_pe^{t+x}(2+|u(t-2,x-3)|)},\ \text{ if }(t,x)\in J:=[0,\infty)\times[0,\infty), \quad (8)$$

$$u(t,0)=t,\ u(0,x)=x^2,\ (t,x)\in J, \quad (9)$$

$$u(t,x)=t+x^2,\ (t,x)\in\tilde{J}, \quad (10)$$

where $\tilde{J} := \mathbb{R}^2 \backslash [0, \infty) \times [0, \infty)$.
Set

$$f(t, x, u_{(t,x)}) = \frac{|u(t-2, x-3)| + 3}{c_p(2 + |u(t-2, x-3)|)}, \quad \text{if } (t, x) \in J,$$

$$g(t, x, u_{(t,x)}) = \frac{2}{c_p e^{t+x}(2 + |u(t-2, x-3)|)}, \quad \text{if } (t, x) \in J,$$

and

$$c_p = \frac{3p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Let $\gamma > 0$, and consider the following phase space

$$\mathcal{B}_\gamma = \{u \in C((-\infty, 0] \times (-\infty, 0], \mathbb{R}) : \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta, \eta) \text{ exists} \in \mathbb{R}\}.$$

The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta) \in (-\infty,0] \times (-\infty,0]} e^{\gamma(\theta+\eta)} |u(\theta, \eta)|.$$

Let

$$E := [0, 1] \times \{0\} \cup \{0\} \times [0, 1],$$

and $u : (-\infty, 1] \times (-\infty, 1] \to \mathbb{R}$ such that $u_{(t,x)} \in \mathcal{B}_\gamma$ for $(t, x) \in E$, then

$$\lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u_{(t,x)}(\theta, \eta) = \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta - t + \eta - x)} u(\theta, \eta)$$

$$= e^{\gamma(t+x)} \lim_{\|(\theta,\eta)\| \to \infty} u(\theta, \eta) < \infty.$$

Hence $u_{(t,x)} \in \mathcal{B}_\gamma$. Finally we prove that

$$\|u_{(t,x)}\|_\gamma = K \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}$$

$$+ M \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E_{(t,x)}\},$$

where $K = M = 1$ and $H = 1$.

If $t + \theta \leq 0,\ x + \eta \leq 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\},$$

and if $t + \theta \geq 0,\ x + \eta \geq 0$, then we have

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}.$$

Thus for all $(t + \theta, x + \eta) \in [0, 1] \times [0, 1]$, we get

$$\begin{aligned} \|u_{(t,x)}\|_\gamma \quad &= \quad \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\} \\ &\quad + \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}. \end{aligned}$$

Then

$$\|u_{(t,x)}\|_\gamma = \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E\} + \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space.

For each $u,\ \overline{u} \in \mathcal{B}_\gamma$ and $(t,x) \in J$, we have

$$|g(t,x,u_{(t,x)}) - g(t,x,\overline{u}_{(t,x)})| \leq \frac{1}{c_p e^{t+x}} \|u - \overline{u}\|_{\mathcal{B}_\gamma}.$$

Hence condition $(H2)$ is satisfied with $\ell_p e^{t+x} = \frac{1}{c_p e^{t+x}}$. Since

$$\ell_p^* = \sup \left\{ \frac{1}{c_p e^{t+x}}, \quad (t,x) \in J \times \mathbb{R} \right\} \leq \frac{1}{c_p}$$

and $K = 1$, we get

$$\frac{k \ell_p^* p^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{3} < 1.$$

Hence condition (4) holds for each $(r_1, r_2) \in (0,1] \times (0,1]$ and all $p \in \mathbb{N}^*$. Also, the function f is continuous on $[0,\infty) \times [0,\infty) \times [0,\infty)$ and

$$|f(t,x,w)| \leq |w| + 3, \text{ for each } (t,x,w) \in [0,\infty) \times [0,\infty) \times \mathcal{B}_\gamma.$$

Thus conditions $(H1)$ and $(H3)$ hold. Consequently Theorem 5.1 implies that problem (8)-(10) has at least one solution defined on $(-\infty,+\infty) \times (-\infty,+\infty)$.

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