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The Obstacle Problem Associated with Nonlinear Elliptic Equations in Generalized Sobolev Spaces

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Abstract: We prove an existence result of entropy solution to the obstacle problem associated with the equation of the type

$$-\mathrm{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f \in L^1(\Omega)$$

in generalized Sobolev spaces, without assuming the sign condition in the nonlinearity g via penalization methods.

Keywords: *generalized Sobolev spaces; boundary value problems; truncations; penalized equations.*

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1 Introduction

The obstacle problem is, roughly speaking, about solving a partial differential equation with the additional constraint that the solution is required to stay above a given function, the obstacle. This leads to a variational inequality. From a minimization point of view, the problem is to find a minimizer with fixed boundary value in the set of functions lying above the obstacle function. Such a set is convex and thus, a unique minimizer exists under reasonable assumptions. The balayage concept of potential theory which is the potential theoretic viewpoint of the obstacle problem is finding the smallest superharmonic function which lies above the obstacle.

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In this paper, we deal with the obstacle problem associated with the following quasi-linear elliptic equations

$$-\operatorname{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f \in L^1(\Omega) \tag{1}$$

with non-standard structural conditions which involve a variable growth exponent $p(.)$. We prove some existence result of entropy solution under the assumption that g has a constant sign. A problem like (1) was studied by Azroul, Benboubker and Rhoudaf in [1], where they proved the existence of entropy solutions by using a decomposition method of the measure μ.

The study of partial differential equations and variational problems involving $p(x)$-growth conditions has received specific attention in recent decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics. Electrorheological fluids and elastic mechanics are two examples of physical fields which benefit from such kinds of studies. In that context, we refer to Diening [7], Ruzicka [18], and the references therein.

Most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant, we recall some papers (and references therein), in which this theory is developed: [1, 5, 6, 11]. For electrorheological fluids, this is not adequate, but rather the exponent p should be able to vary. This situation leads us to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ where $p(.)$ is a real-valued function.

The variable exponent Lebesgue Spaces $L^{p(\cdot)}$, where $p(.)$ is a real-valued function, appeared in the literature for the first time in 1931 in the paper by W.Orlicz [16]. In the 1950s, this study was carried out by Nakano [14] who made the first systematic study of spaces with a variable exponent. Later, Polish and Czechoslovak mathematicians investigated the modular function spaces (see e.g. [13] and [10]). Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In that context, we refer to the work of Tsenov [19] and Zhikov ([22, 23]). The interested reader of the theory of Lebesgue and Sobolev spaces with a variable exponent can find numerous further references in the monograph [8]. Recently, some papers have appeared in the case of the obstacle problem with a variable exponent. See ([15, 17]) for existence and uniqueness of an entropy solution, in the framework of Lewy-Stampacchia inequalities.

A treatment of the obstacle problem (1) in the L^p-case can be found in [3] where the main goal in this work is to obtain a solution with $f \in L^1(\Omega)$ in the general settings of Orlicz-Sobolev spaces. We are interested, in this paper, in the single obstacle problem associated with equation (1), where the techniques used to study this problem are based on the following approximate problems,

$$(\mathcal{P}_\epsilon)\begin{cases} -\operatorname{div}(a(x,u_\epsilon,\nabla u_\epsilon)) + g_\epsilon(x,u_\epsilon,\nabla u_\epsilon) & = f_\epsilon \text{ in } \Omega, \\ u_\epsilon & = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g_\epsilon(x,s,\xi) = \dfrac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$ and f_ϵ is a sequence of regular functions.

Nevertheless, this approximation can not enable to obtain the a priori estimates in our case, this is due to the fact that $u_\epsilon g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)$ has no sign. To overcome this difficulty, one has introduced a doubling approximation, that is we penalized the problem $(\mathcal{P}_\epsilon)$ by

$$(\mathcal{P}^\sigma_\epsilon)\begin{cases} -\operatorname{div}(a(x,u^\sigma_\epsilon,\nabla u^\sigma_\epsilon)) + g^\sigma_\epsilon(x,u^\sigma_\epsilon,\nabla u^\sigma_\epsilon) - \dfrac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u^{\sigma^-}_\epsilon)|^{p(x)-1} & = f_\epsilon \text{ in } \Omega, \\ u^\sigma_\epsilon & = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g_\epsilon^\sigma(x,s,\xi) = \delta_\sigma(s) g_\epsilon(x,s,\xi)$ and where $\delta_\sigma(s)$ is some increasing Lipschitz-function (see Sections 4 and 5). Note also that the obstacle in the problem considered in this paper seems to follow the sign of the nonlinearity g.

As application to the problem considered in this paper, we have the Stefan problem which is a particular kind of boundary value problem for a partial differential equation (PDE), adapted to the case in which a phase boundary can move with time. The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, for example ice passing to water.

Our simplest model is the following $L^{p(\cdot)}$-problem,

$$-\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) + |u|^{r(x)}|\nabla u|^{p(x)} = f \text{ in } \Omega, u = 0 \text{ on } \partial\Omega,$$

generated by the $p(x)$-Laplacian operator.

The paper is organized as follows. In Section 2, we present the preliminaries about Lebesgue and Sobolev spaces with variable exponent. In Section 3, we introduce the assumptions and prove some fundamental lemmas. In Section 4, we prove the existence of entropy solutions to the obstacle problem associated with (1) for the case of positive nonlinearity g. Finally, in Section 5, we prove the existence of entropy solutions to the obstacle problem associated with (1) for the case of negative nonlinearity g.

2 A Framework for Function Spaces

For each open bounded subset Ω of $I\!R^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{p | p \in \mathcal{C}(\overline{\Omega}),\ p(x) > 1 \text{ for any } x \in \bar{\Omega}\}.$$

For every $p \in \mathcal{C}_+(\overline{\Omega})$ we define: $p_+ = \sup_{x\in\Omega} p(x)$ and $p_- = \inf_{x\in\Omega} p(x)$.

We define the variable exponent Lebesgue space by:

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}.$$

The Luxemburg norm on the space $L^{p(x)}(\Omega)$ is defined by

$$\|u\|_{p(x)} = \inf\left\{ \lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1 \right\}.$$

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [9], [21]). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Generalized Hölder inequality

$$\left|\int_\Omega u\,v\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\,\|v\|_{p'(x)},$$

holds true.

Proposition 1 ***(see [9, 21])*** *We denote $\rho(u) = \int_\Omega |u|^{p(x)}\,dx$, $\forall u \in L^{p(x)}(\Omega)$. If $u_n, u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$, then the following assertions hold:*
(i) $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \Leftrightarrow \rho(u) < 1 \quad (resp, = 1, > 1)$,
(ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p_-} \leq \rho(u) \leq \|u\|_{p(x)}^{p_+}$; $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p_+} \leq \rho(u) \leq \|u\|_{p(x)}^{p_-}$,
(iii) $\|u_n\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u_n) \to 0$; $\|u_n\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u_n) \to \infty$.

We define the generalized Sobolev space by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega) \text{ and } |\nabla u|\in L^{p(x)}(\Omega)\}.$$

It is endowed with the following norm

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)} \qquad \forall u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and $p^*(x)=\frac{Np(x)}{N-p(x)}$ for $p(x)<N$.

Proposition 2 ***(see [9]) (i)*** *Assuming $p_->1$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
(ii) *If $q\in\mathcal{C}_+(\bar\Omega)$ and $q(x)<p^*(x)$ for any $x\in\overline{\Omega}$, then $W^{1,p(x)}(\Omega)\hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous.*
(iii) *There is a constant $C>0$, such that*

$$\|u\|_{p(x)}\le C\,\|\nabla u\|_{p(x)} \quad \forall u\in W_0^{1,p(x)}(\Omega),\ \textit{if } p\in\mathcal{C}(\overline{\Omega}).$$

Therefore, $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms in $W_0^{1,p(\cdot)}(\Omega)$.

3 Basic Assumptions and Some Fundamental Lemmas

Let $p\in\mathcal{C}_+(\bar\Omega)$ such that $1<p_-\le p(x)\le p_+<\infty$ and denote $Au=-\mathrm{div}(a(x,u,\nabla u))$, where $a:\Omega\times I\!R\times I\!R^N\to I\!R^N$ is a Carathéodory function satisfying the assumptions :

$$|a(x,s,\xi)|\le\beta[k(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}], \tag{2}$$

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0 \text{ for all } \xi\neq\eta\in I\!R^N, \tag{3}$$

$$a(x,s,\xi)\xi\ge\alpha|\xi|^{p(x)}, \tag{4}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in I\!R\times I\!R^N$, where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta,\alpha>0$.

Furthermore, let $g:\Omega\times I\!R\times I\!R^N\to I\!R$ be a Carathéodory function having a constant sign such that for a.e. x in Ω and for all $s\in I\!R$ and $\xi\in I\!R^N$,

$$|g(x,s,\xi)|\le b(|s|)(c(x)+|\xi|^{p(x)}), \tag{5}$$

$$g(x,0,\xi)=0, \tag{6}$$

where $b:I\!R^+\to I\!R^+$ is a continuous non-decreasing function and $c(.)$ is a positive function which belongs to $L^1(\Omega)$.

We introduce the functional spaces needed later. For $p\in\mathcal{C}_+(\bar\Omega)$, $\mathcal{T}_0^{1,p(x)}(\Omega)$ is defined as the set of measurable functions $u:\Omega\to I\!R$ such that the truncated functions $T_k(u)\in W_0^{1,p(x)}(\Omega)$, where $T_k(s):=\max\{-k,min\{k,s\}\}$, for $s\in I\!R$ and $k>0$.

We give the following lemma which is a generalization of Lemma 2.1 in [5] for generalized Sobolev spaces. Note that its proof is a slight modification of Lemma 2.1 in [5].

Lemma 3.1 *For every $u\in\mathcal{T}_0^{1,p(x)}(\Omega)$, there exists a unique measurable function $v:\Omega\to I\!R^N$ such that $\nabla T_k(u)=v\chi_{\{|u|<k\}}$, a.e. in Ω, for every $k>0$.*

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 3.2 *[4] Let $g \in L^{r(x)}(\Omega)$ and $g_n \in L^{r(x)}(\Omega)$ with $\|g_n\|_{L^{r(x)}(\Omega)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(x)}(\Omega)$.*

Lemma 3.3 *[4] Assume that (2)-(4) hold true, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)dx \to 0. \tag{7}$$

Then, $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

Lemma 3.4 *[2] Let $F : I\!R \longrightarrow I\!R$ be uniformly Lipschitzian with $F(0) = 0$ and $p \in \mathcal{C}_+(\overline{\Omega})$. Let $u \in W_0^{1,p(x)}(\Omega)$. Then $F(u) \in W_0^{1,p(x)}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e.\ in \quad \{x\in\Omega:\ u(x)\notin D\}, \\ \\ 0 & a.e.\ in \quad \{x\in\Omega:\ u(x)\in D\}. \end{cases}$$

Remark that the previous lemma implies that the functions in $W_0^{1,p(x)}(\Omega)$ can be truncated and as a consequence of this lemma we obtain the following result.

Lemma 3.5 *[2] Let $u \in W_0^{1,p(x)}(\Omega)$. Then, $T_k(u) \in W_0^{1,p(x)}(\Omega)$, with $k > 0$. Moreover, we have $T_k(u) \to u$ in $W_0^{1,p(x)}(\Omega)$ as $k \to \infty$.*

Definition 3.1 Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } Y \\ Bu_n \rightharpoonup \chi \text{ in } Y^* \\ \limsup\limits_{n\to\infty}\langle Bu_n, u_n\rangle \leq \langle \chi, u\rangle \end{array}\right\} \implies \chi = Bu \text{ and } \langle Bu_n, u_n\rangle \to \langle \chi, u\rangle.$$

Definition 3.2 Let Y be a reflexive Banach space, a bounded operator B from Y to its dual Y^* is called pseudo-monotone if

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } Y \\ \limsup\limits_{n\to\infty}\langle Bu_n, u_n - u\rangle \leq 0 \end{array}\right\} \implies \liminf\langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \text{ for all } v \in Y.$$

It is clear that the Definition 3.1 is equivalent to the well known Definition 3.2.

4 Statement of the Case of a Positive Nonlinearity g

We first consider the convex set $K_0 = \left\{u \in W_0^{1,p(x)}(\Omega); u \geq 0 \text{ a.e. in } \Omega\right\}$.

Theorem 4.1 *Assume that $(2) - (6)$ hold true and $f \in L^1(\Omega)$. Then there exists at least one solution (entropy solution) to the following unilateral problem,*

$$(\mathcal{P}) \begin{cases} u \in \mathcal{T}_0^{1,p(x)}(\Omega),\ u \geq 0\ a.e.\ in\ \Omega,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx, \\ \forall v \in K_0 \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases}$$

Proof of Theorem 4.1

We consider the following approximated problem

$$(\mathcal{P}_\epsilon)\begin{cases} -\mathrm{div}(a(x,u_\epsilon,\nabla u_\epsilon))+g_\epsilon(x,u_\epsilon,\nabla u_\epsilon) &= f_\epsilon \ \text{ in } \Omega,\\ u_\epsilon &= 0 \ \text{ on } \partial\Omega, \end{cases} \tag{8}$$

where $g_\epsilon(x,s,\xi)=\dfrac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$ and $f_\epsilon=T_{\frac{1}{\epsilon}}(f)$; then $(f_\epsilon)_{\epsilon>0}$ is a sequence of bounded functions which strongly converges to f in $L^1(\Omega)$ and $\|f_\epsilon\|_1\le\|f\|_1$, for all $\epsilon>0$.

Note that $|g_\epsilon(x,s,\xi)|\le|g(x,s,\xi)|\le b(|s|)(c(x)+|\xi|^{p(x)})$ and $|g_\epsilon(x,s,\xi)|\le\frac{1}{\epsilon}$. Nevertheless, it seems difficult to obtain a priori estimates, due to the fact that the quantity $u_\epsilon g_\epsilon(x,u_\epsilon,\nabla u_\epsilon)$ has no constant sign. In order to avoid this inconvenience, we approach the sign function by an increasing Lipschitz function.

Set for $\sigma>0$,

$$\delta_\sigma(s)=\begin{cases} \frac{s-\sigma}{s}, & \text{if } s\ge\sigma>0,\\ 0, & \text{if } |s|\le\sigma,\\ \frac{-s-\sigma}{s}, & \text{if } s<-\sigma<0. \end{cases}$$

Now, we set

$$g_\epsilon^\sigma(x,s,\xi)=\delta_\sigma(s)g_\epsilon(x,s,\xi). \tag{9}$$

Remark that $g_\epsilon^\sigma(x,s,\xi)$ has the same sign as s.

Now, we are in a position to approximate our initial unilateral problem by the following penalized problem

$$(\mathcal{P}_\epsilon^\sigma)\begin{cases} u_\epsilon^\sigma\in W_0^{1,p(x)}(\Omega)\\ \langle Au_\epsilon^\sigma,u_\epsilon^\sigma-v\rangle+\displaystyle\int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)(u_\epsilon^\sigma-v)\,dx-\frac{1}{\epsilon^2}\int_\Omega|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^-})|^{p(x)-1}(u_\epsilon^\sigma-v)\,dx\\ \qquad=\displaystyle\int_\Omega f_\epsilon(u_\epsilon^\sigma-v)\,dx,\qquad\forall v\in W_0^{1,p(x)}(\Omega). \end{cases} \tag{10}$$

We define the operators $G_\epsilon^\sigma,R_\epsilon^\sigma:W_0^{1,p(x)}(\Omega)\longrightarrow W^{-1,p'(x)}(\Omega)$ by,

$$\langle G_\epsilon^\sigma u,v\rangle=\int_\Omega g_\epsilon^\sigma(x,u,\nabla u)v\,dx,\ \langle R_\epsilon^\sigma u,v\rangle=-\frac{1}{\epsilon^2}\int_\Omega|T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}v\,dx.$$

We also denote

$$\langle Au,v\rangle=\int_\Omega a(x,u,\nabla u)\nabla v\,dx.$$

Thanks to the generalized Hölder's inequality, we have for all $u,v\in W_0^{1,p(x)}(\Omega)$,

$$\begin{aligned}\left|\int_\Omega g_\epsilon^\sigma(x,u,\nabla u)v\,dx\right| &\le(\frac{1}{p_-}+\frac{1}{p'_-})\|g_\epsilon^\sigma(x,u,\nabla u)\|_{p'(x)}\|v\|_{p(x)}\\ &\le(\frac{1}{p_-}+\frac{1}{p'_-})\left(\left(1+\frac{1}{\epsilon}\right)^{\frac{p'_+}{p'_-}}(\mathrm{meas}(\Omega)+1)^{\frac{1}{p'_-}}\right)\|v\|_{p(x)}\\ &\le C\|v\|_{1,p(x)}\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Big| -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}v\,dx\Big| &\le \frac{1}{\epsilon^2}\big(\frac{1}{p_-}+\frac{1}{p'_-}\big)\|T_{\frac{1}{\epsilon}}(u^-)^{p(x)-1}\|_{p'(x)}\|v\|_{p(x)}\\ &\le \frac{1}{\epsilon^2}\big(\frac{1}{p_-}+\frac{1}{p'_-}\big)\Big\|\Big(\frac{1}{\epsilon}\Big)^{p(x)-1}\Big\|_{p'(x)}\|v\|_{p(x)}\\ &\le C\|v\|_{1,p(x)}.\end{aligned} \tag{12}$$

We need the following lemma.

Lemma 4.1 *The operator $B^\sigma_\epsilon = A + G^\sigma_\epsilon + R^\sigma_\epsilon$ from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$ is pseudo-monotone. Moreover, B^σ_ϵ is coercive, in the following sense:*

$$\frac{\langle B^\sigma_\epsilon v, v\rangle}{\|v\|_{1,p(x)}} \to +\infty \quad \textit{if} \quad \|v\|_{1,p(x)} \to +\infty.$$

Proof of Lemma 4.1 Using the generalized Hölder's inequality and the growth condition (2) we can show that A is bounded, and by (11) and (12), B^σ_ϵ is bounded in $W_0^{1,p(x)}(\Omega)$. The coercivity follows from (4) and the fact that $g^\sigma_\epsilon(x,s,\xi)s \ge 0$ and $-\dfrac{1}{\epsilon^2}\displaystyle\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}u\,dx \ge 0$. It remains to show that B^σ_ϵ is pseudo-monotone.

Let $(u_k)_{k>0}$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{in} \;\; W_0^{1,p(x)}(\Omega),\\ B^\sigma_\epsilon u_k \rightharpoonup \chi \text{ in } \;\; W^{-1,p'(x)}(\Omega),\\ \limsup\limits_{k\to\infty}\langle B^\sigma_\epsilon u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases} \tag{13}$$

We will prove that $\chi = B^\sigma_\epsilon u$ and $\langle B^\sigma_\epsilon u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k \to +\infty$.

Firstly, since $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, then

$$u_k \to u \text{ in } L^{p(x)}(\Omega) \;\text{ for a subsequence denoted again }\; (u_k)_{k>0}. \tag{14}$$

As $(u_k)_{k>0}$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$, then by (2), $(a(x,u_k,\nabla u_k))_{k>0}$ is bounded in $(L^{p'(x)}(\Omega))^N$. Therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega))^N$ such that

$$a(x,u_k,\nabla u_k) \rightharpoonup \varphi \;\text{ in }\; (L^{p'(x)}(\Omega))^N \;\text{ as }\; k\to\infty. \tag{15}$$

Similarly, it is easy to see that $(g^\sigma_\epsilon(x,u_k,\nabla u_k))_{k>0}$ is bounded in $L^{p'(x)}(\Omega)$ with respect to k, then there exists a function $\psi^\sigma_\epsilon \in L^{p'(x)}(\Omega)$ such that

$$g^\sigma_\epsilon(x,u_k,\nabla u_k) \rightharpoonup \psi^\sigma_\epsilon \;\text{ in }\; L^{p'(x)}(\Omega) \;\text{ as }\; k\to\infty \tag{16}$$

and as $(-\dfrac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k)|^{p(x)-1})_{k>0}$ is bounded in $L^{p'(x)}(\Omega)$, then

$$-\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} \to -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} \text{ in } L^{p'(x)}(\Omega) \text{ as } \; k\to\infty. \tag{17}$$

It is clear that, for all $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}\langle \chi, v\rangle = \lim_{k\to\infty}\langle B_\epsilon^\sigma u_k, v\rangle &= \lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla v dx + \lim_{k\to\infty}\int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k)v dx\\ &\quad + \lim_{k\to\infty} -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} v\, dx\\ &= \int_\Omega \varphi\nabla v\, dx + \int_\Omega \psi_\epsilon^\sigma v\, dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} v\, dx.\end{aligned} \tag{18}$$

On one hand, by (14) we have

$$\int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k)u_k\, dx \to \int_\Omega \psi_\epsilon^\sigma u\, dx \quad \text{as} \quad k\to\infty, \tag{19}$$

$$-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} u_k\, dx \to -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1} u\, dx \text{ as } \ k\to\infty. \tag{20}$$

Consequently, by the hypotheses, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle B_\epsilon^\sigma(u_k), u_k\rangle &= \limsup_{k\to\infty}\Big\{\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx + \int_\Omega g_\epsilon^\sigma(x,u_k,\nabla u_k)u_k\, dx\\ &\qquad -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1}u_k\, dx\Big\}\\ &\le \int_\Omega \varphi\nabla u\, dx + \int_\Omega \psi_\epsilon^\sigma u\, dx - \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}u\, dx.\end{aligned} \tag{21}$$

Therefore,

$$\limsup_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx \le \int_\Omega \varphi\nabla u\, dx. \tag{22}$$

Thanks to (3), we have

$$\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\, dx \ge 0. \tag{23}$$

Then

$$\begin{aligned}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx &\ge -\int_\Omega a(x,u_k,\nabla u)\nabla u\, dx\\ &\quad + \int_\Omega a(x,u_k,\nabla u_k)\nabla u\, dx + \int_\Omega a(x,u_k,\nabla u)\nabla u_k\, dx.\end{aligned}$$

By (15), we get

$$\liminf_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx \ge \int_\Omega \varphi\nabla u\, dx$$

which implies by using (22)

$$\lim_{k\to\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\, dx = \int_\Omega \varphi\nabla u\, dx. \tag{24}$$

By means of (18), (19), (20) and (24), we obtain $\langle B_\epsilon^\sigma u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k\to+\infty$. On the other hand, by (24) and the fact that $a(x,u_k,\nabla u)\to a(x,u,\nabla u)$ in $(L^{p'(x)}(\Omega))^N$, we can deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k) - a(x,u_k,\nabla u))(\nabla u_k - \nabla u)\, dx = 0$$

and so, by virtue of Lemma 3.3 we find $\nabla u_n \to \nabla u$ a.e. in Ω, which concludes

$$a(x,u_k,\nabla u_k) \rightharpoonup a(x,u,\nabla u) \text{ in } (L^{p'(x)}(\Omega))^N,$$

$$g_\epsilon^\sigma(x,u_k,\nabla u_k) \rightharpoonup g_\epsilon^\sigma(x,u,\nabla u) \text{ in } L^{p'(x)}(\Omega)$$

and

$$-\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_k^-)|^{p(x)-1} \rightharpoonup -\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u^-)|^{p(x)-1}.$$

Thus, $\chi = B_\epsilon^\sigma u$.

In view of Lemma 4.1, there exists at least one solution $u_\epsilon^\sigma \in W_0^{1,p(x)}(\Omega)$ to the problem (10), by using the classical theorem in [12]. The continuation of the proof of Theorem 4.1 is divided into several steps.

4.1 Study of the approximate problem with respect to ϵ

4.1.1 A priori estimates

If we take $v = u_\epsilon^\sigma - T_k(u_\epsilon^\sigma)$ as a test function in (10), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_k(u_\epsilon^\sigma)\,dx + \int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_k(u_\epsilon^\sigma)\,dx\\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}T_k(u_\epsilon^\sigma)\,dx = \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma)\,dx.\end{aligned}$$

So, as $u_\epsilon^\sigma = u_\epsilon^{\sigma+} - u_\epsilon^{\sigma-}$, then

$$\begin{aligned}-\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}T_k(u_\epsilon^\sigma) &= -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}T_k(u_\epsilon^\sigma)\chi_{\{u_\epsilon^\sigma\le 0\}}\\ &= \frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}T_k(u_\epsilon^{\sigma-}) \ge 0.\end{aligned} \tag{25}$$

Using the fact that $g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_k(u_\epsilon^\sigma) \ge 0$ and by (25) we get

$$\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_k(u_\epsilon^\sigma)\,dx \le k\|f\|_{L^1(\Omega)}. \tag{26}$$

So, by (4) we get

$$\alpha\|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)}^\gamma \le \alpha\int_\Omega |\nabla T_k(u_\epsilon^\sigma)|^{p(x)}\,dx \le k\|f\|_{L^1(\Omega)} \tag{27}$$

with

$$\gamma = \begin{cases} p_+ & \text{if} \quad \|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)} \le 1,\\ p_- & \text{if} \quad \|\nabla T_k(u_\epsilon^\sigma)\|_{p(x)} > 1.\end{cases}$$

Thanks to Poincaré inequality, we obtain

$$\|T_k(u_\epsilon^\sigma)\|_{1,p(x)} \le Ck^{\frac{1}{\gamma}}, \tag{28}$$

where C does not depend on ϵ. Consequently $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is bounded in $W_0^{1,p(x)}(\Omega)$ uniformly on ϵ and σ.

4.1.2 Convergence in measure of u_ϵ^σ

We prove that u_ϵ^σ converges to some function u^σ in measure. To prove this, we show that u_ϵ^σ is a Cauchy sequence in measure. Let k be large enough. Combining the generalized Hölder's inequality, Poincaré's inequality and (28), one has

$$\begin{aligned} k \text{ meas}(\{|u_\epsilon^\sigma| > k\}) &= \int_{\{|u_\epsilon^\sigma|>k\}} |T_k(u_\epsilon^\sigma)|\,dx \le \int_\Omega |T_k(u_\epsilon^\sigma)|\,dx \\ &\le \left(\tfrac{1}{p_-} + \tfrac{1}{p'_-}\right)(\text{meas}(\Omega)+1)^{\frac{1}{p'_-}} \|T_k(u_\epsilon^\sigma)\|_{p(x)} \\ &\le C_1 \|T_k(u_\epsilon^\sigma)\|_{1,p(x)} \le C_2 k^{\frac{1}{\gamma}}; \end{aligned} \tag{29}$$

which yields

$$\text{meas}(\{|u_\epsilon^\sigma| > k\}) \le \frac{C_2}{k^{1-\frac{1}{\gamma}}} \quad \forall \epsilon > 0,\ \forall k > 0. \tag{30}$$

Hence

$$\text{meas}(\{|u_\epsilon^\sigma| > k\}) \to 0 \text{ as } k \to \infty \text{ (since } 1 - \frac{1}{\gamma} > 0), \tag{31}$$

uniformly in ϵ and σ. Moreover, we have, for every $\delta > 0$,

$$\begin{aligned} \text{meas } (\{|u_n^\sigma - u_m^\sigma| > \delta\}) \le \text{meas } (\{|u_n^\sigma| > k\}) + \text{meas } (\{|u_m^\sigma| > k\}) \\ + \text{meas } (\{|T_k(u_n^\sigma) - T_k(u_m^\sigma)| > \delta\}). \end{aligned} \tag{32}$$

Since $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is bounded in $W_0^{1,p(x)}(\Omega)$, then there exists for $\sigma > 0$ fixed, $v_k^\sigma \in W_0^{1,p(x)}(\Omega)$ such that

$$T_k(u_\epsilon^\sigma) \rightharpoonup v_k^\sigma \qquad \text{in} \quad W_0^{1,p(x)}(\Omega)$$

and by the compact embedding, we have

$$T_k(u_\epsilon^\sigma) \to v_k^\sigma \qquad \text{in} \quad L^{p(x)}(\Omega) \qquad \text{and a.e. in } \Omega. \tag{33}$$

Consequently, we can assume that $(T_k(u_\epsilon^\sigma))_{\epsilon>0}$ is a Cauchy sequence in measure in Ω. Let $\eta > 0$. Then by (30) and (32), there exists some $k(\eta) > 0$ such that $\text{meas}(\{|u_n^\sigma - u_m^\sigma| > \delta\}) < \eta$ for all $n, m \ge n_0(k(\eta), \delta)$. This proves that $(u_\epsilon^\sigma)_{\epsilon>0}$ is a Cauchy sequence in measure and thus, converges almost everywhere to some measurable function u^σ. Therefore, $u_\epsilon^\sigma \to u^\sigma$ a.e. in Ω.

Furthermore,

$$\begin{aligned} &T_k(u_\epsilon^\sigma) \rightharpoonup T_k(u^\sigma) \quad \text{in } W_0^{1,p(x)}(\Omega) \\ \text{and} \\ &T_k(u_\epsilon^\sigma) \to T_k(u^\sigma) \quad \text{in} \quad L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{34}$$

4.1.3 Positivity of u^σ

Taking $v = u_\epsilon^\sigma - T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)$ as a test function in (10), we obtain

$$\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \nabla T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx - \\ \frac{1}{\epsilon^2} \int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1} T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx = \int_\Omega f_\epsilon T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx.$$

Since $\displaystyle\int_\Omega a(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)\nabla T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\,dx \geq 0$ and $g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)T_{\frac{1}{\epsilon}}(u_\epsilon^\sigma)\geq 0$, we get

$$-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\left(-T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^-})\right)\,dx \leq \frac{1}{\epsilon}\|f\|_{L^1(\Omega)}.$$

Thus,

$$\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)}\,dx \leq \epsilon\|f\|_{L^1(\Omega)}.$$

Now, denote by $A=\left\{x\in\Omega \text{ such that } |T_{\frac{1}{\epsilon}}\left(u_\epsilon^{\sigma^-}\right)|=\frac{1}{\epsilon}\right\}$. As ϵ is used to tend to 0, we can take it in $(0,1)$ to get

$$\mathrm{meas}(A)\left(\frac{1}{\epsilon}\right)^{p_-}\leq \int_A |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)}\leq \epsilon\|f\|_{L^1(\Omega)};$$

which implies that (by letting ϵ go to 0)

$$\mathrm{meas}(A)=0.$$

Hence, since $u_\epsilon^\sigma\to u^\sigma$ a.e. in Ω and the fact that $\mathrm{meas}(A)=0$, we conclude that

$$|T_{\frac{1}{\epsilon}}\left(u_\epsilon^{\sigma^-}\right)|^{p(x)}\to |u^{\sigma^-}|^{p(x)} \text{ a.e. in } \Omega.$$

We use again the Fatou's Lemma to obtain

$$\int_\Omega |u^{\sigma^-}|\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)}\,dx \leq \liminf_{\epsilon\to 0}\epsilon\|f\|_{L^1(\Omega)}=0;$$

which yields

$$u^\sigma \geq 0.$$

4.1.4 Almost everywhere convergence of the gradient

For the sake of simplicity we will write $\eta(\epsilon,h)$ for any quantity such that

$$\lim_{h\to+\infty}\lim_{\epsilon\to 0}\eta(\epsilon,h)=0.$$

Finally, by $\eta_h(\epsilon)$ we will denote a quantity that depends on ϵ and h and is such that

$$\lim_{\epsilon\to 0}\eta_h(\epsilon)=0,$$

for any fixed value of h.

Let $h>2k>0$, we shall use in (10) the test function

$$\left\{\begin{array}{lcl} v_\epsilon^{h,\sigma} & = & u_\epsilon^\sigma-\eta\varphi_k(\omega_\epsilon^{h,\sigma})\\ \omega_\epsilon^{h,\sigma} & = & T_{2k}\big(u_\epsilon^\sigma-T_h(u_\epsilon^\sigma)+T_k(u_\epsilon^\sigma)-T_k(u^\sigma)\big)\\ \omega^{h,\sigma} & = & T_{2k}(u^\sigma-T_h(u^\sigma)).\end{array}\right. \tag{35}$$

Let $\varphi_k(t)=te^{\lambda t^2}$, $\lambda=(\frac{b(k)}{2\alpha})^2$, it's obvious to check that (see [6], Lemma 1)

$$\varphi_k'(t)-\frac{b(k)}{\alpha}|\varphi_k(t)|\geq\frac{1}{2},\quad \forall t\in I\!R. \tag{36}$$

It follows that

$$\langle A(u_\epsilon^\sigma), \varphi_k(\omega_\epsilon^{h,\sigma})\rangle + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx -$$
$$\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\varphi_k(\omega_\epsilon^{h,\sigma})\,dx = \int_\Omega f_\epsilon \varphi_k(\omega_\epsilon^{h,\sigma})\,dx,$$

which is equivalent to saying that

$$\begin{aligned}&\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ \varphi_k(\omega_\epsilon^{h,\sigma})\,dx\\ &= \int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx.\end{aligned} \tag{37}$$

Note that, $\nabla\omega_\epsilon^{h,\sigma} = 0$ on the set $\{|u_\epsilon^\sigma| > s = 4k + h\}$, therefore, we get by (37)

$$\begin{aligned}&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ \varphi_k(\omega_\epsilon^{h,\sigma})\,dx\\ &= \int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx.\end{aligned}$$

According to (34), we have $\varphi_k(\omega_\epsilon^{h,\sigma}) \rightharpoonup \varphi_k(\omega^{h,\sigma})$ weakly-* in $L^\infty(\Omega)$ as $\epsilon \to 0$, and then

$$\int_\Omega f_\epsilon\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \to \int_\Omega f\varphi_k(\omega^{h,\sigma})\,dx.$$

Finally, by using Lebesgue's theorem, we can deduce that

$$\int_\Omega f\varphi_k(\omega^{h,\sigma})\,dx \to 0 \text{ as } h \to +\infty.$$

Therefore,

$$\int_\Omega f\varphi_k(\omega_\epsilon^{h,\sigma})\,dx = \eta(\epsilon, h). \tag{38}$$

Note that $\varphi_k(\omega_\epsilon^{h,\sigma})$ and u_ϵ^σ has the same sign in the set $\{x \in \Omega, |u_\epsilon^\sigma| > k\}$, then we have

$$g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma}) \geq 0 \quad\text{and}\quad -\frac{1}{\epsilon^2}|T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\varphi_k(\omega_\epsilon^{h,\sigma}) \geq 0.$$

From (37), we deduce that

$$\begin{aligned}&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\\ &-\frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma-})|^{p(x)-1}\ (u_\epsilon^\sigma - T_k(u^\sigma))\exp(\lambda(\omega_\epsilon^{h,\sigma}))^2\,dx\\ &\leq \eta(\epsilon, h).\end{aligned} \tag{39}$$

Since $u^\sigma \geq 0$, then the third term on the left-hand side of the above inequality is positive, thus,

$$\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx \leq \eta(\epsilon, h). \tag{40}$$

Splitting the first integral on the left-hand side of (40), where $|u_\epsilon^\sigma| \leq k$ and $|u_\epsilon^\sigma| > k$, we can write

$$\begin{aligned}
&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&= \int_{\{|u_\epsilon^\sigma|\leq k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&+ \int_{\{|u_\epsilon^\sigma|>k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx.
\end{aligned} \tag{41}$$

The first term on the right-hand side of the last inequality can be written as

$$\begin{aligned}
&\int_{\{|u_\epsilon^\sigma|\leq k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&= \int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx.
\end{aligned} \tag{42}$$

For the second term on the right-hand side of (41), we can write according to (4),

$$\begin{aligned}
&\int_{\{|u_\epsilon^\sigma|>k\}} a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&\geq -\varphi'(2k)\int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx.
\end{aligned} \tag{43}$$

Since $|a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma)|$ is bounded in $(L^{p'(x)}(\Omega))^N$, if necessary we have

$$|a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma)| \rightharpoonup l_{M,\sigma} \text{ in } (L^{p'(x)}(\Omega))^N \text{ as } \epsilon \to 0, \quad \text{for a subsequence.}$$

Due to $\nabla T_k(u^\sigma)\chi_{\{|u_\epsilon^\sigma|>k\}} \to \nabla T_k(u^\sigma)\chi_{\{|u^\sigma|>k\}}$ in $L^{p(x)}(\Omega)$ as $\epsilon \to 0$, we obtain

$$-\varphi'(2k)\int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx \to$$

$$-\varphi'(2k)\int_{\{|u^\sigma|>k\}} l_{M,\sigma}|\nabla T_k(u^\sigma)|\,dx = 0 \text{ as } \epsilon \to 0.$$

Therefore,

$$-\varphi'(2k)\int_{\{|u_\epsilon^\sigma|>k\}} |a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))||\nabla T_k(u^\sigma)|\,dx = \eta_h(\epsilon). \tag{44}$$

Combining (41) and (44), we deduce that

$$\begin{aligned}
&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi'(\omega_\epsilon^{h,\sigma})\,dx \\
&\geq \int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \eta_h(\epsilon).
\end{aligned} \tag{45}$$

It follows

$$\begin{aligned}
&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi'(\omega_\epsilon^{h,\sigma})\,dx \\
&\geq \int_\Omega [a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma)) - a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))] \\
&\times[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&+\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \eta_h(\epsilon).
\end{aligned} \tag{46}$$

Concerning the second term of the right-hand side of (46) we can write

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx \\
&= \int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))\nabla T_k(u_\epsilon^\sigma)\varphi_k'(T_k(u_\epsilon^\sigma) - T_k(u^\sigma))\,dx \\
&-\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx.
\end{aligned} \tag{47}$$

By the continuity of Nemytskii's operator (cf. [9], [20]), we have

$$a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))\varphi_k'(T_k(u_\epsilon^\sigma) - T_k(u^\sigma)) \to a(x, T_k(u^\sigma), \nabla T_k(u^\sigma))\varphi_k'(0)$$

and $a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma)) \to a(x, T_k(u^\sigma), \nabla T_k(u^\sigma))$ strongly in $(L^{p'(x)}(\Omega))^N$, while $\nabla T_k(u_\epsilon^\sigma) \rightharpoonup \nabla T_k(u^\sigma)$ weakly in $(L^{p(x)}(\Omega))^N$ and $\nabla T_k(u_\epsilon^\sigma)\varphi_k'(\omega_\epsilon^{h,\sigma}) \to \nabla T_k(u^\sigma)\varphi_k'(0)$ strongly in $(L^{p(x)}(\Omega))^N$.
Then, the first and the second term of the right-hand side on (47) tend respectively to

$$\int_\Omega a(x, T_k(u^\sigma), \nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(0)\,dx \text{ as } \epsilon \to 0$$

and

$$-\int_\Omega a(x, T_k(u^\sigma), \nabla T_k(u^\sigma))\nabla T_k(u^\sigma)\varphi_k'(\omega^{h,\sigma})\,dx \text{ as } \epsilon \to 0;$$

therefore,

$$\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx = \eta_h(\epsilon). \tag{48}$$

Combining (46) and (48) yields

$$\begin{aligned}
&\int_\Omega a(x, T_s(u_\epsilon^\sigma), \nabla T_s(u_\epsilon^\sigma))\nabla\omega_\epsilon^{h,\sigma}\varphi'(\omega_\epsilon^{h,\sigma})\,dx \\
&\geq \int_\Omega [a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma)) - a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u^\sigma))] \\
&\times[\nabla T_k(u_\epsilon^\sigma) - \nabla T_k(u^\sigma)]\varphi_k'(\omega_\epsilon^{h,\sigma})\,dx + \eta(\epsilon, h).
\end{aligned} \tag{49}$$

Going back to the second term of the left hand side of (40), we have

$$\begin{aligned}
&\left|\int_{\{|u_\epsilon^\sigma|<k\}} g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\varphi_k(\omega_\epsilon^{h,\sigma})\,dx\right| \\
&\leq b(k)\int_\Omega c(x)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))\nabla T_k(u_\epsilon^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx \\
&\leq \eta(\epsilon, h) + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\epsilon^\sigma), \nabla T_k(u_\epsilon^\sigma))\nabla T_k(u_\epsilon^\sigma)|\varphi_k(\omega_\epsilon^{h,\sigma})|\,dx.
\end{aligned} \tag{50}$$

The last term of the last side of this inequality reads as

$$\begin{aligned}&\frac{b(k)}{\alpha}\int_\Omega[a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))-a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))]\\&[\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx\\&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))[\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx\\&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))\nabla T_k(u^\sigma)|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx.\end{aligned}\tag{51}$$

Reasoning as above, it is easy to see that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))[\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx=\eta_h(\epsilon)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))\nabla T_k(u^\sigma)|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx=\eta(\epsilon,h).$$

Therefore,

$$\begin{aligned}&\left|\int_{\{|u^\sigma_\epsilon|<k\}}g^\sigma_\epsilon(x,u^\sigma_\epsilon,\nabla u^\sigma_\epsilon)\varphi_k(\omega^{h,\sigma}_\epsilon)\,dx\right|\\&\le\frac{b(k)}{\alpha}\int_\Omega[a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))-a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))]\\&\times[\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]|\varphi_k(\omega^{h,\sigma}_\epsilon)|\,dx+\eta(\epsilon,h).\end{aligned}\tag{52}$$

Combining (40), (51) and (52), we obtain

$$\begin{aligned}&\int_\Omega[a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))-a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))]\\&\times[\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)](\varphi'_k(\omega^{h,\sigma}_\epsilon)-\frac{b(k)}{\alpha}|\varphi_k(\omega^{h,\sigma}_\epsilon)|)\,dx\\&\le\eta(\epsilon,h),\end{aligned}\tag{53}$$

which implies by using (36) that

$$\int_\Omega[a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))-a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))][\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]\,dx\le\eta(\epsilon,h).\tag{54}$$

Letting ϵ tend to 0 and h tend to infinity, we deduce that

$$\int_\Omega[a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma_\epsilon))-a(x,T_k(u^\sigma_\epsilon),\nabla T_k(u^\sigma))][\nabla T_k(u^\sigma_\epsilon)-\nabla T_k(u^\sigma)]\,dx\to 0.$$

By Lemma 3.3, we get from convergence above

$$T_k(u^\sigma_\epsilon)\to T_k(u^\sigma)\text{ in }W_0^{1,p(x)}(\Omega).\tag{55}$$

Thus,

$$\nabla u^\sigma_\epsilon\to\nabla u^\sigma\text{ a.e. in }\Omega.\tag{56}$$

4.1.5 Equi-integrability of the nonlinearity g_ϵ^σ

In order to pass to the limit in the approximated equation, we now show that

$$g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) \to g^\sigma(x, u^\sigma, \nabla u^\sigma) \text{ in } L^1(\Omega). \tag{57}$$

In particular, it is enough to prove the equi-integrability of the sequence $\{|g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)|\}$. To this purpose, we take $u_\epsilon^\sigma - T_1(u_\epsilon^\sigma - T_h(u_\epsilon^\sigma)) \geq 0$ as a test function in (10), to obtain

$$\int_{\{|u_\epsilon^\sigma| \geq h+1\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)| \, dx \leq \int_{\{|u_\epsilon^\sigma| > h\}} |f_n| \, dx.$$

Let $\eta > 0$ be fixed. Then, there exists $h(\eta) \geq 1$ such that

$$\int_{\{|u_\epsilon^\sigma| \geq h(\eta)\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)| \, dx < \frac{\eta}{2}. \tag{58}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned} \int_E |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)| \, dx \leq & \int_E b(l(\varepsilon))\big(c(x) + |\nabla T_{h(\eta)}(u_\epsilon^\sigma)|^{p(x)}\big) \, dx \\ & + \int_{\{|u_\epsilon^\sigma| \geq h(\eta)\}} |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)| \, dx. \end{aligned} \tag{59}$$

In view of (55), there exists $\beta(\eta) > 0$ such that

$$\int_E b(h(\eta))\big(c(x) + |\nabla T_{h(\eta)}(u_\epsilon^\sigma)|^{p(x)}\big) \, dx \leq \frac{\eta}{2} \quad \text{for all } E \text{ such that } \operatorname{meas}(E) < \beta(\eta). \tag{60}$$

Finally, by combining (58) and (60), one easily has

$$\int_E |g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)| \, dx \leq \eta \quad \text{for all } E \text{ such that } \operatorname{meas}(E) < \beta(\eta).$$

Then, we deduce that $g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)$ is uniformly equi-integrable in Ω.

4.1.6 Passing to the limit with respect to ϵ

Let $v \in K_0 \cap L^\infty(\Omega)$, we take $u_\epsilon^\sigma - T_k(u_\epsilon^\sigma - v)$ as a test function in (10) to obtain

$$\int_\Omega a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla T_k(u_\epsilon^\sigma - v) \, dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v) \, dx \leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v) \, dx. \tag{61}$$

We deduce that

$$\int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla(u_\epsilon^\sigma - v) dx + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v) dx \leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v) dx, \tag{62}$$

which is equivalent to saying that

$$\begin{aligned} \int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla u_\epsilon^\sigma \, dx - & \int_{\{|u_\epsilon^\sigma - v| \leq k\}} a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)\nabla v \, dx \\ & + \int_\Omega g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) T_k(u_\epsilon^\sigma - v) \, dx \\ & \leq \int_\Omega f_\epsilon T_k(u_\epsilon^\sigma - v) \, dx. \end{aligned} \tag{63}$$

By Fatou's lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u^\sigma_\epsilon), \nabla T_{k+\|v\|_\infty}(u^\sigma_\epsilon)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma)) \text{ in } (L^{p'(x)}(\Omega))^N,$$

we get

$$\begin{aligned} \int_{\{|u^\sigma - v| \le k\}} a(x, u^\sigma, \nabla u^\sigma) \nabla u^\sigma \, dx - \int_{\{|u^\sigma - v| \le k\}} & a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma)) \nabla v \, dx \\ & + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \\ & \le \int_\Omega f T_k(u^\sigma - v) \, dx. \end{aligned} \tag{64}$$

Consequently,

$$\begin{aligned} & \int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_k(u^\sigma - v) \, dx + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \\ & \le \int_\Omega f T_k(u^\sigma - v) \, dx, \qquad \forall v \in K_0 \cap L^\infty(\Omega) \text{ and } \forall k > 0. \end{aligned} \tag{65}$$

4.2 Study of the problem with respect to σ

4.2.1 Estimates with respect to σ

We are going to give some estimates on the sequence $(u^\sigma)_{\sigma > 0}$ identical to (27). For that, we take $v = T_s(u^\sigma - T_k(u^\sigma))$ in (65) and we let $s \to \infty$; then, by the same argument as in section 4.1 we can prove that

$$\alpha \|\nabla T_k(u^\sigma)\|^\gamma_{p(x)} \le \alpha \int_\Omega |\nabla T_k(u^\sigma)|^{p(x)} \, dx \le k \|f\|_{L^1(\Omega)} \quad \text{for all} \quad k > 1. \tag{66}$$

Thus, as in section 4.1.2, there exists u such that $T_k(u) \in W_0^{1,p(x)}(\Omega)$ and

$$\begin{cases} T_k(u^\sigma) \rightharpoonup T_k(u) & \text{in } W_0^{1,p(x)}(\Omega), \\ T_k(u^\sigma) \to T_k(u) & \text{in } \ L^{p(x)}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{67}$$

So, $u^\sigma \ge 0$ a.e. in Ω and we have also $u \ge 0$ a.e. in Ω.

4.2.2 Strong convergence of truncation with respect to σ

Here, in (65) we shall use the test function

$$\begin{cases} v &= T_s(u^\sigma - \varphi_k(\omega^{h,\sigma})), \\ \omega^{h,\sigma} &= T_{2k}\big(u^\sigma - T_h(u^\sigma) + T_k(u^\sigma) - T_k(u)\big), \\ \omega^h &= T_{2k}(u - T_h(u)), \end{cases} \tag{68}$$

where $h > 2k > 0$. It follows that for all $l > 0$,

$$\begin{aligned} & \int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx \\ & + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx \\ & \le \int_\Omega f T_l(u^\sigma - T_s(u^\sigma - \varphi_k(\omega^{h,\sigma}))) \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\{|u^\sigma-\varphi_k(\omega^{h,\sigma})|\le s\}} a(x,u^\sigma,\nabla u^\sigma)\nabla T_l(\varphi_k(\omega^{h,\sigma}))\,dx\\&+\int_\Omega g^\sigma(x,u^\sigma,\nabla u^\sigma)T_l(u^\sigma-T_s(u^\sigma-\varphi_k(\omega^{h,\sigma})))\,dx\\&\le\int_\Omega fT_l(u^\sigma-T_s(u^\sigma-\varphi_k(\omega^{h,\sigma})))\,dx.\end{aligned}$$

Letting $s\to\infty$ and choosing l large enough ($l\ge|\varphi_k(2k)|$), we deduce that

$$\int_\Omega a(x,u^\sigma,\nabla u^\sigma)\nabla\varphi_k(\omega^{h,\sigma})dx+\int_\Omega g^\sigma(x,u^\sigma,\nabla u^\sigma)\varphi_k(\omega^{h,\sigma})dx\le\int_\Omega f\varphi_k(\omega^{h,\sigma})dx. \tag{69}$$

Then, by using the same techniques as in section 4.1.4 we can deduce that

$$T_k(u^\sigma)\to T_k(u) \text{ in } W_0^{1,p(x)}(\Omega) \ \text{ and } \ \nabla u^\sigma\to\nabla u \text{ a.e. in } \Omega. \tag{70}$$

4.2.3 Equi-integrability of the nonlinearity g with respect to σ

Moreover, since g is a Carathéodory function, it is easy to see that

$$g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u) \ \text{ a.e. in } \ \Omega \ \text{ as } \ \sigma\to 0.$$

Then, by assumption (6) (note that this hypothesis is only used here), it is clear that $g^\sigma(x,u^\sigma,\nabla u^\sigma)=\delta_\sigma g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)$ a.e. in $\{x\in\Omega, u(x)\ge 0\}$.

Similarly, we claim that $g^\sigma(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)$ in $L^1(\Omega)$.
Indeed, taking $u^\sigma-T_1(u^\sigma-T_l(u^\sigma))\ge 0$ as test function in (65), we obtain

$$\int_{\{|u^\sigma|\ge l+1\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le\int_{\{|u^\sigma|>l\}}|f|\,dx.$$

Let $\beta>0$ be fixed. Then, there exists $l(\beta)\ge 1$ such that

$$\int_{\{|u^\sigma|\ge l(\beta)\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx<\frac{\beta}{2}. \tag{71}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le&\int_E b(l(\beta))\big(c(x)+|\nabla T_{l(\beta)}(u^\sigma)|^{p(x)}\big)\,dx\\&+\int_{\{|u^\sigma|\ge l(\beta)\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx.\end{aligned} \tag{72}$$

In view of (70), there exists $\alpha(\beta)>0$ such that

$$\int_E b(l(\beta))\big(c(x)+|\nabla T_{l(\beta)}(u^\sigma)|^{p(x)}\big)\,dx\le\frac{\beta}{2} \quad \text{for all } E \text{ such that } \operatorname{meas}(E)<\alpha(\beta). \tag{73}$$

Finally, by combining (71) and (73), one easily has

$$\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\le\beta \quad \text{ for all } E \text{ such that } \operatorname{meas}(E)\le\alpha(\beta).$$

Therefore, we deduce that $g^\sigma(x,u^\sigma,\nabla u^\sigma)$ is uniformly equi-integrable in Ω. So, as in section 4.1.6, we can pass to the limit in σ and conclude. This achieves the proof of Theorem 4.1.

5 Case when the Nonlinearity g is Negative

We consider the convex set $\overline{K}_0 = \{u \in W_0^{1,p(x)}(\Omega); u \leq 0 \text{ a.e. in } \Omega\}$.

Theorem 5.1 *Assume that* $(2) - (6)$ *hold true and that* $f \in L^1(\Omega)$*. Then, there exists at least one solution (entropy solution) to the following unilateral problem,*

$$(\mathcal{P}) \begin{cases} u \in \mathcal{T}_0^{1,p(x)}(\Omega), \;\; u \leq 0 \;\; a.e. \;\; in \;\Omega, \;\; g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \leq \int_\Omega fT_k(u-v)\,dx, \\ \forall v \in \overline{K}_0 \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases}$$

Proof. The same proof as for Theorem 4.1 can be applied with the following changes:
i) We approach the sign function by an increasing Lipschitz function.
ii) The Lipschitz function $\delta_\sigma(s)$ is replaced by:

$$\overline{\delta}_\sigma(s) = \begin{cases} \frac{-s+\sigma}{s}, & \text{if } s \geq \sigma > 0, \\ 0, & \text{if } |s| \leq \sigma, \\ \frac{s+\sigma}{s}, & \text{if } s < -\sigma < 0. \end{cases}$$

iii) The approximated problem becomes:

$$(\overline{\mathcal{P}}_\epsilon^\sigma) \begin{cases} u_\epsilon^\sigma \in W_0^{1,p(x)}(\Omega) \\ \langle Au_\epsilon^\sigma, u_\epsilon^\sigma - v\rangle + \displaystyle\int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)(u_\epsilon^\sigma - v)\,dx + \frac{1}{\epsilon^2}\int_\Omega |T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma^+})|^{p(x)-1}(u_\epsilon^\sigma - v)\,dx \\ \qquad = \displaystyle\int_\Omega f_\epsilon(u_\epsilon^\sigma - v)\,dx, \quad \forall v \in W_0^{1,p(x)}(\Omega). \end{cases} \tag{74}$$

iv) The set K_0 is replaced by $\overline{K}_0 = \{u \in W_0^{1,p(x)}(\Omega); u \leq 0 \text{ a.e. in } \Omega\}$.

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