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NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

AIMS AND SCOPE

Nonlinear Dynamics and Systems Theory is a multidisciplinary journal. It publishes papers focusing on proofs of important theorems as well as papers presenting new ideas and new theory, conjectures, numerical algorithms and physical experiments in areas related to nonlinear dynamics and systems theory. Papers that deal with theoretical aspects of nonlinear dynamics and/or systems theory should contain significant mathematical results with an indication of their possible applications. Papers that emphasize applications should contain new mathematical models of real world phenomena and/or description of engineering problems. They should include rigorous analysis of data used and results obtained. Papers that integrate and interrelate ideas and methods of nonlinear dynamics and systems theory will be particularly welcomed. This journal and the individual contributions published therein are protected under the copyright by International InforMath Publishing Group.

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ABSTRACTING AND INDEXING SERVICES

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Special Issue on the 5th International Conference on Optimization and Control with Applications (OCA5)

The 5th International Conference on Optimization and Control with Applications (OCA5) was held in China University of Petroleum, Beijing, China on December 4-8, 2012. As a continuation of the OCA series, OCA5 provided an international forum for scientists, engineers, researchers, and practitioners to exchange ideas and approaches, to present research findings and state-of-the-art solutions, to share experiences on potentials and limits, and to open new avenues of research and developments, on all issues and topics related to theory and applications of optimization and control. More than 200 representatives from over 20 countries and regions, such as Mainland China, Hong Kong, Taiwan, the United States, Canada, Australia, Russia, and Japan, presented their recent works on theory and applications of optimization and control. Conference participants were invited to submit their revised and expanded papers to be considered for possible publication in a Special Issue for Nonlinear Dynamics and Systems Theory (ND&ST). Seven research papers, having been peer reviewed, are accepted for publication in this special issue.

Firstly, in recognition of Professor Taro Yoshizawa's significant contributions to nonlinear dynamics and systems theory, we include a Personage in Science to introduce his biographical sketch and scientific activities. After that, the first paper, entitled "Coordinating Supply Chains with a Credit Mechanism", solves the supply chain coordination problem with trade credit in two situations: symmetric and asymmetric information. A coordination mechanism through credit contract is proposed to achieve a win-win outcome. In the second paper, entitled "Reduced Order Function Projective Combination Synchronization of Three Josephson Junctions Using Backstepping Technique", a projective combination synchronization scheme of chaotic Josephson junction systems is proposed and studied using the backstepping technique. The third paper, entitled "Stability of Stochastic Interval System with Distributed Delays", is concerned with the stability problem of a stochastic interval system with distributed delays. The existence and uniqueness of solutions are shown, and sufficient criteria for exponential stability are derived. These results are extendable to the stochastic interval systems with multiple time delays. The fourth paper, entitled "Existence of the Solution for Discontinuous Fuzzy Integro-differential Equations and Strong Fuzzy Henstock Integrals", studies the Cauchy problem of discontinuous fuzzy integro-differential equations with strong fuzzy Henstock integral. Some important convergence results and theorems on existence of solutions for strong fuzzy Henstock integral equations are reported. In the fifth paper, entitled "Indirect Adaptive Fuzzy Control of Multivariable Nonlinear Systems Class with Unknown Parameters", a fuzzy adaptive control technique for nonlinear systems is proposed and an adaptive fuzzy control law is designed to ensure the tracking errors and boundedness of the fuzzy logic system parameters be convergent. The obtained results are hence applied to permanent magnet synchronous motors. The sixth paper, entitled

"Designing a Compensator Based on Extended Kalman Filter for Elimination of Noise and Delay Effect in Tracking Loop", investigates a new control scheme to compensate destructive effect of delay and noise in tracking loop. Finally, the seven paper, entitled "The Structure of The Solution of Delay Differential Equations With One Unstable Positive Equilibrium", establishes sufficient conditions to ensure that the solutions will converge to the trivial equilibrium and the positive equilibrium respectively. Other solutions can be divided into three classes according to their eventual tendency. It is noted that the second, fifth, and sixth papers are taken from the Journal's archive as these papers have strong connections with the theme of our conference.

We would like to express our warmest thanks to authors who submitted their papers to be considered for publication in this Special Issue. We highly appreciate the contributions from the reviewers for their careful and critical evaluation of the manuscripts. It is our pleasure to thank Professor A.A. Martynyuk, Editor-in-Chief of ND &ST, for his support and encouragement during the process of editing this Special Issue.

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PERSONAGE IN SCIENCE

Professor Taro Yoshizawa

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Taking into account the great significance of Yoshizawa's work for modern development of nonlinear dynamics and systems theory, the Editorial Board of the journal is publishing a sketch of his life, a brief survey of main directions of his scientific activity and a list of his work published in 1950–1997.

1 Biographical Sketch

Professor Taro Yoshizawa was born in Osaka on August 18, 1919. He received the degree of Bachelor of Science in Mathematics at Kyoto Imperial University where he graduated in December 1941. In August, 1949 he became assistant professor in mathematics at Kyoto University. In August, 1958 he was conferred a Doctor's degree in Mathematics by Kyoto University. He was assigned full professor in mathematics at Nihon University in April, 1959. He joined Tohoku University in August, 1965 and served there as professor of mathematics until his retirement in March, 1983. After that he was with Okayama University of Science as professor until March, 1996.

The main sphere of his scientific interests in mathematics was the stability of differential equations. Starting from September, 1959 he made a two year visit to the Research Institute for Advanced Studies established by Solomon Lefschetz in Baltimore, Md., USA. Since that time until as late as a few months before his death he played central roles in many international conferences and continued to be one of the world leaders in the stability theory of differential equations.

In April, 1993, in recognition of his merits he was decorated with the Third Order of the Rising Sun.

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2   Main Directions of Scientific Activity

Professor Yoshizawa was conducting his scientific research in the period of tremendous upgrowth of the theory of differential and functional differential equations due to its wide application in science and technology. Coming after the brilliant work of Liapunov (1892), the papers by K.P. Persidsky (1946), N.N. Krassovsky (1959), V.I. Zubov (1957), H.A. Antosiewicz (1958), L. Cesari (1963), S. Lefschetz (1963) and others exerted a strong influence on the formation of his scientific interests and determined the directions of his research. We now outline briefly the main avenues of his investigations.

2.1   General questions of the theory of differential equations

This direction was developed in papers [1, 2, 6, 7, 20]. Main results obtained before 1967 were summarized in monograph [33] (in Japanese). The series of works on limiting equations (see [35, 38, 39, 40] presented profound results in the analysis of nonautonomous systems in terms of limiting equations. Also, in these papers conditions of uniform asymptotic stability were established, eventual properties of solutions were studied, converse theorems on bounded properties of solutions of nonautonomous systems were proved as well as theorems on their stability and attraction. Some of these results were included into the generalizing monograph by Kato, A.A. Martynyuk, and A.A. Shestakov, Stability of Motion of Nonautonomous Systems (Method of Limiting Equations), Amsterdam: Gordon and Breach Publishers, 1996.

2.2   Liapunov stability and boundedness of solutions

Papers [9, 11–19, 28] deal with the investigations in this direction. Yoshisawa's monograph "Stability Theory by Liapinov's Second Method" (see [29] in the list of publications below) which followed the famous monographs by V.I. Zubov and N.N. Krassovsky proved to be the most often cited one in the English language literature on the important developments of the theory of stability of motion. Starting with the definitions of stability and boundedness of solutions for nonlinear systems, Yoshizawa set out basic theorems on stability and boundedness of solutions in terms of existence of Liapunov functions with appropriate properties. Completeness of these investigations is supported by the proofs of the converse theorems, i.e. the results showing existence of Liapunov functions with certain properties for certain types of stability of the zero solution.

2.3   Perturbed systems

Main results obtained by Yoshizawa in this direction were published in [24, 25] and then developed in Chapter 6 of monograph [29]. Namely, basing on some results by Gorshin (1936) and Malkin (1944), he proved a series of theorems on stability under persistent perturbations and studied behavior of solutions of perturbed systems both for systems of ordinary differential equations and for equations with small parameter. He also analyzed asymptotic properties of solutions near integral manifolds having developed thereby the results of Hale (1963) and Bogolyubov and Mitropolsky (1963).

2.4   Existence theorems for periodic solutions and almost periodic solutions

This direction is covered by papers [8, 25, 31, 32, 37]. Based on fixed point theorems, Yoshizawa established existence conditions for periodic solutions, which were the gen-

eralizations of some results of Massera (1950), and determined existence conditions for bounded solutions. These results were obtained by means of two Liapunov functions satisfying special conditions. In order to establish existence conditions for almost periodic solutions he applied radially unbounded Liapunov functions to the theory of asymptotic stability. Monograph [37] summarizes Yoshizawa's results obtained up to 1975.

2.5 Theory functional differential equations

Yoshizawa's papers [25, 27, 41-43, 44, 45] refer to this direction of investigations. To prove theorems on existence of solutions of functional differential equations he employed the method of Liapunov-Krassovsky functionals. General results on stability were also proved based on functionals of special type. As an extension of the results of LaSalle (1960), Yoshizawa studied asymptotic behavior of solutions of autonomous systems. He also established boundedness conditions for solutions, including equations with persistent perturbations. For periodic and almost periodic systems of equations the conditions of solution existence, stability and boundedness were obtained. For a survey and developments of some Yoshizawa's ideas and approaches see the monograph by A.Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Orlando: Academic Press, Inc., 1985.

It should be noted that numerous reports made by Prof. Yoshizawa at international conferences and symposia always drew the audience and generated a keen interest.

Taro Yoshizawa was named professor emeritus from Tohoku University. He was one of the former editors of the Tohoku Mathematical Journal. He served as member of the board of directors of the Mathematical Society of Japan as well as other important committees.

Professor Taro Yoshizawa passed away in Kyoto on October 7, 1996.

3 The Life of a Teacher

Professor Yoshizawa was a leading mathematician, speaker, and writer in the broad area of stability theory of differential equations. But he will be remembered most of all for his teaching and mentoring former students and junior colleagues. While we have been unable to find a complete list of his doctoral students and the year in which they received their degrees we would mention the following former students and the year in which their first paper on differential equations was reviewed in the Mathematical Reviews:

1. Tetsuo Furumochi (1971)
2. Yoshihiro Hamaya (1989)
3. Yoshiyuki Hino (1970)
4. Takashi Kaminogo (1975)
5. Junji Kato (1962)
6. Satoru Murakami (1982)
7. Toshiki Naito (1970)
8. Fumio Nakajima (1971).

We have mentioned the major monographs by Prof. Yoshizawa including [29] on Liapunov theory and [37] on periodic and almost periodic solutions of differential equations. His teaching in these areas is strongly reflected in the subsequent work of all of his students. In fact, a logical continuation of his stability theory in [29] is seen in the major monograph by three of his students:

Y. Hino, S. Murakami, and T. Naito, Functional Differential Equations with Infinite Delay, Springer, Tokyo, 1991.

In the same way a logical continuation of his almost periodic theory in [37] is seen in the impressive monograph with two of his students as coauthors:

Y. Hino, T. Naito, Nguyen Van Minh, and Jong Son Shin, Almost Periodic Solutions of Differential Equations in Banach Spaces, Taylor and Francis, London and New York, 2002.

Professor Yoshizawa travelled extensively to meetings all over the world and was almost always accompanied by his wife and some of his former doctoral students. He maintained a life long relationship with his former students and was highly revered by all of them. His life as a teacher is a model which we could all try to emulate.

4   List of Selected Works of T. Yoshizawa

1. On the uniqueness of solutions of a system of ordinary differential equations. *Mem. Coll. Sci. Univ. Kyoto. Ser. A.* **26** (1950), 19–29. (with Kyuzo Hayashi)

2. New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **26** (1951), 225–233. (with Kyuzo Hayashi)

3. Note on the non-increasing solutions of $y'' = f(x, y, y')$. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **27** (1952), 153–162.

4. On the evaluation of the derivatives of solutions of $y'' = f(x, y, y')$. *Mem. Coll. Sic. Univ. Kyoto. Ser. A. Math.* **28** (1953), 27–32.

5. Note on the boundedness of solutions of a system of differential equations. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **28** (1954), 293–298.

6. On the non-linear differential equation. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **28** (1954), 133–141.

7. On the convergence of solutions of the non-linear differential equation. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **28** (1954), 143–151.

8. Note on the existence theorem of a periodic solution of the non-linear differential equation. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **28** (1954), 153–159.

9. On the stability of solutions of a system of differential equations. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **29** (1955), 27–33.

10. Note on the solutions of a system of differential equations. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **29** (1955), 249–273.

11. Note on the boundedness and the ultimate boundedness of solutions of $x' = F(t, x)$. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **29** (1955), 275–291.

12. On the necessary and sufficient condition for the uniform boundedness of solutions of $x' = F(t, x)$. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **30** (1957), 217–226.

13. Appendix to the paper "Note on the boundedness and the ultimate boundedness." Mem. *Coll. Sci. Univ. Kyoto. Ser. A. Math.* **30** (1957), 91–103.

14. Note on the equi-ultimate boundedness of solutions of $x' = F(t, x)$. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **31** (1958), 211–217.

15. On the equiasymptotic stability in the large. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **32** (1959), 171–180.

16. Liapunov's function and boundedness of solutions. *Funkcial. Ekvac.* **2** (1959), 71–103.

17. Stability and boundedness of systems. *Arch. Rational Mech. Anal.* **6** (1960), 409–421.

18. Existence of a bounded solution and existence of a periodic solution of the differential equation of the second order. *Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math.* **33** (1960/1961), 301–308.

19. Liapunov's function and boundedness of solutions. *Bol. Soc. Mat. Mexicana* (2) 5 (1960), 146-151.

20. Asymptotic behavior of solutions of non-autonomous system near sets. *J. Math. Kyoto Univ.* **1** (1961/1962), 303–323.

21. Asymptotic behavior of solutions of a system of differential equations. *Contributions to Differential Equations* **1** (1963), 371–387.

22. Stability of solutions of non-linear differential equations (in Japanese). *Sugaku* **13** (1961/1962), 228–236.

23. Stable sets and periodic solutions in a perturbed system. *Contributions to Differential Equations* **2** (1963), 407–420.

24. Stability of sets and perturbed system. *Funkcial. Ekvac.* **5** (1963), 31–69.

25. Extreme stability and almost periodic solutions of functional differential equations. *Arch. Rational Mech. Anal.* **17** (1964), 148–170.

26. Some notes on stability of sets and perturbed system. *Funkcial. Ekvac.* **6** (1964), 1–11.

27. Asymptotic stability of solutions of an almost periodic system of functional differential equations. *Rend. Circ. Mat. Palermo* (2) **13** (1964), 209–221.

28. Eventual properties and quasi-asymptotic stability of a non-compact set. *Funkcial. Ekvac.* **8** (1966), 79–90.

29. Stability Theory by Liapunov's Second Method. Publications of the Mathematical Society of Japan 9, Math. Soc. of Japan, Tokyo, 1966, viii+223 pp.

30. Linear system and its perturbed system. *Funkcial. Ekvac.* **9** (1966), 9–15. (with Junji Kato)

31. Stability and existence of a periodic solution. *J. Differential Equations* **4** (1968), 121–129.

32. Existence of a globally uniform-asymptotically stable periodic and almost periodic solution. *Tohoku Math. J.* (2) **19** (1967), 423–428.

33. Introduction to Ordinary Differential Equations (in Japanese). Asakura Shoten, Tokyo, 1967.

34. Asymptotically almost periodic solutions of an almost periodic system. *Funkcial. Ekvac.* **12** (1969), 23–40.

35. A relationship between uniformly asymptotic stability and total stability. *Funkcial. Ekvac.* **12** (1969/1970), 233–238. (with Junji Kato)

36. Oscillatory property of solutions of second order differential equations. *Tohoku Math. J.* (2) **22** (1970), 619–634.

37. Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions. Applied Mathematical Sciences 14, Springer-Verlag, New York-Heidelberg, 1975, vii+233 pp.

38. Remarks on global properties in limiting equations. *Funkcial. Ekvac.* **24** (1981), 363–371. (with Junji Kato)

39. Attractivity in nonautonomous systems. *International J. Non-linear Mech.* **20** (1985), 519–528.

40. Total stability property in limiting equations for a functional-differential equation with infinite delay. *Casopis Pest. Mat.* **111** (1986), 62–69, 90. (with Yoshiyuki Hino)

41. Almost periodic solutions in an integrodifferential equation. *Proc. Roy. Soc. Edinburgh. Sect. A.* **114** (1990), 151–159. (with Yoshihiro Hamaya)

42. Relationships between BC-stabilities and p-stabilities and ρ-stabilities in functional-differential equations with infinite delay. *Thoku Math. J.* (2) **44** (1992), 45–57. (with Satoru Murakami)

43. Stability properties and the existence of an almost periodic solution in an integrodifferential equation. Delay and Differential Equations (Ames, I A, 1991), 1–8, World Sci. Publishing, River Edge, NJ, 1992.

44. Asymptotic behavior in a system of Volterra integrodifferential equations with diffusion. *Dynam. Systems Appl.* **3** (1994), 175–188. (with Satoru Murakami)

45. Existence of almost periodic solutions of some functional differential equations with infinite delay in a Banach space. *Tohoku Math. J.* (2) **49** (1997), 133–147. (with Yoshiyuki Hino and Satoru Murakami)

Coordinating Supply Chains with a Credit Mechanism

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Abstract: This paper studies the supply chain coordination with a trade credit under symmetric and asymmetric information, where the retailer has an individual profit target from the business and the vendor is the decision-maker of the supply chain. We propose a coordination mechanism through credit contracts and show that a win-win outcome is achieved by redistributing the cost savings from coordination mechanism under certain constraints. Numerical examples are given to illustrate our results.

Keywords: *supply chain coordination; trade credit; contract; information asymmetry.*

Mathematics Subject Classification (2010): 90B05.

1 Introduction

In the current competitive business environment, trade credit has been widely used and represents an important proportion of firms finance. Rajan and Zingales reported that accounts payable amounted to 15% of the assets for a sample of nonfinancial U.S. firms on Global Vantage while debt in current liabilities accounted for just 7.4% [1]. In China, it was once overused, e.g., powerful buyers like WalMart, Carrefour and Gome used to allow delayed payments to their vendors for as long as one year, so that the government has made a law to ban buyers from delaying payment for more than two months. Since Goyal first developed an economic order quantity (EOQ) model under the conditions of permissible delay in payments [2], there is a great deal of literature dealing with a variety

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of situations such as shortages allowed, partial backlogging, credit-linked demand/order quantity, deterioration etc. Chung utilized the discounted cash-flows (DCF) approach to analyze the optimal inventory policy in the presence of the trade credit [3]. Jamal et al. extended Goyal's model to consider the deteriorating items and to allow for shortages [4]. Teng amended Goyal's model by considering the difference between unit price and unit cost, then established an easy analytical closed-form to the problem [5]. Song and Cai studied the payment time of the retailer and the length of the inventory cycle under permitted delay of payment by wholesaler and derived the optimal joint solution [6]. Chung and Huang extended Goyal's model to the case that the units are replenished at a finite rate [7]. Hu and Liu investigated retailer's optimal replenishment policy under conditions of permissible delay in payments and allowable shortages within the economic production quantity (EPQ) framework [8]. Ouyang et al. developed an inventory model with non-instantaneous receipt where the supplier provides not only a permissible delay but also a cash discount for the retailer [9]. In [10], Ouyang et al. investigated a generalized inventory model for deteriorating items under the delay in payments linked to order quantity. Liang and Zhou developed a two-warehouse inventory model for deteriorating items under permissible delay in payment [11]. With purchasing cost depending on the delay in payments and the order quantity, Krichen et al. proposed a solution approach that generates stable coalition structures for the retailers [12]. Jaggi et al. investigated the impact of credit-linked demand on the retailers optimal replenishment policy under two levels of trade credit policy [13]. Annadurai and Uthayakumar further extended the work of Jaggi et al. [13], by including deteriorating items and backlogging [14].

To the best of our knowledge, Jaber and Osman first used trade credit as a mechanism to coordinate a two-level supply chain [15]. In [16], Sarmah et al. investigated a coordination problem in a single-manufacturer and multiple heterogeneous buyers situation with a credit option. Recently, Duan et al. studied the supply chain coordination policy by delay in payments for the products with fixed lifetime [17]. However, information asymmetry is not considered by the above-cited literature. Luo and Zhang studied the benefit of coordinating a supply chain by a credit contract under both symmetric and asymmetric information [18]. But in their analysis, the vendor gets all the cost savings from the coordination regardless of the buyer's benefits, and the supply chain cannot be coordinated under asymmetric information. However, each member of a supply chain wants a certain fixed amount of cost reduction from the business and hence parties will not be interested in coordinating if their target profit is not achieved. In this paper, we proposed a trade credit contract to coordinate the supply chain from the business where both the parties have their cost targets. Considering credit policy as coordination mechanism between the two parties, our objective is to derive the optimal credit periods under symmetric and asymmetric information, respectively. In addition, we show that using trade credit can coordinate the supply chain in cases with both the symmetric and asymmetric information under some constraints.

2 Model Formulation Under Symmetric Information

2.1 Notation and assumptions

The following notations and assumptions are adopted throughout this paper.

Notation:

S_r	retailer's ordering cost per order;
S_v	vendor's setup cost per production;
D	market demand rate;
h	retailer's inventory holding cost per unit time;
I_r, I_v	retailer's and vendor's unit capital opportunity;
K	integer lot size multiplier per cycle;
$M(K)$	the credit period offered by the vendor;
Q	order quantity of the buyer.

Assumption 2.1 Demand rate is known and constant.

Assumption 2.2 The vendor follows a lot-for-lot manufacturing policy.

Assumption 2.3 Both production and replenishment are instantaneous.

Assumption 2.4 Shortages are not allowed.

2.2   The trade credit under full information

In the absence of any coordination, it is easy to obtain that the retailer's optimal lot size is simply the EOQ, i.e., $Q_r = \sqrt{\frac{2DS_r}{h+I_r}}$, and the retailer's corresponding cost per unit time is

$$TC_r = \sqrt{2DS_r(h + I_r)}. \tag{1}$$

Under the lot-for-lot system, the vendor's cost including only the setup cost is $TC_v = \frac{DS_v}{Q_r}$. For the whole supply chain, the joint cost with coordination is

$$TC(Q) = \frac{D(S_r + S_v)}{Q} + \frac{(h + I_r)Q}{2}. \tag{2}$$

Minimizing (2) yields the optimal lot size for the whole system

$$Q_c = \sqrt{\frac{2D(S_r + S_v)}{h + I_r}}, \tag{3}$$

which is obvious larger than Q_r. Hence, in order to entice the buyer to alter his current EOQ by a factor $K(K > 0)$, i.e., the retailer's new ordering quantity is KQ_r, the vendor offers a delay period $M(K)$ (the delay period M is dependent on the retailer's order size) to the buyer to compensate the retailer for his increased inventory cost, and possibly provides an additional saving, such that the vendor can benefit from higher order quantity. It is assumed that the buyer is willing to coordinate as long as his target cost is no larger than $[1 - G(K)]TC_r$, where $G(K)(\geqslant 0)$ is the reduction factor of retailer's target cost from supply chain coordination , such that the retailer can also benefit from the supply chain coordination. For simplification, we assume that $G(K)$ is a linear function of K, i.e, $G(K) = \alpha K + \beta$.

Under the above trade credit contract, the buyer's inventory cost per unit time, denoted by $\widetilde{TC}_r(Q, K)$, is

$$\widetilde{TC}_r(Q, K) = \frac{DS_r}{KQ_r} + \frac{(h + I_r)KQ_r}{2} - DI_r M(K), \tag{4}$$

and the vendor's cost per unit time is

$$\widetilde{TC}_v(Q,K) = \frac{DS_v}{KQ_r} + DI_vM(K). \tag{5}$$

For achieving the retailer's target cost, it is obvious that the trade credit should satisfy

$$\frac{DS_r}{KQ_r} + \frac{(h+I_r)KQ_r}{2} - DI_rM(K) \leqslant (1-\beta-\alpha K)TC_r. \tag{6}$$

Simplifying the above inequality, we have

$$M(K) \geqslant \frac{TC_r}{2DI_r}\left[\frac{1}{K} + K - 2(1-\beta-\alpha K)\right]. \tag{7}$$

Next, we will determine the vendor's optimal lot size. From (5), we can see that the vendor's cost is minimized only if the trade credit gets the smallest value. Hence, substituting $M(K) = \frac{TC_r}{2DI_r}\left[\frac{1}{K} + K - 2(1-\beta-\alpha K)\right]$ into (5), yields

$$\widetilde{TC}_v(Q,K) = \frac{DS_v}{KQ_r} + \frac{I_vTC_r}{2I_r}\left[\frac{1}{K} + K - 2(1-\beta-\alpha K)\right]. \tag{8}$$

It is easy to verify that $\widetilde{TC}_v(Q,K)$ in (8) is convex in K, so the optimal value of K is determined by the first-order condition as follows

$$\frac{\partial \widetilde{TC}_v(Q,K)}{\partial K} = \frac{(1+2\alpha)I_vTC_r}{2I_r} - \frac{TC_r}{2K^2}\left(\frac{S_v}{S_r} + \frac{I_v}{I_r}\right) = 0. \tag{9}$$

From (9), we get

$$K^* = \sqrt{\frac{\frac{S_v}{S_r}\frac{I_r}{I_v} + 1}{1+2\alpha}}.$$

Then $Q_c = K^*Q_r$ if $\alpha^* = \frac{S_v(I_r-I_v)}{2(S_r+S_v)I_v}$. Furthermore, from the perspective of vendor, it is obvious that β should satisfy

$$\frac{DS_v}{KQ_r} + \frac{I_vTC_r}{2I_r}\left[\frac{1}{K^*} + K^* - 2(1-\beta-\alpha^*K^*)\right] \geqslant \frac{DS_v}{Q_r},$$

if supply chain coordination can be achieved, i.e., $\beta \geqslant 1 - \left[(1+2\alpha^*)K^* - \frac{S_vI_r}{2S_rI_v}\right]$. Hence, we have the following proposition.

Proposition 1 *If*

$$\alpha^* = \frac{S_v(I_r-I_v)}{2(S_r+S_v)I_v}, \tag{10}$$

then the proposed trade credit contract can not only achieves the supply chain coordination but achieve the win-win outcome for any given

$$\beta \geqslant 1 - \left[(1+2\alpha^*)K^* - \frac{S_vI_r}{2S_rI_v}\right]. \tag{11}$$

In reality, according to the firm's bargaining powers, the benefits allocation can be achieved by changing β, e.g., a simple effective way to set the benefits allocated is to divide them equitably between the two firms.

3   Model Formulation Under Asymmetric Information

In this section, we consider the situation that the capital cost of the buyer is his private information. Under asymmetric information, the vendor will offer a menu of contracts for the sake of stimulating the buyer to reveal his information via contract selection. This so-called screening game in the game theory, where the vendor is the first to move, can derive the optimal contract by using the approach developed in [19].

We assume that the vendor knows the random variable I_r characterized by a prior distribution $F(I_r)$ and the corresponding density function $f(I_r)$ on its domain $[\underline{I_r}, \overline{I_r}]$, where $0 \leqslant \underline{I_r} < \overline{I_r} < \infty$. In the case of information asymmetry, the vendor now offers a menu of contracts $\{K, M(K)\}$. Then the buyer chooses a specific pair from the contract menu, and the vendor can infer buyer's I_r from his selection.

Based on the above arguments, we can present the vendor's contract design problem as the following optimization program

$$\min_{M,K} \widetilde{TC}_v^A = \int_{\underline{I_r}}^{\overline{I_r}} \left[\frac{DS_v}{KQ_r} + DI_v M(K) \right] f(I_r) dI_r \tag{12}$$

such that

$$\frac{DS_r}{KQ_r} + \frac{(h+I_r)KQ_r}{2} - DI_r M(K)] \leqslant \frac{DS_r}{\tilde{K}Q_r} + \frac{(h+I_r)\tilde{K}Q_r}{2} - DI_r M(\tilde{K}), \tag{12a}$$

$$\frac{DS_r}{KQ} + \frac{(h+I_r)KQ}{2} - DwM(K) \leqslant (1 - \beta - \alpha K)TC_r^0, \tag{12b}$$

$$\underline{I_r} \leqslant I_r \leqslant \overline{I_r}. \tag{12c}$$

The constraint condition in (12a) indicates that the retailer will choose the optimal contract menu to minimize his cost, and the constraint condition in (12b) requires that the buyer's cost with trade credit must be not higher than his reservation cost, for ensuring his participation. TC_r^0 is an exogenous variable that is retailer's reservation cost (e.g., it can be set as the retailer's cost without coordination).

The following Proposition 2 gives the vendor's optimal menu of credit contracts.

Proposition 2 *When the retailer has the personal information about his capital cost I_r, the vendor's optimal contract menu is*

$$M_A^* = \frac{1}{DI_r} \left[\frac{(h+I_r)K_A^* Q_r}{2} + \frac{DS_r}{K_A^* Q_r} \right] - \frac{(1 - \beta - \alpha K_A^*)TC_r^0}{D\overline{I_r}}, \tag{13}$$

where K_A^* is given as follows

$$K_A^* = \begin{cases} \frac{1}{Q_r} \sqrt{\frac{2DS_v I_r + 2DS_r[I_v - F(I_r)/f(I_r)]}{h[I_v - F(I_r)/f(I_r)] + I_r I_v + 2\alpha I_r TC_r^0/Q_r}}, if I_v \geqslant \\ \max\left\{ \frac{hS_v}{S_r} - \frac{2\alpha TC_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha I_r TC_r^0/Q_r}{h+I_r} \right\} \\ or \ I_v \leqslant \min\left\{ \frac{F(I_r)}{f(I_r)} - \frac{S_v I_r}{S_r}, \frac{hS_v}{S_r} - \frac{2\alpha TC_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha I_r TC_r^0/Q_r}{h+I_r} \right\}; \\ \frac{1}{Q_r} \sqrt{\frac{2DS_r}{h}}, otherwise. \end{cases} \tag{14}$$

Proof. Let $U(I_r) = \frac{DS_r}{KQ_r} + \frac{(h+I_r)KQ_r}{2} - DI_rM(K)$, then $U'(I_r) = -\frac{DS_rK'}{K^2Q_r} + \frac{(h+I_r)Q_rK'}{2} + \frac{KQ_r}{2} - DI_rM' - DM$. Hence, we have

$$U(I_r) = U(\overline{I_r}) - \int_{I_r}^{\overline{I_r}} \left(\frac{KQ_r}{2} - DM \right) dI_r. \tag{15}$$

Furthermore, the constraint in (12a) is equal to

$$\left. \frac{dU(\tilde{I}_r)}{d\tilde{I}_r} \right|_{\tilde{I}_r = I_r} = 0, \tag{16}$$

$$\left. \frac{d^2U(\tilde{I}_r)}{d\tilde{I}_r^2} \right|_{\tilde{I}_r = I_r} \geqslant 0. \tag{17}$$

That is, the buyer will incur the lowest cost when he chooses $\tilde{I}_r = I_r$. It follows from (16)–(17) that

$$\frac{DS_rK'}{K^2Q_r} - \frac{(h+I_r)Q_rK'}{2} + DI_rM' = 0, \tag{18}$$

$$-\frac{2DS_r(K')^2}{K^3Q_r} + \frac{DS_rK''}{K^2Q_r} - \frac{(h+I_r)Q_rK''}{2} + DI_rM'' \geqslant 0. \tag{19}$$

Differentiating both sides of (18) with respect to I_r yields

$$-\frac{2DS_r(K')^2}{K^3Q_r} + \frac{DS_rK''}{K^2Q_r} - \frac{(h+I_r)Q_rK''}{2} - \frac{Q_rK'}{2} + DI_rM'' + DM' = 0. \tag{20}$$

From (19) and (20), we obtain

$$DM' - \frac{Q_rK'}{2} \leqslant 0. \tag{21}$$

Since the objective function $\widetilde{TC}_v^A$ can be rewritten as

$$\widetilde{TC}_v^A = \int_{\underline{I_r}}^{\overline{I_r}} \left\{ \frac{D(A_v + A_r)}{KQ_r} + D(I_v - I_r)M(K) + \frac{(h+I_r)KQ_r}{2} - U(I_r) \right\} f(I_r) dI_r, \tag{22}$$

the vendor's expected cost is decreasing with $U(I_r)$. Hence the condition $U(\overline{I_r}) = (1 - \beta - \alpha K)TC_r^0$ must be satisfied for any given K.

Under the above discussion, the optimal problem in (12) can be simplified as

$$\min_{M,K} \widetilde{TC}_v^A = \int_{\underline{I_r}}^{\overline{I_r}} \left\{ \frac{D(S_v + S_r)}{KQ_r} + D(I_v - I_r)M + \left[\frac{(h+I_r)Q_r}{2} + \alpha TC_r^0 \right] K \right.$$
$$\left. + \left(\frac{KQ_r}{2} - DM \right) \frac{F(I_r)}{f(I_r)} \right\} f(I_r) dI_r - (1 - \beta)TC_r^0 \tag{23}$$

such that

$$DM' - \frac{Q_r K'}{2} \leqslant 0, \tag{23a}$$

$$\underline{I_r} \leqslant I_r \leqslant \overline{I_r}. \tag{23b}$$

Next, we first neglect the constraint conditions. Then the optimal contract must satisfy the following first-order condition with respect to K

$$-\frac{D(S_v + S_r)}{K^2 Q_r} + D(I_v - I_r)\frac{dM}{dK} + \frac{(h + I_r)Q_r + 2\alpha T C_r^0}{2} + \left(\frac{Q_r}{2} - D\frac{dM}{dK}\right)\frac{F(I_r)}{f(I_r)} = 0. \tag{24}$$

Since (18) implies that $\frac{dM}{dK} = \frac{1}{DI_r}\left[\frac{(h+I_r)Q_r}{2} - \frac{DS_r}{K^2 Q_r}\right]$, we have

$$M = \frac{1}{DI_r}\left[\frac{(h + I_r)KQ_r}{2} + \frac{DS_r}{KQ_r}\right] - \frac{(1 - \beta - \alpha K)TC_r^0}{D\overline{I_r}}. \tag{25}$$

Combining (24) and (25) leads to the optimal order multiple

$$K_A^* = \frac{1}{Q_r}\sqrt{\frac{2DS_v I_r + 2DS_r[I_v - F(I_r)/f(I_r)]}{h[I_v - F(I_r)/f(I_r)] + I_r I_v + 2\alpha I_r T C_r^0/Q_r}}. \tag{26}$$

Now, taking into account the neglected constraint in (23a) leads to

$$K^2 \leqslant \frac{2DS_r}{hQ_r^2}. \tag{27}$$

Solving inequity (27), we get

$$I_v \geqslant \max\left\{\frac{hS_v}{S_r} - \frac{2\alpha T C_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha I_r T C_r^0/Q_r}{h + I_r}\right\} \tag{28}$$

or

$$I_v \leqslant \min\left\{\frac{F(I_r)}{f(I_r)} - \frac{S_v I_r}{S_r}, \frac{hS_v}{S_r} - \frac{2\alpha T C_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha I_r T C_r^0/Q_r}{h + I_r}\right\}. \tag{29}$$

When inequalities (28)–(29) are not fulfilled, K_A^* does not satisfy the constraint in (23a) and hence it is not the optimal solution of optimization problem (12). But we can easily prove that $\widetilde{TC}_v^A$ is decreasing on the interval $\left[0, \frac{1}{Q_r}\sqrt{\frac{2DS_r}{h}}\right]$ (see [18]), and hence the vendor has the minimum cost at $K_A^* = \frac{1}{Q_r}\sqrt{\frac{2DS_r}{h}}$ in this case. $\square$

From Proposition 2, if

$$I_v \leqslant \min\left\{\frac{F(I_r)}{f(I_r)} - \frac{S_v I_r}{S_r}, \frac{hS_v}{S_r} - \frac{2\alpha_A^* T C_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha_A^* I_r T C_r^0/Q_r}{h + I_r}\right\}$$

or

$$I_v \geqslant \max\left\{\frac{hS_v}{S_r} - \frac{2\alpha_A^* T C_r^0}{Q_r}, \frac{hF(I_r)/f(I_r) - 2\alpha_A^* I_r T C_r^0/Q_r}{h + I_r}\right\}$$

for all $I_r \in [\underline{I_r}, \overline{I_r}]$, then the supply chain can be coordinated by choosing an appropriate value of β, where

$$\alpha_A^* = \frac{Q_r}{I_r TC_r^0} \left\{ \frac{DS_v I_r + DS_r[I_v - F(I_r)/f(I_r)]}{1 + S_v/S_r} - \frac{h}{2}\left[I_v - \frac{F(I_r)}{f(I_r)}\right] - \frac{I_r I_v}{2} \right\}. \quad (30)$$

As a result, the joint menu of contracts $(\alpha_A^*, K_A^*, M(K_A^*))$ achieves chain coordination between the retailer and the vendor.

4 Numerical Examples

In this section, we first study a full information case to examine the division of surplus generated due to coordination. The basic parameters are set as follows: $D = 1500 \text{ units/year}, S_v = 2000 \text{ /order}, S_r = 500 \text{ /order}, h = 10 \text{ /unit/year}$.

Example 1. If $I_r = 4$ and $I_v = 3$, then by Proposition 1, $(\alpha^*, K^*) = (0.13, 2.24)$. For given $\beta = 0.02$, the retailer's cost and the vendor's cost are 3124.66 and 6366.41, i.e., decrease is 46.66% and 43.96% from without coordination, respectively.

Example 2. If $I_r = 5$ and $I_v = 8$, then by Proposition 1, $(\alpha^*, K^*) = (-0.15, 2.24)$. For given $\beta = 0.45$, the retailer's cost and the vendor's cost are 4199.87 and 7705.19, i.e., decrease is 12.94% and 23.12% from without coordination, respectively.

Example 3. Assume that $I_v = 2$ and $I_r \sim U(2,6)$ (uniformly distribution), then by Proposition 2, the optimal value of K is $K_A^* = \sqrt{1 + 0.1I_r}$ and the corresponding credit menu is $M_A^* = \sqrt{5/3}(0.1 + 2/I_r) + 0.02\sqrt{1 + 0.1I_r} - 0.34$ for given I_r. Hence the vendor's cost is 6971.33. The costs for the vendor and the retailer decrease are 15.63% and 20.91% from without coordination, respectively.

Example 4. Assume that $I_v = 10$ and $I_r \sim U(3,5)$, then by Proposition 2, the joint menu of contracts $(\alpha_A^*, K_A^*, M(K_A^*))$, which is determined by (13), (14) and (30), can coordination the supply chain. For fixed $\beta = 0.12$ and $TC_r^0 = 4000$, the costs for the vendor and the retailer decrease are 25.43% and 16.17% from without coordination, respectively.

5 Conclusion

In this paper, through credit contracts, we study the division of benefits sharing for the supply chain coordination from the business under both symmetric and asymmetric information. To some extent, the results of this paper may be applied to some practice business. The wide usage of trade credit has shown that it can effectively reduce the cost of supply chain members. In real practice, the trade credit policy may also be more attractive than other policies like quantity discount to the retailer. In addition, by using a quantity-dependent trade credit to entice the retailer to alter his order quantity, the vendor can get much more benefits from the supply chain coordination, and the vendor also prefers such a policy when he is financially strong.

Acknowledgment

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Reduced Order Function Projective Combination Synchronization of Three Josephson Junctions Using Backstepping Technique

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Abstract: In this paper, a new synchronization scheme, combination synchronization, is used to realize reduced order function projective synchronization among three chaotic Josephson junction systems using backstepping technique. In the first case, function projective synchronization of two (2) third order drive systems with a single second order Josephson junction is considered while in the second case, a single third order system is synchronized with two (2) second order system using backstepping. Controllers are designed and simulated to show the efficacy of combination synchronization scheme.

Keywords: *function projective; reduced order synchronization; Josephson junction; combination synchronization.*

Mathematics Subject Classification (2010): 34H10, 93C10.

1 Introduction

Synchronization between two chaotic systems has evolved greatly since its proposition by Pecora and Caroll [1]. Many types of synchronization schemes have been proposed and implemented including complete synchronization (CS) [1], projective synchronization (PS) [2, 3], lag synchronization (LS) [4], modified projective synchronization [5] while techniques such as adaptive control method [6], active control [7], active backstepping [2] and feedback control [8] have been used for design of controllers. Backstepping scheme has been efficient in the design technique for stabilization, tracking and synchronization

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of chaotic systems. According to Tian et al., [9], some of the advantages of the method include: applicability to a variety of chaotic systems irrespective of whether they contain external excitation or not; needs only one controller to realize synchronization of chaotic systems and finally there is no derivative in the controller. Backstepping technique offers faster and better transient error dynamics convergence and synchronization time than the active control technique [2]. The results of chaos synchronization are utilized in biological sciences [10], economics and finance [11], chemical reactions, secure communication [12, 13] and cryptography and data encryption. Recently, synchronization between fractional order and integer order system was reported in [14]

Projective Synchronization (PS) refers to the dynamical behavior in which the responses of two identical systems synchronize up to a constant scaling factor $\alpha \in \Re$ [15]. When $\alpha = 1$ we have complete synchronization and $\alpha = -1$ gives antisynchronization of the systems. Function projective synchronization in which the scaling factor is not a constant value was proposed by Du et al., [16]. A form of projective synchronization referred to as hybrid projective synchronization, in which the different state variables can synchronize up to different scaling factors was implemented by Hu et al., [7, 17]. The hybrid function projective synchronization was extended to different systems with time varying parameters [18], fractional order system [19] and hyperchaotic system [20]. The scaling constant in projective synchronization gives faster communication, hence, the popularity of the scheme.

Until recently, synchronization has been applied to two systems of the same dimension (identical or non-identical), however, natural and practical systems tend to involve systems of different order. As pointed out in [2], there are real situations where systems of different order need to be synchronized e.g the order of the thalamic neurons can be different from the hippocampal neurons, the synchronization between heart and lungs, the synchronization in neuron systems and certain biomechanical systems (such as biological implants), mechanical systems [21]. This motivated the implementation of increased order [22–24] and reduced order synchronization [25, 26].

It was also proposed by Runzi and Yinglan [27] that information signal be transmitted by two different drive systems. For example, we split the transmitted signals into several parts, each part loaded in different systems; or divide time into different intervals, the signals in different intervals loaded in different systems. If this is really so, then the transmitted signals may have stronger anti-attack ability and anti-translated capability than that transmitted by the usual transmission model. Furthermore, in a communication network, there are many users (slave) but one control (master) which connects different users to one another. There is the need to implement a synchronization scheme whereby many users can be connected to and routed through a single master securely. Increased order and reduced order combination synchronization of three different nonlinear systems was implemented using active backstepping design [19]. Synchronization between combination of two drive systems and combination of two response systems in drive-response synchronization model was investigated by Sun et al., [28] in a new scheme referred to as combination-combination synchronization.

To the best of our knowledge, research into reduced order function projective combination synchronization has not been carried out, hence, we set forth in this paper to investigate it. From the aforementioned, we implement a reduced order projective synchronization of (i) two (2) 3-dimensional system and one slave (ii) two (2) one 3-dimensional master system and two (2) 2-dimensional slave system using the backstepping technique. The remainder of the paper is arranged as follows:

2   Reduced-order Function Projective Combination Synchronization of Two Third Order and One Second Order Josephson Junctions

2.1   Design of controller via active backstepping technique

In this section, two third order Josephson junctions in (1) and (2) are taken as the drive systems while one second order non-autonomous Josephson junction (3) is taken as the response system in order to achieve generalized reduced order combination synchronization among the three chaotic Josephson junctions

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3), \\
\dot{x}_3 &= \frac{1}{\beta_L}(x_2 - x_3),
\end{aligned}
\tag{1}
$$

the second drive system is

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= \frac{1}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3), \\
\dot{x}_3 &= \frac{1}{\beta_L}(y_2 - y_3),
\end{aligned}
\tag{2}
$$

while the response system is given as

$$
\begin{aligned}
\dot{z}_1 &= z_2 + u_1, \\
\dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b\sin \omega t + u_2,
\end{aligned}
\tag{3}
$$

where $u_i(t)$, $i = 1, 2$ are the controllers to be designed. We define the error systems as follows

$$
\begin{aligned}
e_1 &= z_1 - (\alpha_1(t)x_1 + \beta_1(t)y_1 + \alpha_3(t)x_3 + \beta_3(t)y_3), \\
e_2 &= z_2 - (\alpha_2(t)x_2 + \beta_2(t)y_2).
\end{aligned}
\tag{4}
$$

Using the error systems defined in (4) with systems defined in (1), (2) and (3) yields the following error dynamics

$$
\dot{e}_1 = z_2 + u_1 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3)
$$
$$
- \dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3
$$

$$
= e_2 + \alpha_2(t)x_2 + \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + u_1
$$
$$
- \dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3,
$$

$$
\dot{e}_2 = -\alpha z_2 - \sin z_1 + a + b\sin \omega t + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)
$$
$$
- \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t)
$$

$$= -\alpha(e_2 + \alpha_2(t)x_2 + \beta_2(t)y_2) - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t) - \sin z_1$$

$$+ a + b\sin\omega t + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Thus, the error dynamics of the system can be written as

$$\dot{e}_1 = e_2 + u_1 + A_1, \tag{5}$$

$$\dot{e}_2 = -\alpha e_2 + u_2 + A_2, \tag{6}$$

where

$$A_1 = \alpha_2(t)x_2 + \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) - \dot{\alpha}_1(t)x_1$$

$$- \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3,$$

$$A_2 = -\alpha(\alpha_2(t)x_2 + \beta_2(t)y_2) - \sin z_1 + a + b\sin\omega t - \dot{\alpha}_2(t)x_2 - \dot{\beta}_2(t) - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2$$

$$- \sin x_1 - x_3) - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Our goal is to find the control functions which will enable the systems (1), (2) and (3) realize generalized reduced order function projective combination synchronization by active backstepping technique. The design procedure includes three steps as shown below:

Step 1. Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + u_1 + A_1, \tag{7}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the following Lyapunov function $v_1 = \frac{1}{2}q_1^2$. The time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + u_1 + A_1). \tag{8}$$

Suppose $\alpha_1(q_1) = 0$ and the control function u_1 is chosen as

$$u_1 = -(A_1 + kq_1), \tag{9}$$

then $\dot{v}_1 = -kq_1^2 < 0$, where k is positive constant which represent the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the following (q_1, q_2)-subsystems

$$\dot{q}_1 = q_2 - kq_1,$$
$$\dot{q}_2 = -\alpha q_2 + u_2 + A_2. \tag{10}$$

Step 2. In order to stabilize subsystem (10), a Lyapunov function can be chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. The time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + u_2 + A_2). \tag{11}$$

If the control function u_2 is chosen as

$$u_2 = \alpha q_2 - q_1 - A_2 - kq_2, \tag{12}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$, where k is a positive constant which represent the feedback gain. Then, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (10) is asymptotically stable. This implies that generalized reduced order function projective combination synchronization of the drive systems (1) and (2) and the response system (3) is achieved.

Finally, we have the following subsystems

$$\dot{q}_1 = q_2 - kq_1,$$
$$\dot{q}_2 = -q_1 - kq_2. \tag{13}$$

Now the generalized reduced order function projective combination synchronization is achieved, the following can be obtained.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then we have Case 1.

Case 1: If the controllers are chosen as

$$
\begin{aligned}
u_1 &= (\beta_1(t) - \beta_2(t))y_2 + \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + \dot{\beta}_1(t)y_1 + \dot{\beta}_3(t)y_3 - kq_1, \\
u_2 &= (\alpha - k)q_2 - q_1 + \alpha\beta_2(t)y_2 + \sin z_1 - a - b\sin\omega t + \\
&\quad \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \dot{\beta}_2(t)y_2,
\end{aligned} \tag{14}
$$

where $q_1 = z_1 - \beta_1(t)y_1 - \beta_3(t)y_3$, $q_2 = z_2 - \beta_2(t)y_2$, then the drive system (2) achieves reduced order modified function projective synchronization with the response system (3). Let $\beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Case 2.

Case 2: If the controllers are chosen as

$$
\begin{aligned}
u_1 &= (\alpha_1(t) - \alpha_2)x_2 + \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - kq_1 + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3, \\
u_2 &= (\alpha - k)q_2 - q_1 + \alpha\alpha_2(t)x_2 + \sin z_1 - a - b\sin\omega t + \\
&\quad \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + \dot{\alpha}_2(t)x_2,
\end{aligned} \tag{15}
$$

where $q_1 = z_1 - \alpha_1(t)x_1 - \alpha_3(t)x_3$, $q_2 = z_2 - \alpha_2(t)x_2$, then the drive system (1) achieves reduced order modified function projective synchronization with the response system (2).

Suppose $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Case 3.

Case 3: If the controllers are chosen as

$$
\begin{aligned}
u_1 &= -kq_1, \\
u_2 &= (\alpha - k)q_2 - q_1 + \sin z_1 - a - b\sin\omega t,
\end{aligned} \tag{16}
$$

where $q_1 = z_1$, $q_2 = z_2$, then the equilibrium point $(0, 0, 0)$ of the response system (3) is asymptotically stable.

Suppose $\beta_1(t) = \beta_2(t) = \beta_3(t) = \alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \gamma(t)$, then we obtain Case 4.

Case 4: If the controllers are chosen as

$$
\begin{aligned}
u_1 &= \frac{\gamma(t)}{\beta_L}(x_2 - x_3 + y_2 - y_3) + \dot{\gamma}(t)(x_1 + y_1 + x_3 + y_3) - kq_1, \\
u_2 &= (\alpha - k)q_2 - q_1 + \alpha\gamma(t)(x_2 + y_2) + \sin z_1 - a - b\sin\omega t + \dot{\gamma}(t)(x_2 + y_2) \\
&\quad - \frac{\gamma(t)}{\beta_C}(g(x_2)x_2 + g(y_2)y_2 + \sin x_1 + \sin y_1 + x_3 + y_3 - 2i),
\end{aligned}
\tag{17}
$$

where $q_1 = z_1 - \gamma(t)(x_1 + y_1 + x_3 + y_3)$, $q_2 = z_2 - \gamma(t)(x_2 + y_2)$, then the drive systems (1) and (2) achieve reduced order function projective combination synchronization with the response system (3).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$, then we obtain Case 5.

Case 5: If the controllers are chosen as

$$
\begin{aligned}
u_1 &= -\alpha_2(t)x_2 - \beta_2(t)y_2 + \alpha_1(t)x_2 + \beta_1(t)y_2 + \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) \\
&\quad + \dot{\alpha}_1(t)x_1 + \dot{\beta}_1(t)y_1 + \dot{\alpha}_3(t)x_3 + \dot{\beta}_3(t)y_3 - kq_1, \\
u_2 &= \alpha(\alpha_2(t)x_2 + \beta_2(t)y_2) + \sin z_1 - a - b\sin\omega t + \dot{\alpha}_2(t)x_2 \\
&\quad + \dot{\beta}_2(t)(y_2) + (\alpha - k)q_2 - q_1 + \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) \\
&\quad + \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3),
\end{aligned}
\tag{18}
$$

$q_1 = z_1 - (\alpha_1(t)x_1 + \beta_1(t)y_1 + \alpha_3(t)x_3 + \beta_3(t)y_3)$, $q_2 = z_2 - (\alpha_2(t)x_2 + \beta_2(t)y_2)$, then the drive systems (1) and (2) achieve reduced order modified function projective combination synchronization with the response system (3).

2.2 Numerical simulation results

The designed controllers are verified in our numerical simulation using the in-built fourth order Runge-Kutta (ode45) routine in Matlab. In the numerical simulation procedure we used the systems parameters within the chaotic region and controllers are chosen in accordance with Case 4. The initial conditions of the drive systems and response system are given as $(x_1, x_2, x_3) = (0, 0, 0)$, $(y_1, y_2, y_3) = (111)$, $(z_1, z_2) = (0, 1)$, $\gamma(t) = 2.0 + 0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Figure 1 shows the dynamics of the error variables when the controllers are deactivated for $0 \leq t \leq 200$. Figure 2 shows that reduced order combination synchronization among systems (1), (2) and (3) is achieved as indicated by the convergence of the error state variables to zero as soon as the controllers are switch on for $t \geq 80$. Figure 3 shows that the state variables of the drive and the response systems follow the same trajectory when the controllers are activated for $t \geq 80$, this also confirms reduced order combination synchronization among systems (1), (2) and (9). Evidence of reduced order function projective combination synchronization is presented in Figure 4.
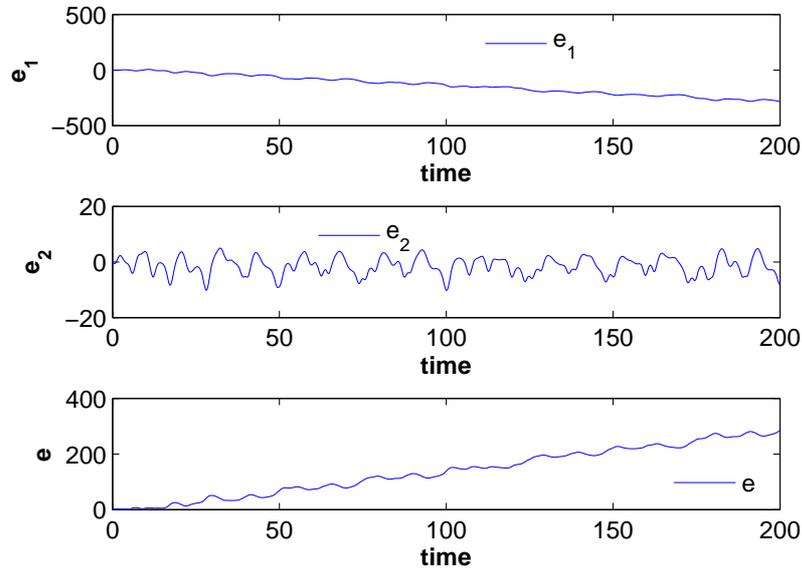
Figure 1: Error dynamics among systems (1), (2) and (3) with control activated at $t \geq 80$.
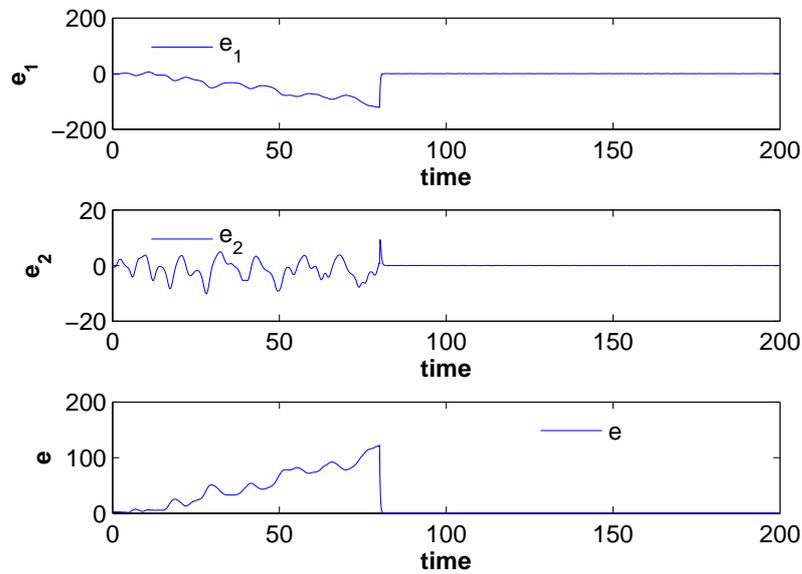

Figure 2: Error dynamics among systems (1), (2) and (9) with control activated activated at $t \geq 80$.
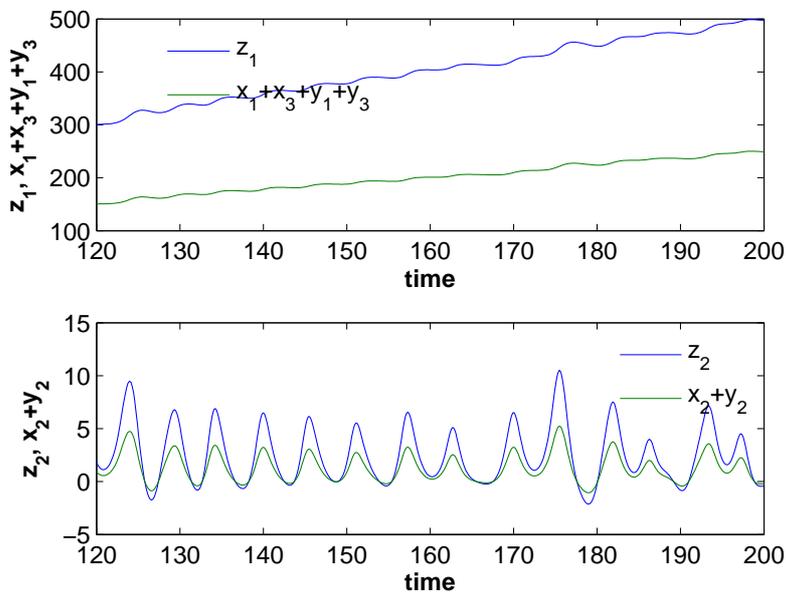
**Figure 3**: Dynamics of state variables with control applied.
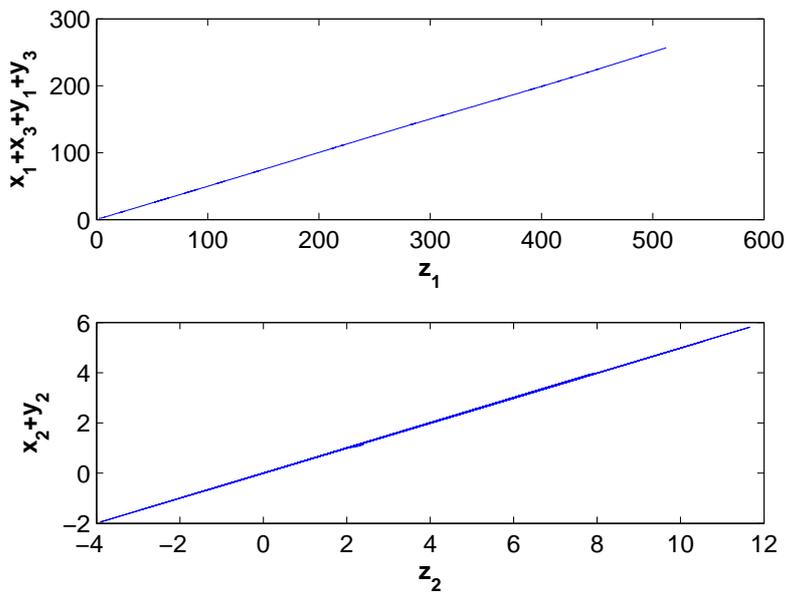
Figure 4: Evidence of projective synchronization.

3 Generalized Reduced-order Function Projective Combination Synchronization of One Third Order and Two Second Order Josephson Junctions

3.1 Design of controller via active backstepping technique

In this section, one third order Josephson junction in (1) is taken as the drive system while two second order non-autonomous Josephson junctions in (19) and (20) are taken as the response systems in order to achieve reduced order function projective combination synchronization among the three chaotic Josephson junctions

$$
\begin{aligned}
\dot{y}_1 &= y_2 + u_1, \\
\dot{y}_2 &= -\alpha y_2 - \sin y_1 + a + b \sin \omega t + u_2,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\dot{z}_1 &= z_2 + u_3, \\
\dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b \sin \omega t + u_4,
\end{aligned}
\tag{20}
$$

where u_1, u_2, u_2 and u_4 are the controllers to be designed. We define the error systems as follows

$$
\begin{aligned}
e_1 &= z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t)x_3), \\
e_2 &= z_2 + y_2 - \alpha_2 x_2.
\end{aligned}
\tag{21}
$$

Using the error systems defined in (21) with systems defined in (1), (19) and (20) yields the following error dynamics

$$
\begin{aligned}
\dot{e}_1 &= z_2 + y_2 + u_3 - \alpha_1(t)x_2 + u_1 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3 \\
&= e_2 + (\alpha_2(t) - \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + u_3 + u_1 - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3,
\end{aligned}
$$

$$
\begin{aligned}
\dot{e}_2 &= -\alpha z_2 - \sin z_1 + a + b \sin \omega t + u_4 - \alpha y_2 - \sin y_1 + a + b \sin \omega t \\
&\quad + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) - \dot{\alpha}_2(t)x_2 \\
&= -\alpha(e_2 + \alpha_2(t)x_2) - \sin z_1 + a + b \sin \omega t) - \dot{\alpha}_2(t)x_2 \\
&\quad + u - \sin y_1 + a + b \sin \omega t) + u_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3).
\end{aligned}
$$

Thus, the error dynamics of the system can be written as:

$$
\dot{e}_1 = e_2 + U_1 + B_1,
\tag{22}
$$

$$
\dot{e}_2 = -\alpha e_2 + U_2 + B_2,
\tag{23}
$$

where

$$
B_1 = (\alpha_2(t) - \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3,
$$

$$
B_2 = -\alpha\alpha_2(t)x_2 - \sin z_1 + 2a + 2b \sin \omega t - \sin y_1 - \dot{\alpha}_2(t)x_2 - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3),
$$

$$U_1 = u_1 + u_3, \quad U_2 = u_2 + u_4.$$

Our goal is to find the control functions which will enable the systems (7), (19) and (20) realize reduced order function projective combination synchronization by active backstepping technique. The design procedure includes three steps as shown below:

Step 1. Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + U_1 + B_1, \tag{24}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the following Lyapunov function $v_1 = \frac{1}{2}q_1^2$ and its time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + U_1 + B_1). \tag{25}$$

Suppose $\alpha_1(q_1) = 0$ and the control function U_1 is chosen as

$$U_1 = -(B_1 + kq_1), \tag{26}$$

then $\dot{v}_1 = -kq_1^2 < 0$, where k is positive constant which represents the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the following (q_1, q_2)-subsystems

$$\dot{q}_1 = q_2 - kq_1,$$
$$\dot{q}_2 = -\alpha q_2 + U_2 + B_2. \tag{27}$$

Step 2. In order to stabilize system (27), a Lyapunov function can be chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. The time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + U_2 + B_2). \tag{28}$$

If the control function u_2 is chosen as

$$U_2 = -B_2 - kq_2 + \alpha q_2 - q_1, \tag{29}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$, where k is positive constant which represents the feedback gain. Then, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (27) is asymptotically stable. This implies that the drive system (1) and the response systems (19) and (20) achieve reduced order function projective combination synchronization. Finally, we have the following subsystems

$$\dot{q}_1 = q_2 - kq_1,$$
$$\dot{q}_2 = -q_1 - kq_2. \tag{30}$$

Here we limit our results to only two major Corollaries.

Let $\alpha_1 = \alpha_2 = \alpha_3$, $u_1 = u_3$ and $u_2 = u_4$, then we have Case 6.

Case 6.: If the controllers are chosen as

$$u_1 = u_3 = \frac{1}{2}(\frac{\alpha_1(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)(x_1 + x_3) - kq_1),$$

$$u_2 = u_4 = \frac{1}{2}(\alpha - k)q_2 - q_1 + (\alpha\alpha_1(t) + \dot{\alpha}_1(t))x_2 + \sin z_1$$

$$+ \sin y_1 - 2a - 2b\sin\omega t - \frac{\alpha_1(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3),$$

where $e_1 = z_1 - \alpha_1(x_1 + x_3)$, $e_2 = z_2 - \alpha_1 x_2$, then the drive system (1) achieves reduced order function projective combination synchronization with the response systems (19) and (20).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ with $u_1 = u_3$ and $u_2 = u_4$, then we have Case 7.

Case 7: If the controllers are chosen as

$$u_1 = u_3 = \frac{1}{2}((\alpha_1(t) - \alpha_2(t))x_2 + (\frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3 - kq_1),$$

$$u_2 = u_4 = \frac{1}{2}(\alpha - k)q_2 - q_1 + (\alpha\alpha_2(t) + \dot{\alpha}_2(t)x_2)x_2 + \sin z_1 + \sin y_1$$

$$- 2a - 2b\sin\omega t - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3),$$

where $e_1 = z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t))$, $e_2 = z_2 + y_2 - \alpha_2(t)x_2$, then reduced order modified function projective combination synchronization is achieved between the drive system (1) and the response systems (19) and (20).

3.2   Numerical simulation results

The designed controllers are verified in our numerical simulation using the in-built fourth order Runge-Kutta (ode45) routine in Matlab. In the numerical simulation procedure we used the systems parameters within the chaotic region and controllers are chosen in accordance with Case 6. The initial conditions of the drive systems and response system are given as $(x_1, x_2, x_3) = (0, 0, 0)$, $(y_1, y_2, y_3) = (111)$, $(z_1, z_2) = (0, 1)$, $\gamma(t) = 2.0 + 0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Figure 5 shows the dynamics of the error variables when the controllers were deactivated. Figure 6 shows that reduced order function projective combination synchronization among systems (1), (19) and (20) is achieved as indicated by the convergence of the error state variables to zero as soon as the controllers are switch on for $t \geq 80$. Figure 7 shows that the state variables of the drive and response systems follow the same trajectory when the controllers are activated for $t \geq 80$, this also confirms reduced order function projective combination synchronization among systems (1), (19) and (20). Figure 8 presents evidence of reduced order function projective synchronization among the systems.

4   Conclusion

Reduced order function projective combination synchronization of three chaotic systems consisting of: (i) two third order chaotic Josephson junctions as drives and one second order chaotic Josephson junction as response system; (ii) one third order chaotic Josephson junction as the drive and two second order chaotic Josephson junctions as the slaves via active backstepping technique has been achieved. We showed from the theoretical analysis that various controllers which is suitable for different type of synchronization scheme can be obtained from the general results. Furthermore, reduced order function projective combination synchronization has more potential application to secure communication systems and biological systems.
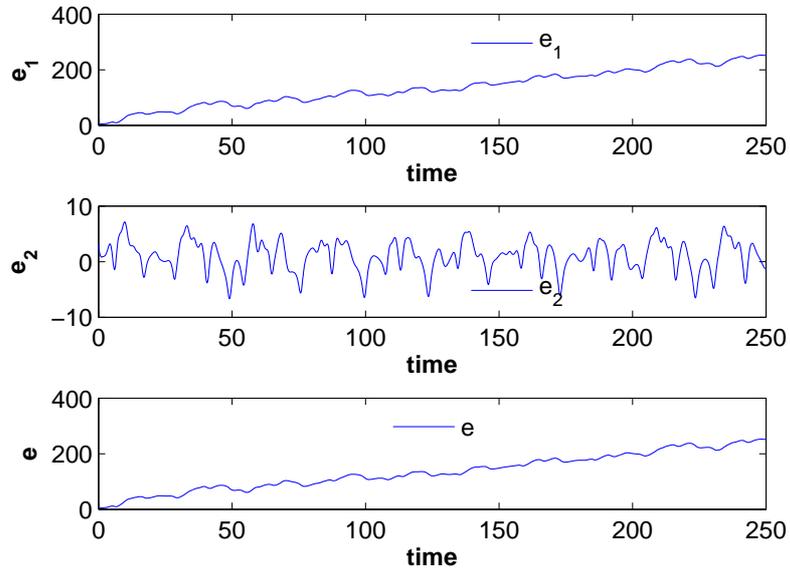
Figure 5: Error dynamics without control activated.
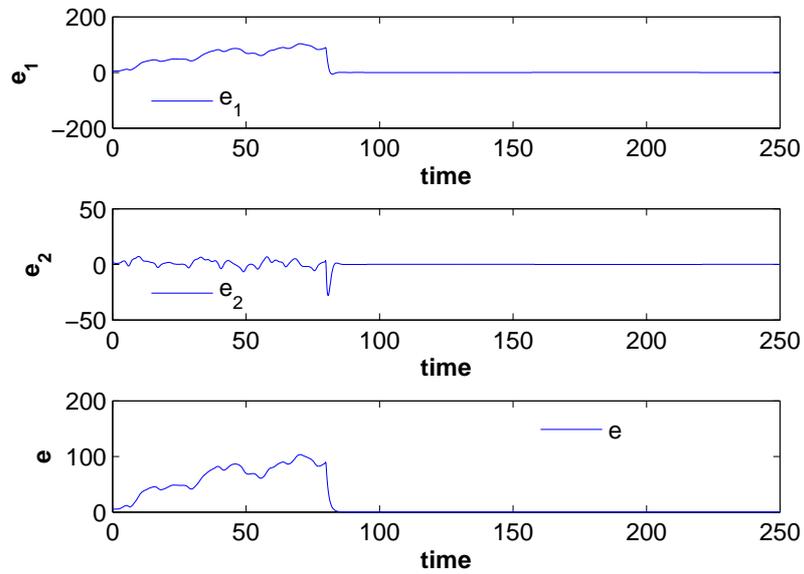
Figure 6: Error dynamics with control activated.
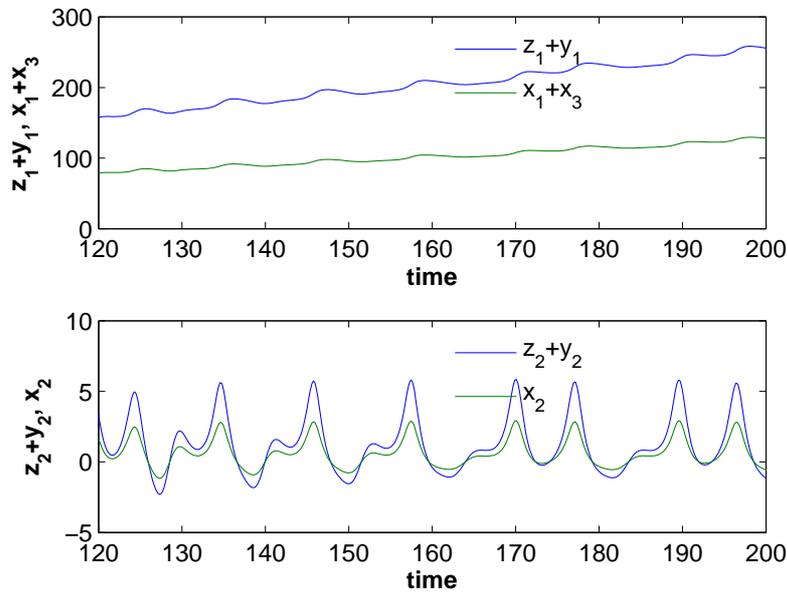
Figure 7: Dynamics of state variables with control activated.
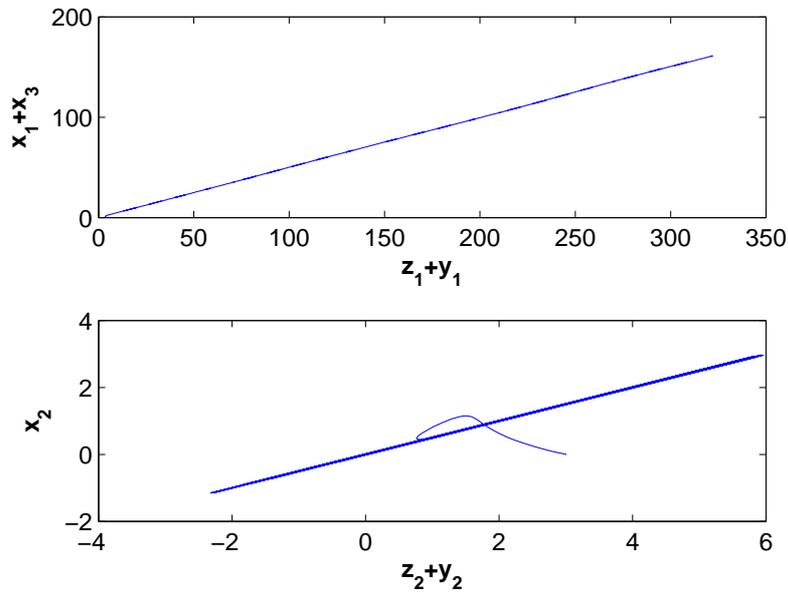

Figure 8: Evidence of projective synchronization.

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Stability of Stochastic Interval System with Distributed Delays

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Abstract: In this paper, we study the stability problem of a stochastic interval system with distributed delays. Firstly, we prove that the solution of such system exists and is unique, and then a sufficient criterion of exponential stability is obtained and such result can be generalized to the systems with multiple time delays. Finally, an example is given to illustrate the result.

Keywords: *exponential stability; stochastic interval system.*

Mathematics Subject Classification (2010): 65C30, 60H10.

1 Introduction

Stochastic modelling has come to play an important role in many branches of science and industry, such as neural network and automatic control of stochastic system and so on, see [1–9]. One of the most useful stochastic models which are often used in practice is stochastic differential delay equation [10–15]. However, in many practical models, it is difficult to determine the parameters with a fixed value and instead of obtaining some estimation – the parameters are changed in an interval. Such a system can be described by stochastic interval system.

In the past decades, a lot of work on stochastic differential interval systems could be found in [16–18] and the results are generalized to Markov switched system by [19, 20]. Motivated by these works, in this paper, we study stochastic interval system with distributed delays. Consider the following stochastic system

$$
\begin{aligned}
dx(t) = {} & [A_0 x(t) + A_1 x(t-\tau) + A_2 \textstyle\int_{-\tau}^{0} x(t+\theta) d\mu(\vartheta)] dt \\
& + [B_0 x(t) + B_1 x(t-\tau) + B_2 \textstyle\int_{-\tau}^{0} x(t+\theta) d\nu(\vartheta)] dB_t,
\end{aligned}
\tag{1}
$$

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where $A_0, A_1, A_2, B_0, B_1, B_2$ are constant matrices and μ, ν denote probability measures, τ is a positive constant. This system is incorporated with time delays, which would be appropriate in circumstances where a process is dependent not only upon the present state but also upon the state at all the times of some interval in the past. In practice, the matrix coefficients must be estimated from empirical data, and chosen as confidence intervals under a statistical method, so we use interval matrices instead of the coefficients of system (1), and this results in an interval system of the form

$$
\begin{aligned}
dx(t) &= [(A_0 + \Delta A_0)x(t) + (A_1 + \Delta A_1)x(t - \tau) \\
&+ (A_2 + \Delta A_2)\int_{-\tau}^{0} x(t + \theta)d\mu(\vartheta)]dt + [(B_0 + \Delta B_0)x(t) \\
&+ (B_1 + \Delta B_1)x(t - \tau) + (B_2 + \Delta B_2)\int_{-\tau}^{0} x(t + \theta)d\nu(\vartheta)]dB_t,
\end{aligned}
\tag{2}
$$

where $\Delta A_i, \Delta B_i, A_i, B_i$ are constant matrices and $\Delta A_i \in [-A_{im}, A_{im}]$, $\Delta B_i \in [-B_{im}, B_{im}]$, $i = 0, 1, 2$, where A_{im}, B_{im}, $im = 0, 1, 2$ are constant matrices and μ, ν denote probability measures, τ is a positive constant.

The study of stochastic interval system (2) becomes more difficult than that of stochastic system (1), since the parameters of (2) belong to intervals. Since the coefficients of such system have the property of uncertainty, we always treat interval systems as uncertain systems. Because the coefficients as interval matrices are not well-performing to preserve the stability properties, so it is useful and helpful to study the stability behaviour of such systems. There are many stability properties to be studied, but this paper will focus on the study of exponential stability of stochastic interval systems (2) with distributed time delay.

In the next section, we will give some notations used throughout this paper. In Section 3, we discuss a particular type of stochastic interval system with distributed delays, a stability criterion is given which will be applied to examine the stability of stochastic interval system with distributed delays, and then we generalize these results to a stochastic interval system with multiple distributed delays. An example is given to illustrate our result.

2 Preliminaries

Let R^n be Euclidean space and $| \bullet |$ be the Euclidean norm in R^n. If A is a matrix, its transpose is denoted by A^T and define a norm of A as $\| A \| = sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{max}(AA^T)}$. If A is a symmetric matrix, let $\lambda_{max}(A)$ and $\lambda_{min}(A)$ represent its largest and smallest eigenvalue respectively. Obviously, if A is a symmetric matrix, then $\lambda_{max}(A) \leq \|A\|$.

If $A^m = [a_{ij}^m]_{n \times n}$ and $A^M = [a_{ij}^M]_{n \times n}$ are matrices and $a_{ij}^m \leq a_{ij}^M$, $\forall 1 \leq i, j \leq n$, the interval matrix $[A^m, A^M]$ is defined by

$$
[A^m, A^M] = \{A = [a_{ij}]_{n \times n} : a_{ij}^m \leq a_{ij} \leq a_{ij}^M, \ \forall 1 \leq i, j \leq n\}.
$$

For $A, A_m \in R^{n \times n}$, where A_m is a nonnegative matrix, we note that any interval matrix $[A^m, A^M]$ has a unique representation of the form $[A - A_m, A + A_m]$, where $A = \frac{1}{2}(A^m + A^M)$, and $A_m = \frac{1}{2}(A^M - A^m)$.

In this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t \geq 0}$ satisfying the usual conditions. Let B_t denote a Brownian motion defined on the probability space. Let τ be a positive number and $C([-\tau, 0]; R^n)$ be the family of all continuous R^n-valued functions on $[-\tau, 0]$ with the values in R^n. We define a norm

as $\|y\|_\tau = \sup_{-\tau \le s \le 0} |y(t)|$ for any $y \in C([-\tau, 0]; R^n)$. Let $L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$ represent all $\mathcal{F}_{t_0}$-measurable $C([-\tau, 0]; R^n)$-valued random variables ξ with $E\|\xi\|_\tau^2 < \infty$ and we write L^2 for short unless otherwise specified. If $x(t), t \ge t_0 - \tau$ is an n-dimensional continuous stochastic process, we denote $x_t = x(t + s) : -\tau \le s \le 0$ as a $C([-\tau, 0]; R^n)$-valued process on $t \ge 0$. For any initial data $\hat{x}(t_0) = \xi \in L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$, there exists a unique global solution of (1) which is denoted by $x(t, t_0, \xi)$.

Definition 2.1 The system (1) is said to be

(a) exponentially stable in $L^2(\Omega, C([-\tau, 0]; R^n))$, if there exist positive constants M and γ such that for all $t_0 \ge 0$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$,

$$E\|\hat{x}(t, t_0, \xi)\|_\tau^2 \le M e^{-\gamma(t-t_0)} E\|\xi\|_\tau^2.$$

(b) almost surely exponentially stable if

$$\lim_{t \to \infty} \sup \frac{1}{t} \log |x(t, t_0, \xi)| < 0. \qquad a.s.$$

3   Main Results

In this section, we will study the stability properties of system (2). For system (2), we can't make sure that the solution of such system exists and is unique under the condition of linear growth and Lipschitz condition. In this paper, we affirm that the solution of system (2) exists and is unique; here we only need to prove the results for system (1), because the norm of the matrix coefficient of system(2) is bounded as in system (1), which is an essential step in the proof of the following theorems.

Let

$$\Im = \{\phi | \phi(\theta) : -\tau \le \theta \le 0 \text{ is random variable}; \phi \in \mathcal{F}_{t_0} \cap C([-\tau, 0]; R^n), \ E\|\phi\|^2 < \infty\}.$$

Lemma 3.1 *If $x(t)$ is a solution of equation (1), then for any $T > t_0$, $\exists C > 0$, such that*

$$E(\sup_{t_0 - \tau \le t \le T} |x(t)|^2) < C.$$

In particular, $x(t)$ belongs to $L^2([t_0 - \tau, T]; R^n))$.

Theorem 3.1 *For any $\xi \in \Im$, there exists a unique solution $x(t)$ of system (1), (2) and $x_{t_0} = \xi$.*

Proof. This can be easily proved by using the method of [21].

In order to study the stability of system (2), firstly, we consider system (1).

Theorem 3.2 *Assume that there exists a symmetric positive-definite matrix Q such that*

$$2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}} A_1^T Q A_1 Q^{-\frac{1}{2}})} + 2\sqrt{\varpi_A(\lambda) \lambda_{max}(Q^{-\frac{1}{2}} A_2^T Q A_2 Q^{-\frac{1}{2}})}$$

$$+ (\sqrt{\lambda_{max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}})} + \sqrt{\lambda_{max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}})} \qquad (3)$$

$$+ \sqrt{\varpi_B(\lambda) \lambda_{max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}})})^2 < -\lambda_{max}(Q^{-\frac{1}{2}}(Q A_0 + A_0^T Q) Q^{-\frac{1}{2}}) \triangleq \lambda,$$

where $\varpi_A(\lambda) = \int_{-\tau}^0 e^{-\lambda\theta} d\mu(\theta)$, $\quad \varpi_B(\lambda) = \int_{-\tau}^0 e^{-\lambda\theta} d\nu(\theta)$.

Then system (1) is exponentially stable in $L^2(\Omega, C([-\tau, 0]; R^n))$ and moreover, it is almost surely exponentially stable.

Proof. Firstly, we note that $Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}$ must be negative definite.
Set

$$\lambda = -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}) > 0. \tag{4}$$

By the condition of Theorem 3.2, we can find a constant $\gamma \in (0, \lambda)$ such that

$$
\begin{aligned}
&(1 + e^{\gamma\tau})\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}A_1^T QA_1 Q^{-\frac{1}{2}})} \\
&+(1 + e^{\gamma\tau})\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}B_0^T QB_0 Q^{-\frac{1}{2}})\lambda_{max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}})} \\
&+2\sqrt{\varpi_A(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}A_2^T QA_2 Q^{-\frac{1}{2}})} \\
&+2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}B_0^T QB_0 Q^{-\frac{1}{2}})\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}})} \\
&+(1 + e^{\gamma\tau})\sqrt{\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}})\lambda_{max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}})} \\
&+\lambda_{max}(Q^{-\frac{1}{2}}B_0^T QB_0 Q^{-\frac{1}{2}}) + \varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}}) \\
&+e^{\gamma\tau}\lambda_{max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}}) < \lambda - \gamma.
\end{aligned}
\tag{5}
$$

We claim that there exists a constant $C > 0$ such that

$$\int_{t_0}^\infty e^{\gamma t} E(x(t)^T Qx(t)) dt \le Ce^{\gamma t_0} E\|\xi^T Q\xi\|, \tag{6}$$

for all $t_0 \ge 0$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$.

In addition, we also affirm that there exists another constant $C' > 0$ such that

$$E\|x_t^T Qx_t\| \le C'e^{-\gamma(t-t_0)} E\|\xi^T Q\xi\|, \tag{7}$$

which is held in $L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$. It follows from (6) that (2) is almost surely exponentially stable.

Next, we will give proofs of (6) and (7).

Fix $t_0 \ge 0$ and ξ, write $x(t) = x(t, t_0, \xi)$, then Ito's formula yields that

$$
\begin{aligned}
e^{\lambda t}E(x(t)^T Q x(t)) =\ & e^{\lambda t_0}E(x(t_0)^T Q x(t_0)) + \lambda \int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds \\
&+2\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q A_0 x(s))ds + 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q A_1 x(s-\tau))ds \\
&+2\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q A_2 \int_{-\tau}^{0} x(s+\vartheta)d\mu(\vartheta))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T Q B_0 x(s))ds + \int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T Q B_1 x(s-\tau))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^T B_1^T Q B_0 x(s))ds + \int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^T B_1^T Q B_1 x(s-\tau))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^T B_1^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta) B_2^T Q B_0 x(s))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta) B_2^T Q B_1 x(s-\tau))ds \\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta) B_2^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds.
\end{aligned}
\tag{8}
$$

We note that by (4)

$$
2x(t)^T Q A_0 x(t) \le -\lambda x(t)^T x(t).
$$

For any $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 > 0$,

$$
\begin{aligned}
& 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q A_1 x(s-\tau))ds \\
\le\ & \varepsilon_1 \int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds + \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_1} E(x(s-\tau)^T A_1^T Q A_1 x(s-\tau))ds, \\
& 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q A_2 \int_{-\tau}^{0} x(s+\vartheta)d\mu(\vartheta))ds \\
=\ & 2\int_{t_0}^{t} e^{\lambda s}\int_{-\tau}^{0} E(x(s)^T Q A_2 x(s+\vartheta))d\mu(\vartheta))ds \\
\le\ & \int_{t_0}^{t} e^{\lambda s}\int_{-\tau}^{0} (\varepsilon_3 E(x(s)^T Q x(s)) + \frac{E(x(s+\theta)^T A_2^T Q A_2 x(s+\theta))}{\varepsilon_3})d\mu(\theta)ds \\
=\ & \varepsilon_3 \int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds + \int_{t_0}^{t}\int_{-\tau}^{0} \frac{e^{\lambda s}}{\varepsilon_3} E(x(s+\theta)^T A_2^T Q A_2 x(s+\theta))d\mu(\theta)ds \\
\le\ & \varepsilon_3 \int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds + \int_{t_0-\tau}^{t} \frac{e^{\lambda s}}{\varepsilon_3} E(x(s)^T A_2^T Q A_2 x(s))ds \int_{-\tau}^{0} e^{-\lambda\theta}d\mu(\theta) \\
=\ & \varepsilon_3 \int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds + \varpi_A(\lambda) \int_{t_0-\tau}^{t_0} \frac{e^{\lambda s}}{\varepsilon_3} E(\xi^T A_2^T Q A_2 \xi)ds \\
& + \varpi_A(\lambda) \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_3} E(x(s)^T A_2^T Q A_2 x(s))ds,
\end{aligned}
\tag{9}
$$

where $\varpi_A(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d\mu(\theta)$.

We can proceed in a similar fashion for other forms in (5).

$$
\begin{aligned}
e^{\lambda t}E(x(t)^T Q x(t)) \leq{}& e^{\lambda t_0}E\left\|\xi^T Q\xi\right\| + \varepsilon_1\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds \\
&+ \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_1}E(x(s-\tau)^T A_1^T Q A_1 x(s-\tau))ds + \varepsilon_3\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds \\
&+ \varpi_A(\lambda)[\int_{t_0-\tau}^{t_0} \frac{e^{\lambda s}}{\varepsilon_3}E(\xi^T A_2^T Q A_2\xi)ds + \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_3}E(x(s)^T A_2^T Q A_2 x(s))ds] \\
&+ (1+\varepsilon_2)\int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T Q B_0 x(s))ds \\
&+ \int_{t_0}^{t} \frac{(1+\varepsilon_2)e^{\lambda s}}{\varepsilon_2}E(x(s-\tau)^T B_1^T Q B_1 x(s-\tau))ds \\
&+ \varepsilon_4\int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T Q B_0 x(s))ds \\
&+ \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_4}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+ \varepsilon_5\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^T B_1^T Q B_1 x(s-\tau))ds \\
&+ \int_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_5}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+ \int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds.
\end{aligned}
\tag{10}
$$

Here we mention that

$$
\begin{aligned}
&\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T Q B_2 \int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&= \int_{t_0}^{t} e^{\lambda s} \int_{-\tau}^{0}\int_{-\tau}^{0} E(x(s+\phi)^T B_2^T Q B_2 x(s+\vartheta))d\nu(\phi)d\nu(\vartheta)ds \\
&\leq \int_{t_0}^{t} \frac{e^{\lambda s}}{2} \int_{-\tau}^{0}\int_{-\tau}^{0} [E(x(s+\phi)^T B_2^T Q B_2 x(s+\phi)) \\
&\quad + E(x(s+\vartheta)^T B_2^T Q B_2 x(s+\vartheta))]d\nu(\phi)d\nu(\vartheta)ds \\
&\leq \int_{t_0}^{t} e^{\lambda s} \int_{-\tau}^{0} E(x(s+\vartheta)^T B_2^T Q B_2 x(s+\vartheta))d\nu(\vartheta)ds \\
&\leq \varpi_B(\lambda)[\int_{t_0-\tau}^{t_0} e^{\lambda s}E(\xi^T B_2^T Q B_2\xi)ds + \int_{t_0}^{t} e^{\lambda s}E(x(s)^T B_2^T Q B_2 x(s))ds],
\end{aligned}
\tag{11}
$$

where $\varpi_B(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d\nu(\theta)$. Hence,

$$
\begin{aligned}
&e^{\lambda t}E(x(t)^T Q x(t)) \\
&\leq e^{\lambda t_0}E\left\|\xi^T Q\xi\right\| + \int_{t_0-\tau}^{t_0} \frac{\varepsilon_4+\varepsilon_5+\varepsilon_4\varepsilon_5}{\varepsilon_4\varepsilon_5}\varpi_B(\lambda)e^{\lambda s}E(\xi^T B_2^T Q B_2\xi)ds \\
&\quad + \int_{t_0-\tau}^{t_0} \frac{\varpi_A(\lambda)}{\varepsilon_3}e^{\lambda s}E(\xi^T A_2^T Q A_2\xi)ds + (\varepsilon_1+\varepsilon_3)\int_{t_0}^{t} e^{\lambda s}E(x(s)^T Q x(s))ds
\end{aligned}
\tag{12}
$$

$$+ \int\limits_{t_0}^{t} \frac{e^{\lambda s}}{\varepsilon_1} E(x(s-\tau)^T A_1^T Q A_1 x(s-\tau)) ds$$

$$+ \int\limits_{t_0}^{t} \frac{\varpi_A(\lambda)}{\varepsilon_3} e^{\lambda s} E(x(s)^T A_2^T Q A_2 x(s)) ds$$

$$+ (1 + \varepsilon_2 + \varepsilon_4) \int\limits_{t_0}^{t} e^{\lambda s} E(x(s)^T B_0^T Q B_0 x(s)) ds$$

$$+ \int\limits_{t_0}^{t} \frac{1+\varepsilon_2+\varepsilon_2\varepsilon_5}{\varepsilon_2} e^{\lambda s} E(x(s-\tau)^T B_1^T Q B_1 x(s-\tau)) ds$$

$$+ \varpi_B(\lambda) \int\limits_{t_0}^{t} \frac{\varepsilon_4+\varepsilon_5+\varepsilon_4\varepsilon_5}{\varepsilon_4\varepsilon_5} e^{\lambda s} E(x(s)^T B_2^T Q B_2 x(s)) ds$$

$$\leq C_1 e^{\lambda t_0} E \left\| \xi^T Q \xi \right\| + C_2 \int\limits_{t_0}^{t} e^{\lambda s} E(x(s)^T Q x(s)) ds$$

$$+ C_3 \int\limits_{t_0}^{t} e^{\lambda s} E(x(s-\tau)^T Q x(s-\tau)) ds,$$

where

$$
\begin{aligned}
C_1 &= 1 + (\tfrac{1}{\varepsilon_4} + \tfrac{1}{\varepsilon_5} + 1)\varpi_B(\lambda)\tau\lambda_{\max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}}) \\
&\quad + \tfrac{1}{\varepsilon_3}\varpi_A(\lambda)\tau\lambda_{\max}(Q^{-\frac{1}{2}} A_2^T Q A_2 Q^{-\frac{1}{2}}), \\
C_2 &= \varepsilon_1 + \varepsilon_3 + \tfrac{1}{\varepsilon_3}\varpi_A(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}} A_2^T Q A_2 Q^{-\frac{1}{2}}) \\
&\quad + (1 + \varepsilon_2 + \varepsilon_4)\lambda_{\max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}}) \\
&\quad + (\tfrac{1}{\varepsilon_4} + \tfrac{1}{\varepsilon_5} + 1)\varpi_B(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}}), \\
C_3 &= \tfrac{1}{\varepsilon_1}\lambda_{\max}(Q^{-\frac{1}{2}} A_1^T Q A_1 Q^{-\frac{1}{2}}) + (1 + \tfrac{1}{\varepsilon_2} + \varepsilon_5)\lambda_{\max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}}).
\end{aligned}
$$

Then

$$
\begin{aligned}
E(x(t)^T Q x(t)) &\leq C_1 e^{-\lambda(t-t_0)} E \left\| \xi^T Q \xi \right\| + C_2 \int\limits_{t_0}^{t} e^{-\lambda(t-s)} E(x(s)^T Q x(s)) ds \\
&\quad + C_3 \int\limits_{t_0}^{t} e^{-\lambda(t-s)} E(x(s-\tau)^T Q x(s-\tau)) ds.
\end{aligned}
$$

For any $T > t_0$,

$$\int\limits_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t)) dt$$

$$\leq C_1 \int\limits_{t_0}^{T} e^{\gamma t} e^{-\lambda(t-t_0)} E \left\| \xi^T Q \xi \right\| dt + C_2 \int\limits_{t_0}^{T} e^{\gamma t} \int\limits_{t_0}^{t} e^{-\lambda(t-s)} E(x(s)^T Q x(s)) ds dt$$

$$+ C_3 \int\limits_{t_0}^{T} e^{\gamma t} \int\limits_{t_0}^{t} e^{-\lambda(t-s)} E(x(s-\tau)^T Q x(s-\tau)) ds dt$$

$$\leq \frac{C_1 + C_3 \tau e^{\gamma\tau}}{\lambda - \gamma} e^{\gamma t_0} E \left\| \xi^T Q \xi \right\| + \frac{C_2 + C_3 e^{\gamma\tau}}{\lambda - \gamma} \int\limits_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t)) dt.$$

If we let

$$\frac{1}{\lambda - \gamma}(C_2 + C_3 e^{\gamma\tau}) < 1, \tag{13}$$

which implies that there exists a constant $C > 0$ such that (6) holds.

Next, we will give a proof of (7).

Using Ito's formula, we have

$$
e^{\gamma(t-\tau)}E\left\|\hat{x}(t)^T Q\hat{x}(t)\right\| \le E(\sup_{t-\tau \le r \le t} e^{\gamma r}x(r)^T Qx(r))
$$

$$
\le C_1 e^{\gamma t_0}E\left\|\xi^T Q\xi\right\| + C_2 \int\limits_{t_0}^{t} e^{\gamma s}E(x(s)^T Qx(s))ds
$$

$$
+C_3 \int\limits_{t_0}^{t} e^{\gamma s}E(x(s-\tau)^T Qx(s-\tau))ds
$$

$$
+2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}x^T(s)QB_0 x(s)dB_s)
$$

$$
+2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}x^T(s)QB_1 x(s-\tau)dB_s)
$$

$$
+2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}[x^T(s)QB_2 \int\limits_{-\tau}^{0} x(s+\vartheta)d\mu(\vartheta)]dB_s).
$$

Here we used B-D-G inequality [21]. For any $\varepsilon_6, \varepsilon_7, \varepsilon_8 > 0$,

$$
2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}x^T(s)QB_0 x(s)dB_s)
$$

$$
\le 2\sqrt{32}E(\int\limits_{t-\tau}^{t} e^{2\gamma s}\left|x^T(s)Qx(s)\right|\left|x^T(s)B_0^T QB_0 x(s)\right|ds)^{1/2}
$$

$$
\le 2\sqrt{32}E(\|x_s^T Qx_s\|_\tau \int\limits_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}}B_0^T QB_0 Q^{-\frac{1}{2}})e^{2\gamma s}\|x_s^T Qx_s\|_\tau ds)^{\frac{1}{2}}
$$

$$
\le \varepsilon_6 e^{\gamma(t-\tau)}E\|x_s^T Qx_s\|_\tau
$$

$$
+\tfrac{32}{\varepsilon_6}e^{-\gamma(t-\tau)}\lambda_{\max}(Q^{-1/2}B_0^T QB_0 Q^{-1/2})\int\limits_{t-\tau}^{t} e^{2\gamma s}\|x_s^T Qx_s\|_\tau ds,
$$

$$
2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}x^T(s)QB_1 x(s-\tau)dB_s)
$$

$$
\le 2\sqrt{32}E(\int\limits_{t-\tau}^{t} e^{2\gamma s}|x^T(s)Qx(s)||x^T(s-\tau)B_1^T QB_1 x(s-\tau)|ds)^{\frac{1}{2}}
$$

$$
\le 2\sqrt{32}E(\|x_s^T Qx_s\|_\tau \int\limits_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}})e^{2\gamma s}\|x_s^T Qx_s\|_\tau ds)^{\frac{1}{2}}
$$

$$
\le \varepsilon_7 e^{\gamma(t-\tau)}E\|x^T{}_s Qx_s\|_\tau
$$

$$
+\tfrac{32}{\varepsilon_7}e^{-\gamma(t-\tau)}\lambda_{\max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}})\int\limits_{t-\tau}^{t} e^{2\gamma s}\|x_s{}^T Qx_s\|_\tau ds,
$$

$$
2E(\sup_{t-\tau \le r \le t} \int\limits_{t_0}^{r} e^{\gamma s}[x^T(s)QB_2 \int\limits_{-\tau}^{0} x(s+\theta)d\mu(\theta)]dB_s)
$$

$$
\le 2\sqrt{32}E(\int\limits_{t-\tau}^{t} e^{2\gamma s}\left|x^T(s)Qx(s)\right|\left|\int\limits_{-\tau}^{0} x^T(s+\theta)d\mu(\theta)B_2^T QB_2 \int\limits_{-\tau}^{0} x(s+\theta)d\mu(\theta)\right|ds)^{\frac{1}{2}}
$$

$$
\le 2\sqrt{32}E(\|x_s{}^T Qx_s\|_\tau \int\limits_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}})e^{2\gamma s}\|x^T{}_s Qx_s\|_\tau ds)^{\frac{1}{2}}
$$

$$
\le \varepsilon_8 e^{\gamma(t-\tau)}E\|x^T{}_s Qx_s\|_\tau
$$

$$
+\tfrac{32}{\varepsilon_8}e^{-\gamma(t-\tau)}\lambda_{\max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}})\int\limits_{t-\tau}^{t} e^{2\gamma s}\|x^T{}_s Qx_s\|_\tau ds.
$$

If $t \geq t_0 + \tau$,

$$(1 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8)e^{\gamma(t-\tau)}E\|x_t^T Q x_t\|_\tau$$
$$\leq C_1 e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau + (C_2 + C_3)\int_{t_0}^t e^{\gamma s}E\|x_s^T Q x_s\|_\tau ds$$
$$+ C_4 e^{-\gamma(t-\tau)}\int_{t_0}^t e^{2\gamma s}E\|x_s^T Q x_s\|_\tau ds,$$

where

$$C_4 = 32(\tfrac{1}{\varepsilon_6}\lambda_{\max}(Q^{-1/2}B_0^T Q B_0 Q^{-\frac{1}{2}}) + \tfrac{1}{\varepsilon_7}\lambda_{\max}(Q^{-\frac{1}{2}}B_1^T Q B_1 Q^{-\frac{1}{2}})$$

$$+ \tfrac{1}{\varepsilon_8}\lambda_{\max}(Q^{-\frac{1}{2}}B_2^T Q B_2 Q^{-\frac{1}{2}})).$$

If $t_0 \leq t \leq t_0 + \tau$.

$$e^{\gamma(t-\tau)}E\|x_t^T Q x_t\|_\tau \leq e^{\gamma t_0}E(\|\xi^T Q\xi\|_\tau + \sup_{t_0 \leq r \leq t}|x^T(r)Q x(r)|)$$

$$\leq e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau + E\sup_{t_0 \leq r \leq t}(e^{\gamma r}|x^T(r)Q x(r)|)$$

$$\leq (1 + C_1)e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau + (C_2 + C_3)\int_{t_0}^t e^{\gamma s}E\|x_s^T Q x_s\|_\tau ds$$

$$+ C_4 e^{-\gamma(t-\tau)}\int_{t_0}^t e^{2\gamma s}E\|x_s^T Q x_s\|_\tau ds + C_4 e^{-\gamma(t_0-\tau)}\int_{t_0-\tau}^{t_0} e^{2\gamma s}E\|x_s^T Q x_s\|_\tau ds$$

$$+ (\varepsilon_6 + \varepsilon_7 + \varepsilon_8)e^{\gamma(t-\tau)}E\|x_t^T Q x_t\|_\tau.$$

Set $\varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 1/6$, we obtain

$$e^{\gamma(t-\tau)}E\|x_t^T Q x_t\|_\tau \leq M_1 + M_2\int_{t_0}^t e^{\gamma s}E\|x_s^T Q x_s\|ds \leq M,$$

where

$$M_1 = \begin{cases} 2(1 + C_1)e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau, & t_0 \leq t \leq t_0 + \tau, \\[2mm] 2(1 + C_1)e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau \\ \quad + 2C_4 e^{-\gamma(t_0-\tau)}\int_{t_0-\tau}^{t_0} e^{2\gamma s}E\|x_s^T Q x_s\|ds, & t \geq t_0 + \tau, \end{cases}$$

$$M_2 = 2(C_2 + C_3 + C_4 e^{\gamma\tau}),$$

$$M = M_1 + M_2 C e^{\gamma t_0}E\|\xi^T Q\xi\|_\tau.$$

And right now we complete the proof of this theorem.

Apply Theorem 3.2. Now we are capable to cope with such system with interval matrix coefficient, just for all matrices belong to the interval the sufficient condition of Theorem (3.2) must be satisfied and here we prove the following result.

Theorem 3.3 *If there exists a symmetric positive-definite matrix Q such that*

$$2[\lambda_{max}(Q^{-\frac{1}{2}}A_1^T Q A_1 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|A_1\|\|A_{1m}\| + \|A_{1m}\|^2)]^{\frac{1}{2}}$$

$$+2[(\lambda_{max}(Q^{-\frac{1}{2}}A_2^T Q A_2 Q^{-\frac{1}{2}})) + \tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|A_2\|\|A_{2m}\| + \|A_{2m}\|^2)\varpi_A(\lambda)]^{\frac{1}{2}}$$

$$+\{[\lambda_{max}(Q^{-\frac{1}{2}}B_0^T Q B_0 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_0\|\|B_{0m}\| + \|B_{0m}\|^2)]^{\frac{1}{2}}$$

$$+[\lambda_{max}(Q^{-\frac{1}{2}}B_1^T Q B_1 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_1\|\|B_{1m}\| + \|B_{1m}\|^2)]^{\frac{1}{2}} \qquad (14)$$

$$+[(\lambda_{max}(Q^{-\frac{1}{2}}B_2^T Q B_2 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_2\|\|B_{2m}\| + \|B_{2m}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2$$

$$\leq -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}) - \tfrac{2\|A_{0m}\|\|Q\|}{\lambda_{min}(Q)}.$$

Then equation (2) is exponentially stable in $L^2(\Omega, C([-\tau,0]; R^n))$ and moreover, it is almost surely exponentially stable.

Before proving the theorem, we first give some lemmas [18].

Lemma 3.2 *Let Q be a positive-definite symmetric matrix. Then*

$$\|Q^{-\frac{1}{2}}\|\|Q^{\frac{1}{2}}\| \leq \frac{\|Q\|}{\lambda_{min}(Q)}.$$

Lemma 3.3 *Let Q be a positive-definitive matrix and A be an $n \times n$ matrix. Then*

$$\lambda_{max}(Q^{-\frac{1}{2}}(QA + A^T Q)Q^{-\frac{1}{2}}) \leq \frac{2\|A\|\|Q\|}{\lambda_{min}(Q)}.$$

Lemma 3.4 *If $\Delta A \in [-A_m, A_m]$, then $\|\Delta A\| \leq \|A_m\|$.*

Lemma 3.5 *Let Q be a positive-definitive matrix, B be an $n \times n$ matrix and $\Delta B \in [-B_m, B_m]$. Then*

$$\lambda_{\max}(Q^{-1/2}(B^T Q \Delta B + (\Delta B)^T QB + (\Delta B)^T Q(\Delta B))Q^{-1/2})$$

$$\leq \tfrac{2\|B\|\|Q\|\|B_m\|}{\lambda_{\min}(Q)} + \tfrac{\|Q\|\|B_m\|^2}{\lambda_{\min}(Q)}.$$

Proof of Theorem 3.3. In order to guarantee the exponential stability of the interval system (2), we should show that the condition (3) of Theorem 3.2 holds for all the matrix coefficients $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta A_2 \in [-A_{2m}, A_{2m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$, $\Delta B_2 \in [-B_{2m}, B_{2m}]$, i.e.

$$2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(A_1 + \Delta A_1)^T Q(A_1 + \Delta A_1)Q^{-\frac{1}{2}})}$$

$$+2\sqrt{\varpi_A(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}(A_2 + \Delta A_2)^T Q(A_2 + \Delta A_2 Q^{-\frac{1}{2}})}$$

$$+(\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(B_0 + \Delta B_0)^T Q(B_0 + \Delta B_0)Q^{-\frac{1}{2}})}$$

$$+\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(B_1 + \Delta B_1)^T Q(B_1 + \Delta B_1)Q^{-\frac{1}{2}})}$$

$$+\sqrt{\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}(B_2 + \Delta B_2)^T Q(B_2 + \Delta B_2)Q^{-\frac{1}{2}}))^2}$$

$$< -\lambda_{max}(Q^{-\frac{1}{2}}(Q(A_0 + \Delta A_0) + (A + \Delta A_0)^T Q)Q^{-\frac{1}{2}}),$$

where $\varpi_A(\lambda) = \int_{-\tau}^0 e^{-\lambda\theta}d\mu(\theta)$, $\varpi_B(\lambda) = \int_{-\tau}^0 e^{-\lambda\theta}d\nu(\theta)$.

According to Lemma 3.3 and Lemma 3.4, we note that

$$-\lambda_{max}(Q^{-\frac{1}{2}}(Q(A_0 + \Delta A_0) + (A + \Delta A_0)^T Q)Q^{-\frac{1}{2}})$$

$$\geq -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}) - \frac{2\|A_{0m}\|\|Q\|}{\lambda_{min}(Q)} \triangleq I_1.$$

Using Lemma 3.4 and Lemma 3.5, we have

$$\lambda_{max}(Q^{-\frac{1}{2}}(A_1 + \Delta A_1)^T Q(A_1 + \Delta A_1)Q^{-\frac{1}{2}})$$
$$< \lambda_{max}(Q^{-\frac{1}{2}}A_1^T QA_1 Q^{-\frac{1}{2}}) + \lambda_{max}(Q^{-\frac{1}{2}}(A_1^T Q\Delta A_1 + (\Delta A_1)^T QA_1$$
$$+(\Delta A_1)^T Q\Delta A_1)Q^{-\frac{1}{2}})$$
$$\leq \lambda_{max}(Q^{-\frac{1}{2}}A_1^T QA_1 Q^{-\frac{1}{2}}) + \frac{\|Q\|}{\lambda_{min}(Q)}(2\|A_1\|\|A_{1m}\| + \|A_{1m}\|^2).$$

Then
$$2\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(A_1 + \Delta A_1)^T Q(A_1 + \Delta A_1)Q^{-\frac{1}{2}})}$$
$$+2\sqrt{\varpi_A(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}}(A_2 + \Delta A_2)^T Q(A_2 + \Delta A_2)Q^{-\frac{1}{2}})}$$
$$+(\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(B_0 + \Delta B_0)^T Q(B_0 + \Delta B_0)Q^{-\frac{1}{2}})}$$
$$+\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(B_1 + \Delta B_1)^T Q(B_1 + \Delta B_1)Q^{-\frac{1}{2}})}$$
$$+\sqrt{\varpi_B(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}}(B_2 + \Delta B_2)^T Q(B_2 + \Delta B_2)Q^{-\frac{1}{2}}))^2}$$

$$\leq 2[\lambda_{\max}(Q^{-\frac{1}{2}}A_1^T QA_1 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_1\|\|A_{1m}\| + \|A_{1m}\|^2)]^{\frac{1}{2}}$$

$$+2[(\lambda_{\max}(Q^{-\frac{1}{2}}A_2^T QA_2 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_2\|\|A_{2m}\| + \|A_{2m}\|^2))\varpi_A(\lambda)]^{\frac{1}{2}}$$

$$+\{[\lambda_{\max}(Q^{-\frac{1}{2}}B_0^T QB_0 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\|B_{0m}\| + \|B_{0m}\|^2)]^{\frac{1}{2}}$$

$$+[\lambda_{\max}(Q^{-\frac{1}{2}}B_1^T QB_1 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_1\|\|B_{1m}\| + \|B_{1m}\|^2)]^{\frac{1}{2}}$$

$$+[(\lambda_{\max}(Q^{-\frac{1}{2}}B_2^T QB_2 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_2\|\|B_{2m}\| + \|B_{2m}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2 \triangleq I_2.$$

If $I_1 \geq I_2$, we can conclude that the matrix coefficients of interval type satisfy the condition of Theorem 3.2, which leads to the exponential stability of the stochastic interval system.

It is not hard to generalize the result of the theorems to the multiple time delays case. Without any details of proof, we directly present the conclusion as follows.

Theorem 3.4 *Consider the following system*

$$
\begin{aligned}
dx(t) &= [(A_0 + \Delta A_0)x(t) + \sum_{i=1}^{N}(A_{1i} + \Delta A_{1i})x(t - \tau_i) \\
&+ (A_{2i} + \Delta A_{2i}) \int_{-\tau_i}^{0} x(t + \theta)d\mu(\theta)]dt + [(B_0 + \Delta B_0)x(t) \\
&+ \sum_{i=1}^{N}(B_{1i} + \Delta B_{1i})x(t - \tau_i) + (B_{2i} + \Delta B_{2i}) \int_{-\tau_i}^{0} x(t + \theta)d\nu(\theta)]dB_t.
\end{aligned}
\tag{15}
$$

If there exists a symmetric positive-definite matrix Q such that

$$
2\sum_{i=1}^{N}\{[\lambda_{\max}(Q^{-\frac{1}{2}}A_{1i}^T Q A_{1i} Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_{1i}\|\|A_{1im}\| + \|A_{1im}\|^2)]^{\frac{1}{2}}
$$

$$
+ [(\lambda_{\max}(Q^{-\frac{1}{2}}A_{2i}^T Q A_{2i} Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_{2i}\|\|A_{2im}\| + \|A_{2im}\|^2))\varpi_A(\lambda)]^{\frac{1}{2}}\}
$$

$$
+ \{[\lambda_{\max}(Q^{-\frac{1}{2}}B_0^T Q B_0 Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\|B_{0m}\| + \|B_{0m}\|^2)]^{\frac{1}{2}}
$$

$$
+ \sum_{i=1}^{N}[\lambda_{\max}(Q^{-\frac{1}{2}}B_{1i}^T Q B_{1i} Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_{1i}\|\|B_{1im}\| + \|B_{1im}\|^2)]^{\frac{1}{2}}
$$

$$
+ [(\lambda_{\max}(Q^{-\frac{1}{2}}B_{2i}^T Q B_{2i} Q^{-\frac{1}{2}}) + \tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_{2i}\|\|B_{2im}\| + \|B_{2im}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2
$$

$$
\leq -\lambda_{\max}(Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}) - \tfrac{2\|A_{0m}\|\|Q\|}{\lambda_{\min}(Q)}.
$$

Then the system (15) is exponentially stable in $L^2(\Omega, C([-\tau, 0], R^n))$ and moreover, it is almost surely exponentially stable.

4   Example

In this section, we'll give a simple example to illustrate our result of Theorem 3.3.

Consider the following system

$$
\begin{aligned}
dx(t) &= [(A_0 + \Delta A_0)x(t) + (A_1 + \Delta A_1)x(t - \tau) \\
&+ (A_2 + \Delta A_2)\int_{-\tau}^{0} x(t + \theta)d\mu(\vartheta)]dt + [(B_0 + \Delta B_0)x(t) \\
&+ (B_1 + \Delta B_1)x(t - \tau) + (B_2 + \Delta B_2)\int_{-\tau}^{0} x(t + \theta)d\nu(\vartheta)]dB_t,
\end{aligned}
\tag{16}
$$

where $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta A_2 \in [-A_{2m}, A_{2m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$, $\Delta B_2 \in [-B_{2m}, B_{2m}]$. In order to simplify the com-

putation, we set $Q = I$, in consequence, condition (14) can be simplify as

$$2[\lambda_{max}(A_1^T A_1) + (2\|A_1\|\|A_{1m}\| + \|A_{1m}\|^2)]^{1/2}$$

$$+2[(\lambda_{max}(A_2^T A_2) + (2\|A_2\|\|A_{2m}\| + \|A_{2m}\|^2)\varpi_A(\lambda)]^{1/2}$$

$$+\{[\lambda_{max}(B_0^T B_0) + (2\|B_0\|\|B_{0m}\| + \|B_{0m}\|^2)]^{1/2}$$

$$+[\lambda_{max}(B_1^T B_1) + (2\|B_1\|\|B_{1m}\| + \|B_{1m}\|^2)]^{1/2} \tag{17}$$

$$+[(\lambda_{max}(B_2^T B_2) + (2\|B_2\|\|B_{2m}\| + \|B_{2m}\|^2))\varpi_B(\lambda)]^{1/2}\}^2$$

$$\leq -\lambda_{max}(A_0 + A_0^T) - 2\|A_{0m}\|,$$

where

$$A_0 = \begin{pmatrix} -50 & -3 \\ 3 & -22 \end{pmatrix}, A_{0m} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0.5 \\ 0.25 & 0.75 \end{pmatrix},$$

$$A_{1m} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.5 \end{pmatrix}, A_2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, A_{2m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_{0m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_1 = \begin{pmatrix} 0.25 & 0.75 \\ 0.25 & 0.25 \end{pmatrix},$$

$$B_{1m} = \begin{pmatrix} 0.25 & 0.25 \\ 0.5 & 0.25 \end{pmatrix}, B_2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_{2m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}.$$

It can easily be computed that $\lambda_{\max}(A_0 + A_0^T) = -44, \|A_{0m}\| = 1.281, \|A_1\| = 1.279, \|A_{1m}\| = 0.354, \|A_2\| = 0.966, \|A_{2m}\| = 0.655, \|B_0\| = 0.966, \|B_{0m}\| = 0.655, \|B_1\| = 1.189, \|B_{1m}\| = 0.655, \|B_2\| = 0.966, \|B_{2m}\| = 0.655$, and we set $\mu(\theta) = \nu(\theta) = \frac{\theta}{\tau}$, then $\varpi_A(\lambda) = \varpi_B(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d(\frac{\theta}{\tau}) = \frac{e^{\lambda\tau}-1}{\lambda\tau}$, here we note that $\lambda = -\lambda_{\max}(A_0 + A_0^T) = 44$, so we can choose a sufficient small delay τ such that $\varpi_A(\lambda) = \varpi_B(\lambda) = 1.1$. Hence condition (14) is satisfied. Therefore we can conclude that (16) is exponentially sable in $L^2(\Omega, C([-\tau, 0]; R^n))$ and moreover, it is almost surely exponentially stable.

5 Conclusion

In this paper, exponential stability of stochastic interval systems with time delays is studied. Using Itô formula and inequality techniques, some sufficient conditions are derived and at last, a simple example is given to illustrate our result.

Acknowledgment

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Existence of the Solution for Discontinuous Fuzzy Integro-differential Equations and Strong Fuzzy Henstock Integrals

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Abstract: In this paper, we use convergence theorem and the properties of strong fuzzy Henstock integrals to establish some existence theorems of solution for a kind of the discontinuous fuzzy integro-differential equations. The results are generalizations of earlier investigation for continuous fuzzy systems.

Keywords: *fuzzy number; existence of solution; discontinuous fuzzy integro-differential equations; strong fuzzy Henstock integrals.*

Mathematics Subject Classification (2010): 03E72, 28A15, 34A07; 45J05.

1 Introduction

Differential equations are used for modeling of various physical phenomena. Unfortunately, many problems are dynamical and too complicated and accurate differential equation model for such problems requires complex and time consuming algorithms hardly implementable in practice. Thus, a usage of fuzzy mathematics seems to be appropriate. In recent years, the fuzzy set theory introduced by Zadeh [5] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of science such as physical, mathematical, differential equations and information science.

The Cauchy problems for fuzzy differential equations have been studied by several authors [16,21–23,25,26] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding

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level sets. In [23], the author has proved the Cauchy problem has a uniqueness result if f was continuous and bounded. In [16, 22], the authors presented a result for uniqueness of solution when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent book by Lakshmikantham and Mohapatra [27] and references therein. In 2002, Xue and Fu [28] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous systems in which the right-hand side functions $f : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [16] on certain intervals and their solutions are not absolute continuous functions. So, in this paper, we will use the strong fuzzy Henstock integral, which is nonabsolute integrable.

It is well known that the Henstock integral is designed to integrate highly oscillatory functions which the Lebesgue integral fails to do. It is known as nonabsolute integration and is a powerful tool. It is well-known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue and Newton integrals [2, 3]. Though such an integral was defined by Denjoy in 1912 and also by Perron in 1914, it was difficult to handle using their definitions. But with the Riemann-type definition introduced more recently by Henstock [2] in 1963 and also independently by Kurzweil [3], the definition is now simple and furthermore the proof involving the integral also turns out to be easy. For more detailed results about the Henstock integral, we refer to [4]. Recently, Wu and Gong [14, 15] have combined the fuzzy set theory and nonabsolute integration theory, and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [16] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, we [17, 18] has defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem.

In this paper, according to the idea of [1, 29] and using the concept of generalized differentiability [19], the operator j which is the isometric embedding from (E^n, D) onto its range in the Banach space X and the controlled convergence theorems for the fuzzy Henstock integrals, we will deal with the Cauchy problem of discontinuous fuzzy integro-differential equations as following:

$$\begin{cases} x'(t) = \tilde{f}(t, x(t), \int_0^t \tilde{k}(t, s, x(s)) \mathrm{d}s), \\ x(0) = x_0, \quad t \in I_a = [0, a], a > 0, x_0 \in E^n, \end{cases} \tag{1}$$

where the integral is taken in the sense of strong fuzzy Henstock integral.

To make our analysis possible, we will first recall some basic results of fuzzy numbers and give some definitions of absolutely continuous fuzzy-number-valued function. In addition, we present the concept of generalized differentiability and we present the concept of fuzzy Henstock integrals and the controlled convergence theorem for the fuzzy Henstock integrals. In Section 3, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations. And in Section 4, we present some concluding remarks.

2 Preliminaries

2.1 Fuzzy number theory

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff

metric [30]:
$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| b - a \|\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)–(4) below$\}$ is a fuzzy number space. where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semi-continuous,

(4) $[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [30]:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u, v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0, 1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy to see that D is a metric in E^n. Using the results [31], we know that:

(1) (E^n, D) is a complete metric space,

(2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E^n$,

(3) $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r, x) = \sup_{\alpha \in [u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$ (see [31]).

Theorem 2.1 [31] *There exists a real Banach space X such that E^n can be imbedding as a convex cone C with vertex 0 into X. Furthermore the following conclusions hold:*

(1) the imbedding j is isometric,

(2) addition in X induces addition in E^n,

(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,

(4) $C - C$ is dense in X,

(5) C is closed.

It is well-known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [25] and it is based on the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $\tilde{f} : [a, b] \to \mathbb{R}_{\mathcal{F}}$ defined by $\tilde{f}(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar

multiplication (in the fuzzy context) and $g : [a, b] \to \mathbb{R}^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [19, 20]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [19].

Definition 2.1 [19] Let $\tilde{f} : (a, b) \to E^n$ and $x_0 \in (a, b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that:

(1) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(3) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(4) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)}{h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h} \cdot$ and $-\frac{1}{h} \cdot$, respectively).

2.2 The strong Henstock integrals of fuzzy-number-valued functions in E^n

In this section we define the strong Henstock integrals of fuzzy-number-valued functions in fuzzy number space E^n and we give some properties of this integral.

Definition 2.2 [18] A fuzzy-number-valued function $\tilde{f}$ will be termed piecewise additive on $[a, b]$ if there exists a division $T : a = a_0 < a_1 < \cdots < a_n = b$, such that $\tilde{f}(x)$ is additive on each $[a_i, a_{i+1}]$ $(i = 0, 1, \cdots, n - 1)$.

Definition 2.3 [17, 18] A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a piecewise additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{[x_{i-1}, x_i], \xi_i\}_{i=1}^n$ of $[a, b]$, we have

$$\sum_{i \in K_n} D(\tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{F}([x_{i-1}, x_i])) + \sum_{j \in I_n} D(\tilde{f}(\xi_j)(x_j - x_{j-1}), (-1) \cdot \tilde{F}([x_j, x_{j-1}])) < \varepsilon,$$

where $K_n = \{i \in \{1, 2, \cdots, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \cdots, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a, b]$.

Definition 2.4 [18]A fuzzy-number-valued function $\tilde{F}$ defined on $X \subset [a, b]$ is said to be $AC^*(X)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite sequence of non-overlapping intervals $\{[a_i, b_i]\}$, satisfying $\Sigma_{i=1}^n |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i we have

$$\sum \omega(\tilde{F}, [a_i, b_i]) < \varepsilon,$$

where ω denotes the oscillation of $\tilde{F}$ over $[a_i, b_i]$, i.e.,

$$\omega(\tilde{F}, [a_i, b_i]) = \sup\{D(\tilde{F}(y), \tilde{F}(x)); x, y \in [a_i, b_i]\}.$$

Definition 2.5 [18] A fuzzy-number-valued function $\tilde{F}$ is said to be ACG^* on X if X is the union of a sequence of closed sets $\{X_i\}$ such that on each X_i, $\tilde{F}$ is $AC^*(X_i)$.

For the strong fuzzy Henstock integrable we have the following theorems.

Theorem 2.2 *Let $\tilde{f} : [a, b] \to E^n$. If $\tilde{f} = 0$ a.e. on $[a, b]$, then $\tilde{f}$ is SFH integrable on $[a, b]$ and $\int_a^b \tilde{f}(t)\mathrm{d}t = 0$.*

Theorem 2.3 *Let $\tilde{f} : [a, b] \to E^n$ be SFH integrable on $[a, b]$ and let $\tilde{F}(x) = \int_a^x \tilde{f}(t)\mathrm{d}t$ for each $x \in [a, b]$. Then*
(a) the function $\tilde{F}$ is continuous on $[a, b]$;
(b) the function $\tilde{F}$ is differentiable a.e. on $[a, b]$ and $\tilde{F}' = f$;
(c) $\tilde{f}$ is measurable.

Theorem 2.4 *(Controlled Convergence Theorem) [18] Suppose $\{\tilde{f}_n\}$ is a sequence of SFH integrable functions on $[a, b]$ satisfying the following conditions:*
(1) $\tilde{f}_n(x) \to \tilde{f}(x)$ almost everywhere (a.e.) in $[a, b]$ as $n \to \infty$;
(2) the primitives $\tilde{F}_n$ of $\tilde{f}_n$ are ACG^* uniformly in n;
(3) the primitives $\tilde{F}_n$ converge uniformly on $[a, b]$;
then $\tilde{f}$ is also SFH integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b \tilde{f}_n(x)\mathrm{d}x = \int_a^b \tilde{f}(x)\mathrm{d}x.$$

3 The Existence of Solutions for Discontinuous Fuzzy Integro-differential Equations

In this section we prove the existence theorem for the problem (1).

For any bounded subset A of the Banach space X we denote by $\alpha(A)$ the Kuratowski measure of non-compactness of A, i.e. the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [24] for example.

Lemma 3.1 [24] *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions. Then*

$$\alpha(H) = \sup_{t \in I_\gamma} \alpha(H(t)) = \alpha(H(I_\gamma)),$$

where $\alpha(H)$ denotes the Kuratowski measure of non-compactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 3.1 [24] *Let D be a closed convex subset of X, and let F be a continuous function from D into itself. If for $x \in D$ the implication*

$$\bar{V} = c\bar{o}n(\{x\} \cup F(V)) \Rightarrow V$$

is relatively compact, then F has a fixed point.

We now give some useful definitions and results, which we will use throughout this paper.

Definition 3.1 A fuzzy-number-valued function $\tilde{f} : I_a \times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:
   (1) the fuzzy mapping $(x,y) \in E^n \times E^n$ is measurable for all $t \longrightarrow \tilde{f}(t,x,y)$;
   (2) the fuzzy mapping $t \in I_a$ is continuous for all $(x,y) \longrightarrow \tilde{f}(t,x,y)$.

Definition 3.2 A fuzzy-number-valued function $\tilde{k} : I_a \times I_a \times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:
   (1) the fuzzy mapping $(t,s) \longrightarrow \tilde{k}(t,s,y)$ is measurable for all $y \in E^n$;
   (2) the mapping $y \longrightarrow \tilde{k}(t,s,y)$ is continuous for all $(t,s) \in I_a \times I_a$.

Definition 3.3 A fuzzy-valued function $(t,s,z) \rightarrow h(t,s,z)$ defined on $\tilde{k} : I_a \times I_a \times E^n$ is a fuzzy Kamke function if h satisfied Carathéodory conditions, and for each fixed t,s, the function $z \rightarrow h(t,s,z)$ is nondecreasing and for each $q, 0 < q < a$, the function identically equal to zero is the unique continuous of the integral equation $z(t) = \int_0^t h(t,s,z(s))\mathrm{d}s$ defined on $[0,q)$.

Theorem 3.2 *If the fuzzy-number-valued function $\tilde{f} : I_a \longrightarrow E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I),$$

where $\overline{conv}\tilde{f}(I)$ is the closure of the convex of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I.

Proof. Because $j \circ \tilde{f}$ is abstract (H) integrable in a Banach space, by using the mean valued theorem of (H) integrals, we have

$$(H)\int_I j \circ \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}j \circ \tilde{f}(I) = |I| \cdot j \circ \overline{conv}\tilde{f}(t).$$

In additional, there exists $(H)\int_I j \circ \tilde{f}(t)\mathrm{d}t = j \circ \int_I \tilde{f}(t)\mathrm{d}t$.

So, we have $j \circ \int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}j \circ \tilde{f}(I)$. And the set $\{|I| \cdot \overline{conv}\tilde{f}(I)\}$ is a closed convex set, we have

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I).$$

We shall consider the problem

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z,s,x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t) = x_0 + (-1) \cdot \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s)) \mathrm{d}s) \mathrm{d}z, \quad t \in I_a, x_0 \in E^n, \qquad (2)$$

where integrals are taken in the sense of (SFH).

To obtain the existence results it is necessary to define a notion of a solution.

Definition 3.4 An ACG^* fuzzy-valued function $x : I_a \to E^n$ is said to be a solution of problem (1) if it satisfies the following conditions: (1) $x(0) = x_0$;
(2) $x'(t) = \tilde{f}(t, x(t), \int_0^t \tilde{k}(t, s, x(s)) \mathrm{d}s)$ for a.e. $t \in I_a$.

Definition 3.5 A continuous fuzzy-valued function $x : I_a \to E^n$ is said to be a solution of problem (2) if it satisfies

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s)) \mathrm{d}s) \mathrm{d}z$$

or

$$x(t) = x_0 + (-1) \cdot \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s)) \mathrm{d}s) \mathrm{d}z$$

for all $t \in I_a$.

Theorem 3.3 *Each solution $x(t)$ of problem* (1) *is equivalent to the solution of problem* (2).

Proof. Let $x(t)$ be a continuous solution of (1). By the definition, $x(t)$ is ACG^* function and $x(0) = x_0$. Since, for a.e. $t \in I_a$, we have $x'(t) = \tilde{f}((t, x(t), \int_0^t \tilde{k}(t, s, x(s)) \mathrm{d}s)$ and the last is (SFH) integrable, it is differentiable *a.e.* Moreover,

$$\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s)) \mathrm{d}s) \mathrm{d}z = \int_0^t x'(s) \mathrm{d}s = x(t) -_H x_0.$$

Thus (2) is satisfied.

In addition, we assume that $y(t)$ is ACG^* function and it is clear that $y(0) = x_0$. By the definition of (SFH) integrals there exists an ACG^* fuzzy-valued function $\tilde{g}$ such that $\tilde{g}(0) = x_0$, and $\tilde{g}'(t) = \tilde{f}((t, y(t), \int_0^t \tilde{k}(t, s, y(s)) \mathrm{d}s), a.e.$

Hence

$$y(t) = x_0 + \int_0^t \tilde{f}(z, y(z), \int_0^z \tilde{k}(z, s, y(s)) \mathrm{d}s) \mathrm{d}z$$

$$= x_0 + \int_0^t \tilde{g}'(s) \mathrm{d}s = x_0 + \tilde{g}(t) -_H \tilde{g}(0) = \tilde{g}(t).$$

We have $y = \tilde{g}$ and then $y'(t) = \tilde{f}(z, y(z), \int_0^z \tilde{k}(z, s, y(s) \mathrm{d}z)$.

For $x \in C(I_a, E^n)$, we define the metric of x by

$$H(x, \tilde{0}) = \sup_{t \in I_a} D(x, \tilde{0}).$$

Let

$$B = \{x \in C(I_a, E^n) | H(x, \tilde{0}) \le H(x, \tilde{0}) + p, p > 0\}.$$

Obviously the set B is closed and convex. We define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by:

$$F(x)(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \quad t \in I_a, x_0 \in E^n.$$

Let $\Gamma = \{F(x) \in C(I_a, E^n) | x \in B\}$.

Now we present the existence theorems for the problem (1) in a fuzzy number space E^n.

Theorem 3.4 *Assume that, for each ACG^* fuzzy-valued function $x : I_a \to E^n$, a fuzzy-number-valed function $\tilde{k}(\cdot, s, x(s)), \tilde{f}(\cdot, x(\cdot), \int_0^{(\cdot)} \tilde{k}(\cdot, s, x(s))\mathrm{d}s)$ are (SFH) integrable, $\tilde{f}$ and $\tilde{k}$ is L^1-Carathéodory function. Suppose that there exists a constant d such that*

$$\alpha(j \circ \tilde{f}(t, A, C)) \leq d \cdot \max\{\alpha(j \circ A), \alpha(j \circ C)\} \tag{3}$$

for each bounded subset $A, C \subset E^n$ and $t \in I_a$. Where α denotes the measure of non-compactness. Assume that there exists a continuous $g : I_a \times I_a \to R^+$ such that

$$\alpha(j \circ \tilde{k}(I, I, X)) \leq \sup_{s \in I} g(t, s)\alpha(j \circ X) \tag{4}$$

for each bounded subset $X \subset E^n$, and $t, s \in I, I \subset I_a$, and the zero function is the unique continuous solution of the inequality

$$p(t) \leq d \cdot c \cdot \sup_{z \in I_c} \int_0^c g(z, s)p(s)\mathrm{d}s \quad on \quad I_c. \tag{5}$$

Moreover, let Γ be equicontinuous, equibounded and uniformly ACG^* on I_a. Then there exists a solution of problem (1) on I_c for some $0 < c \leq a, d \cdot c < 1$.

Proof. By equicontinuous and equiboundedness of Γ, there exists a number $c, 0 < c \leq a$ such that

$$H(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$\leq p$$

for fixed $p > 0, x \in B, t \in I_c$. By the assumption on the operator F, we have

$$H(F(x)(t), \tilde{0}) = H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(x_0, \tilde{0}) + \sup D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$\leq \sup_{t \in I_c} D(x_0, \tilde{0}) + l.$$

Using Theorem 2.3 we have F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for some $x \in B$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since V is equicontinuous, by Lemma 3.1, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$ we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. And for fixed $z \in [0, t]$, we divide the interval $[0, z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m, j = 0, 1, 2, \cdots, m$.

Let $V([z_j, z_{j+1}]) = \{u(s) | u \in V, z_j \leq s \leq z_{j+1}\}, j = 0, 1, 2, \cdots, m - 1$. By Lemma 3.1 and the continuity of v, there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(j \circ V([z_j, z_{j+1}])) = \sup_{t \in I_c}\{\alpha(j \circ V(s)) | z_j \leq s \leq z_{j+1}\} := v(s_j).$$

By Theorem 3.2 and the properties of the (SFH) integral we have

$$F(u)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, u(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} \tilde{k}(z, s, u(s))\mathrm{d}s)\mathrm{d}z$$

$$\leq x_0 + \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\tilde{f}(z, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}]))).$$

Using (4), (5) and the properties of measure of noncompactness α, we have

$$\alpha(j \circ F(V(t)))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i)\alpha(j \circ \tilde{f}(z, V(I_i), \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}]))))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \max\{\alpha(j \circ V(I_i)),$$

$$\alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}])))\}.$$

We observe that

(1) if $\alpha(j \circ V(I_i)) > \alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}])))$, then

$$\alpha(j \circ V) = \alpha(j \circ \overline{conv}(\{x\} \cup g(V))) \leq \alpha(j \circ F(V)) < d \cdot c \cdot \alpha(j \circ V),$$

Because $d \cdot c < 1$, so $\alpha(j \circ V) < \alpha(j \circ V)$ is a contradiction;

(2) if $\alpha(j \circ V(I_i)) < \alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}])))$, then

$$\alpha(j \circ V) < \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}])))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j)\alpha(j \circ \tilde{k}(z, I_j, V([z_j, z_{j+1}])))$$

$$\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j)\sup_{s \in I_j} g(z, s)\alpha(j \circ V([z_j, z_{j+1}]))$$

$$= d \cdot c \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j) \cdot g(z, p_j)j \circ (v(s_j))$$

$$= d \cdot c \cdot [\sum_{j=0}^{m-1}(z_{j+1} - z_j) \cdot g(z, p_j)j \circ (v(p_j))$$

$$+ \sum_{j=0}^{m-1}(z_{j+1} - z_j)(g(z, p_j)(j \circ v(s_j) - j \circ v(p_j)))].$$

By continuity of v we have $j \circ v(s_j) - j \circ v(p_j) < \varepsilon$ and $\varepsilon \to 0$ if $m \to \infty$, so

$$j \circ v(t) = \alpha(j \circ V(t)) \leq d \cdot c \cdot \sup_{z \in I_c} \int_0^c g(z, s)j \circ v(s)\mathrm{d}s.$$

By (6) we have $j \circ v(t) = \alpha(j \circ V(t)) = 0$ for $t \in I_c$.

Using Arzel$\acute{a}$-Ascoli theorem, we have V is relatively compact. By Theorem 3.1 the operator F has a fixed point. This means that there exists a solution of the problem (1).

Theorem 3.5 *Assume that, for each ACG^* fuzzy-valued function $x : I_a \to E^n$, a fuzzy-number-valed function $\tilde{k}(\cdot, s, x(s)), \tilde{f}(\cdot, x(\cdot), \int_0^{(\cdot)} \tilde{k}(\cdot, s, x(s))\mathrm{d}s)$ are (SFH) integrable, $\tilde{f}$ and $\tilde{k}$ is L^1-Carath$\acute{e}$odory function. Suppose that there exists a constant d such that*

$$\alpha(j \circ \tilde{f}(t, A, C)) \leq d \cdot \max\{\alpha(j \circ A), \alpha(j \circ C)\} \tag{6}$$

for each bounded subset $A, C \subset E^n$ and $t \in I_a$. Where α denotes the measure of non-compactness. Assume that

$$\alpha(j \circ \tilde{k}(t, s, j \circ X)) \leq h(t, s, \alpha(j \circ X)) \tag{7}$$

for each bounded subset $X \subset E^n$, and $0 \leq s \leq t \leq a$, where h is a Kamke function.

Moreover, Let Γ be equicontinuous, equibounded and uniformly ACG^* on I_a. Then there exists at least one solution of problem (1) on I_c for some $0 < c \leq a, d \cdot c < 1$.

Proof. By equicontinuous and equiboundedness of Γ, there exists a number $c, 0 <$

$c \leq a$ such that

$$H(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$\leq p$$

for fixed $p > 0, x \in B, t \in I_c$. By the assumption on the operator F, we have

$$H(F(x)(t), \tilde{0}) = H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$= \sup_{t \in I_c} D(x_0, \tilde{0}) + \sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})$$

$$\leq \sup_{t \in I_c} D(x_0, \tilde{0}) + p.$$

Using Theorem 2.3 we have F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for some $x \in B$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since V is equicontinuous, by Lemma 3.1, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$ we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. We denote $T_i = [t_i, t_{i+1}]$ and fix $z \in I_c$. Let $\int_0^z \tilde{K}(s)\mathrm{d}s = \{\int_0^z x(s) : x \in \tilde{K}\}$ for any $\tilde{K} \subset C(I_c, E^n)$ and let $\tilde{k_z}$ denote the mapping defined by $\tilde{k_z}(x(s)) = \tilde{k}(z, s, x(s))$ for each $x \in B$ and $s \in I_c$. Obviously, $\tilde{k_z}(j \circ V(s)) = \tilde{k}(z, s, j \circ V(s))$.

Let

$$\tilde{F}(j \circ V(t)) = \{\tilde{F}(x)(t) \in C(I_c, E^n) : x \in V, t \in I_c\}$$

$$= \{x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z : x \in V, t \in I_c\}.$$

By Theorem 3.2 and the properties of the (SFH) integral we have

$$F(x)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z$$

$$\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}\tilde{f}(z, V(T_i), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s).$$

Therefore, $\tilde{F}(j \circ V(t)) \subset x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}\tilde{f}(z, V(T_i), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s)$.

Using (6), (7) and the properties of measure of noncompactness α, we have

$$\alpha(j \circ F(V(t)))$$

$$\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i)\alpha j \circ (\tilde{f}(z, V(T_i), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s))$$

$$\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d \cdot \max\{\alpha(j \circ V(T_i)), \alpha(\int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s)\}.$$

We observe that
(1) if $\alpha(j \circ V(T_i)) > \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)$, then

$$\alpha(j \circ V) = \alpha(j \circ \overline{conv}(\{x\} \cup F(V))) \leq \alpha(j \circ F(V)) < d \cdot c \cdot \alpha(j \circ V),$$

Because $d \cdot c < 1$, so $\alpha(j \circ V) < \alpha(j \circ V)$ is a contradiction;
(2)if $\alpha(j \circ V(T_i)) < \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)$, then

$$\alpha(j \circ V) < \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d \cdot \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s) \leq 2dc\alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)$$

$$\leq 2dc \int_0^Z \alpha(j \circ \tilde{k}(z, s, V(s))\mathrm{d}s) \leq 2dc \int_0^z h(z, s, V(s))\mathrm{d}s,$$

since $V = \overline{conv}(\{x\} \cup j \circ F(V))$, we have

$$v(t) = 2dc \int_0^z h(z, s, v(s))\mathrm{d}s.$$

Now, we apply a theorem of differential inequalities. We have $v(t) = \alpha(j \circ V(t)) = 0$. By Arzelá-Ascoli theorem, we have V is relatively compact. By Theorem 3.1 the operator F has a fixed point. This means that there exists a solution of the problem (1).

4   Conclusion

In this paper, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrability. Our result improves the result given in [16, 23, 26] and [28] (where uniform continuity was required), as well as those referred therein.

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Indirect Adaptive Fuzzy Control of Multivariable Nonlinear Systems Class with Unknown Parameters

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Abstract: This paper develops an adaptive fuzzy control of nonlinear system class. In this method, we investigated the possibilities offered by the fuzzy systems of Takagi-Sugeno type in terms of approximation capacity of the continuous nonlinear functions and we exploited the Lyapunov theory to establish a parametric adaptation law, ensuring the total stability of the system. Finally, simulation results are presented to show the effectiveness of this kind of controller.

Keywords: *fuzzy systems; fuzzy adaptive law; permanent magnet synchronous motors; current controlled inverter.*

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1 Introduction

The vast majority of conventional control techniques have been devised for linear time-invariant systems that are assumed to be completely known and well understood. In most practical instances, however, the systems to be controlled are nonlinear, time-varying and the basic physical processes in them are not completely known a priori. These types of model uncertainties are extremely difficult to manage, even with the conventional techniques. For these systems, the linear control exhibits generally poor performances and the recourse to a nonlinear adaptive control can be a judicious solution. Besides, the theory of fuzzy logic has also been applied successfully for the control of nonlinear systems. In general the control strategy used for fuzzy logic controller is based on expert knowledge, so the fuzzy logic controller has the ability to emulate the human strategies control [1–5] and [6]. Moreover, it would be necessary that the control strategy can perform the control objectives even if the parameters of the system evolve or are badly

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known. In order to solve this problem, we develop an adaptive fuzzy controller which is able to modify the control law according to the evolution of the system parameters.

During the past two decades, many works have been devoted to the development of the fuzzy logic controllers [7–10] based on the Takagi−Sugeno (T−S) model or the fuzzy dynamic models. The basic idea of these methods is: 1) To represent the complex nonlinear system by a family of local linear models, each linear model exhibits the dynamics of the complex system in one local region. Then, to construct a global nonlinear model by aggregating all the local models through the fuzzy membership functions. 2) To design the local controllers based mainly on each local model, which is much easier than on the global region for the nonlinear system. Then, the global controller can be aggregated from the local controllers. It was also proven that the fuzzy system is capable to approximate any nonlinear functions over a convex compact region [11]. Based mainly on this property, the fuzzy logic system is applied in the area of the adaptive control where the unknown nonlinear functions are approximated by a fuzzy basis functions and its parameters are updated on line to cope with uncertainties.

Many researchers have considered adaptive fuzzy control of nonlinear dynamical systems. The methods appeared in the literature could be derived into two groups: indirect methods and direct methods. An indirect adaptive controller tries to identity the dynamics of the system, and then generates a control input based on certainty equivalent principle [13,14,16,30–32]. Direct adaptive controller, on the other hand, directly adjusts the parameters of a controller to archive control objectives [12,30–32]. The approaches presented in [12, 15, 17–19, 31, 32] are limited to SISO nonlinear systems with constant control gain. In the paper [30], the authors propose a direct adaptive control by using either fuzzy systems or neural networks for uncertain SISO nonlinear systems with state dependent control gain. In their proof of stability, the authors assume that the time derivative of the control gain is bounded from above by a known function, which is difficult to find for unknown systems.

In this paper, we develop a new stable fuzzy adaptive control for a class of MIMO nonlinear systems in order to confer high robustness to the controller in the presence of parametric uncertainties and dominant uncertain nonlinearities. The fuzzy systems are used to approximate the model of controlled system. The approximation theory and the Lyapunov method are used together to construct, in first stage, the fuzzy adaptive control law and to establish, in second stage, the convergence of the tracking error and the boundedness of the adaptive. The method is applied by simulation to the problem of tracking speed or position of the permanent magnet synchronous motor (PMSM).

This paper is organized as follows: in Section 2 the used fuzzy logic system is briefly described and Section 3 is devoted to the problem statement. The proposed fuzzy adaptive scheme for a class of MIMO nonlinear system is developed in Section 4. In Section 5, the performances of the proposed scheme are evaluated by simulation for the case of PMSM. The conclusion is presented in Section 5.

2   Description of the Used Fuzzy Logic System

The fuzzy logic system incorporates generally four principal components: fuzzifier, fuzzy rules base, inference engine and defuzzifier [20–22]:

   – Fuzzifier maps crisp points in the input space into fuzzy sets in the input space.

   – Fuzzy rules base contains the fuzzy rules interpreting the behavior of a given system; it is the central element from which the other components interpret and combine these

rules to form the final output.

   – Inference engine exploits an approximate reasoning procedure in order to map fuzzy sets in the input space into fuzzy sets in the output space.

   – Defuzzifier extracts crisp points in the output space from fuzzy sets in the output space.

   A FLS can be seen as a mathematical application of the input towards the output. This application is very rich in its mathematical formulation by the existence of various mathematical interpretations concerning the fuzzy rules, the fuzzy inference and the defuzzifier. It is significant to note that the implementation of the adaptive fuzzy control, in real time, requires that the mathematical model of the FLS must be simple. In our case, we are interested in the FLS of Sugeno-Takagi model, developed initially to model system from numerical data [22]. In this case the consequences rules are numerical functions, which depend on the values of the crisp input variables.

   In this section, we give a mathematical formulation of used fuzzy systems and fuzzy basis functions in the case of Sugeno-Takagi model. Denote by $x_{sf_1}, \ldots, x_{sf_n}$ the inputs of the FLS, and by y_{sf} its output. Each variable x_{sf_i} is related to m_i fuzzy sets F_i^j defined on U_i. Moreover, it is assumed that for any value of x_{sf_i} on U_i, there exists at least one fuzzy set among F_i^j ($i = 1, \cdots, n$ and $i = 1, \cdots, m$) for which the membership degree is non null. The rule base of the FLS incorporates $\prod_1^n m_i$ rules of the form:

$$R_l : \text{if } x_{sf_1} \text{ is } F_1^{l_1} \text{ and} \ldots \text{and } x_{sf_i} \text{ is } F_i^{l_i} \text{ and} \ldots \text{and } x_{sf_n} \text{ is } F_n^{l_n}, \\ \text{then } y_{sf_l}(x) = a_0^l + a_1^l x_{sf_1} + \ldots + a_n^l x_{f_n} \tag{1}$$

with $l = 1, ..., M$; $i = 1, ..., n$ and $1 \le l_i \le m_i$.

Indeed, the base of fuzzy rules contains all the combinations of the fuzzy sets related to the input variables. By considering the rules of the form (1) and using the product to interpret the fuzzy implication and the T-norm, therefore the expression of the FLS output is involved by the following relation [21, 24, 25]:

$$y_{sf} = \frac{\sum\limits_{l=1}^{M} \mu_l \cdot y_{sf_l}}{\sum\limits_{l=1}^{M} \mu_l}, \tag{2}$$

where μ_l stands for the firing strength of the R_l rule, which is given by:

$$\mu_l = \prod_{i=1}^{n} \mu_{F_i^{l_i}}(x_i), \quad 1 \le l_i \le m_i, \tag{3}$$

and $\mu_{F_i^{l_i}}$ is the membership function of variable x_i associated to fuzzy set $F_i^{l_i}$. This function is selected as Gaussian function:

$$\mu_{F_i^j}(x_{sf_i}) = \exp\left\{-0.5\left(v_i^j\left(x_{sf_i} - c_i^j\right)\right)\right\}, \tag{4}$$

where c is the average, v is the inverse of the variance. If the premises parameters are fixed *apriori*, only the conclusion parameters can be freely adjustable. Thus, the final output can be rewritten in the form:

$$y_{sf} = W(x_{sf}) \cdot A, \tag{5}$$

where A is a vector gathering the parameters a_i^j and $W(x_{sf})$ is a vector of basis fuzzy functions.

3 Problem Statement

Consider a class of MIMO non-linear systems described by the following set of differential equations (for $i = 1, \cdots, m$):

$$\dot{x}_i = f_i\left(x\right) + g_i\left(x\right) \cdot u_i,$$
$$y_i = x_i, \tag{6}$$

where $f_i\left(x\right)$ and $g_i\left(x\right)$ are unknown functions, whereas $x = [x_1, \ldots, x_m]^T$, $u = [u_1, \ldots, u_m]$ and $y = [y_1, \ldots, y_m]$ are the system state, the control input and the plant output respectively. The control objective is to force the output vector $x = [x_1, \ldots, x_m]^T$ to follow the specified desired trajectory $x_d = [x_{d1}, \ldots, x_{dm}]^T$. Define the tracking error vector $e(t)$ as:

$$e(t) = x(t) - x_d(t).$$

Therefore, we should design a fuzzy adaptive control law $u(t)$ such that $e(t)$ converges to a small neighbourhood of zero. To this end, the following assumptions are assumed:

Assumption 3.1 Let:
- $f_i(x) \in \mathbb{R}$ and $g_i(x) \in \mathbb{R}$ are bounded smooth nonlinear functions,
- The state vector is available,
- The reference signal x_d and its derivation $\dot{x}_d$ are known bounded signals.

If the functions $f_i(x)$ and $g_i(x)$ are well known the control input can be taken as [26]:

$$u^* = \frac{v_i - f_i\left(x\right)}{g_i\left(x\right)} \tag{7}$$

with

$$v_i = \dot{x}_{id} + \lambda_i e_i, \tag{8}$$

$$e_i = x_{id} - x_i, \lambda_i > 0, \quad i = 1, \ldots, m. \tag{9}$$

Introducing (7) into (6) leads to the tracking error dynamic equation:

$$\dot{e}_i + \lambda_i \cdot e = 0. \tag{10}$$

Since the coefficients λ_i are imposed such that $p + \lambda_i$ polynomial is Hurwitz, the tracking error vector $e(t)$ converges asymptotically to zero. In the case where the functions $f_i\left(x\right)$ and $g_i\left(x\right)$, involved in the dynamic model (6), are badly known, the implementation of the control law (7) is inoperative since it requires a precise model. To solve this problem an approach by a fuzzy logic system (FLS) is proposed. Our objective is to develop a model of identification and an adaptation law where the functions $f_i\left(x\right)$ and $g_i\left(x\right)$ are replaced by FLS. For this purpose, the dynamic of the system is rewritten, first of all, in the following form:

$$\dot{x}_i = \hat{f}_i\left(x; \theta_{f_i}\right) + \hat{g}_i\left(x; \theta_{g_i}\right)\ u_i + \varepsilon_i. \tag{11}$$

Let $\hat{f}_i\left(x; \hat{\theta}_{fi}\right)$ and $\hat{g}_i\left(x; \hat{\theta}_{gi}\right)$ be the estimated of the functions $f_i\left(x\right)$ and $g_i\left(x\right)$, where $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$ are a parameter vectors, whereas ε_i is a reconstruction error, it is given by:

$$\varepsilon_i = \left[f_i\left(x\right) - \hat{f}_i\left(x; \theta_{f_i}\right)\right] + [g_i\left(x\right) - \hat{g}_i\left(x; \theta_{g_i}\right)]\ u_i \tag{12}$$

such as:
$$\|\varepsilon_i\| \le \bar{\varepsilon}_i.$$

Therefore, one can construct the following control input $u(t)$:

$$u(t) = \frac{v_i - \hat{f}_i\left(x; \hat{\theta}_{fi}\right)}{\hat{g}_i\left(x; \hat{\theta}_{gi}\right)}. \tag{13}$$

The control law (13) requires a FLS for reconstructing the functions $\hat{f}_i\left(x; \hat{\theta}_{fi}\right)$ and $\hat{g}_i\left(x; \hat{\theta}_{gi}\right)$ and an adaptation mechanism for the parameters $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$ in order that this control ensures the convergence of the tracking error $e(t)$ to zero and the boundedness of all signals of the plant.

4   Control Synthesis

The FLS is principally used to estimate on line the nonlinear function given in (11). To this end, the functions $f_i(x)$ and $g_i(x)$ are ideally approximated by FLS such that:

$$\begin{aligned}
f_i(x) &= W_{f_i}(x)\,\theta_{f_i i} + \varepsilon_{fi}, \\
g_i(x) &= W_{g_i}(x)\,\theta_{g_i i} + \varepsilon_{gi},
\end{aligned} \tag{14}$$

where $W_{f_i}(x)$ and $W_{g_i}(x)$ are basis functions [20], θ_{f_i} and θ_{g_i} are vectors of optimal parameters, while ε_{f_i} and ε_{g_i} are the unavoidable reconstruction errors satisfying the condition [27, 28]:

$$\begin{aligned}
\|\varepsilon_{fi}\| &\le \bar{\varepsilon}_{fi}, \quad \bar{\varepsilon}_{fi} > 0, \\
\|\varepsilon_{gi}\| &\le \bar{\varepsilon}_{gi}, \quad \bar{\varepsilon}_{gi} > 0,
\end{aligned} \qquad 1 \le i \le m. \tag{15}$$

Consequently, the functions $\hat{f}_i\left(x; \hat{\theta}_{fi}\right)$ and $\hat{g}_i\left(x; \hat{\theta}_{gi}\right)$ which are the approximation of $f_i(x)$ and $g_i(x)$ can be defined under the form:

$$\begin{aligned}
\hat{f}_i\left(x; \hat{\theta}_{fi}\right) &= W_{f_i}(x)\,\hat{\theta}_{f_i}, \\
\hat{g}_i\left(x; \hat{\theta}_{gi}\right) &= W_{g_i}(x)\,\hat{\theta}_{g_i}.
\end{aligned} \tag{16}$$

Proposition 4.1 *We use the following serial-parallel identification model:*

$$\dot{\hat{x}}_i = -\alpha_i\,\hat{x}_i + \alpha_i\,x + \hat{f}_i\left(x; \theta_{f_i}\right) + \hat{g}_i\left(x; \theta_{g_i}\right)\,u_i \tag{17}$$

where α_i is given positive scalar. The whole identification scheme is shown in Figure 1.

The goals of identification are the following: Specify the fuzzy systems $\hat{f}_i\left(x; \hat{\theta}_{fi}\right)$ and $\hat{g}_i\left(x; \hat{\theta}_{gi}\right)$, and develop an adaptive law for the parameters $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$ such that: a) all signals involved in the identification model must be uniformly bounded, i.e., it must be guaranteed that $\hat{x} \in L\infty$, $\left(\hat{\theta}_f \theta_f^T\right) \le M_f$, and $\left(\hat{\theta}_g \theta_g^T\right) \le M_g$ (the input u and the system state x are uniformly bounded by assumption), and b) the error $e_i = x_i - \hat{x}_i$ should be as small as possible.
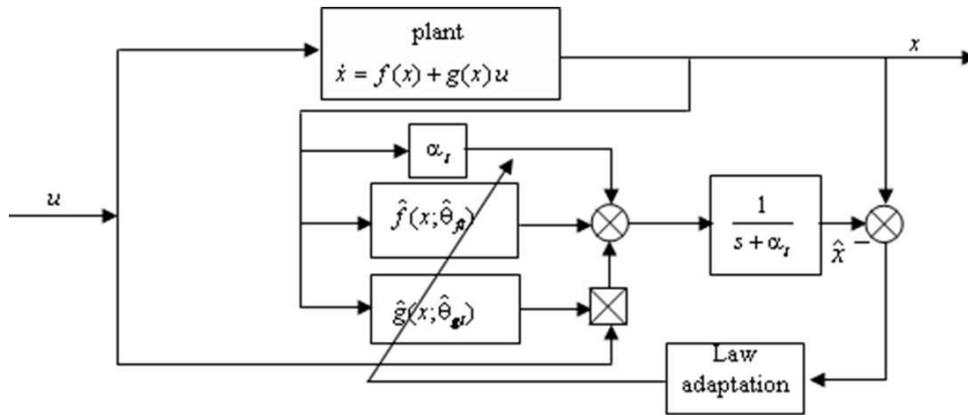
Figure 1: Identification model using fuzzy systems.

While having the fuzzy model (11), under the assumptions (A-3.1), and if the system (6) is conducted by the control law:

$$u(t) = \frac{v_i - \hat{f}_i\left(x; \hat{\theta}_{fi}\right)}{\hat{g}_i\left(x; \hat{\theta}_{gi}\right)} \tag{18}$$

and the parameters $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$ are updated under the law:

$$
\begin{aligned}
\dot{\hat{\theta}}_{f_i} &= \eta_{f_i} W_{f_i}^T(x)\, e_i - k_i\, \|e_i\|\, \hat{\theta}_{f_i} \\
\dot{\hat{\theta}}_{g_i} &= \eta_{g_i} W_{g_i}^T(x)\, e_i u_i - k_i\, \|e_i\|\, \hat{\theta}_{g_i}
\end{aligned}
\tag{19}
$$

where η_{fi} , η_{gi} and k_i are positive constants. Therefore, the tracking error converges asymptotically to zero and the state vector $\hat{x}$ and the parameters $\hat{\theta}_{fi}$ and $\hat{\theta}_{gi}$ are bounded.

Remark 4.1 The adaptation law (19) updates only the conclusion parameters. Indeed in our case, the designer specifies, in advance, the structure of FLS, the input variables, the fuzzy sets (or membership functions) and the number of rules. In practice, to make the "good choice" for all these FLS parameters, in advance, is a difficult task, apart for a skilled operator in the area of the controlled system. A common practice is an arbitrary defining the membership functions to cover the interest subset of the input space [12, 30–32]. One can think that this adaptation law also compensates, in a certain manner, for the inadequacy of the fuzzy sets and the insufficiency of the rules number.

5   Application to Permanent Magnet Synchronous Motor

5.1   Mathematical model of PMSM

The model of the permanent magnet synchronous motors (PMSM) is considered in the case of the usually allowed simplifying assumptions i.e.:
   – The spatial distribution of stator winding is sinusoidal.

– The saturation is neglected.
– The damping effect is neglected.

Thus, in the synchronous $d - q$ reference form, the dynamic of PMSM is represented as follows:

$$
\begin{aligned}
v_d &= R_s i_d + L_d \frac{di_d}{dt} - p L_q \Omega i_q, \\
v_q &= R_s i_q + L_q \frac{di_q}{dt} + p L_d \Omega i_d + p \Omega \Phi_f, \\
J \frac{d\Omega}{dt} &= T_{em} - T_r - F_c \Omega, \\
T_{em} &= \tfrac{3}{2} p \left(\Phi_f i_q + (L_d - L_q) \; i_d i_q \right),
\end{aligned}
\tag{20}
$$

where:

v_d, v_q	Stator voltage in $d - q$-axis;
i_d, i_q	Stator current in $d - q$-axis;
L_d, L_q	Stator inductance in $d - q$-axis;
R_s	Stator resistance;
p	Number of pole pairs;
Ω	Mechanical speed of motor
Φ_f	Flux created by the rotor magnets;
F_c	Viscous friction coefficient;
J	Total moment of inertia of the motor and load;
T_{em}, T_r	Electromagnetic torque and load torque;

5.2   Speed Control

In the case of surface-mounted PMSM ($L_d = L_q$), the electromagnetic torque depends solely on current in the q axis. For a given torque, the transferred power is optimized if the current in the direct axis is null ($i_d = 0$) [29]. Hence, the control objective is to force the current i_d to zero and to impose the demanded torque by controlling the current i_q. Physically by this strategy, the linked stator flux is maintained in quadrature with flux produced by the rotor magnets. The proposed schema of indirect adaptive fuzzy control, about the speed tracking of the PMSM, appears in Fig. 2.

From the reference speed Ω_{ref} and measured speed Ω, the fuzzy adaptive controller provides the desired current. The three-phase reference current is obtained from the ($d - q$) stator reference current (i_{dref}, $i_{qref} = 0$) by using the inverter Park transformation. The actual stator current (i_a, i_b, i_c) is restricted in hysteresis bandwidth Δi around the three-phase reference currents by using an appropriate switching of the inverter legs. By using the equilibrium equation between the motor torque and the torque opposed by the mechanical part of the system, we can write:

$$
\frac{d\Omega}{dt} = f(\Omega) + g(\Omega) \cdot i_q.
\tag{21}
$$

Using equation (30), the fuzzy identifier becomes:

$$
\dot{\hat{\Omega}} = -\alpha \cdot \hat{\Omega} + \alpha \cdot \Omega + \hat{f}(\Omega; \hat{\theta}_f) + \hat{g}(\Omega; \hat{\theta}_g) \cdot i_q.
\tag{22}
$$

However, the estimated $\hat{f}(.)$ and $\hat{g}(.)$ of the function $f(.)$ and $g(.)$ can be generated in the form:

$$
\begin{aligned}
\hat{f}(\Omega; \hat{\theta}_f) &= W_f(\Omega) \hat{\theta}_f, \\
\hat{g}(\Omega; \hat{\theta}_g) &= W_g(\Omega) \hat{\theta}_g.
\end{aligned}
\tag{23}
$$
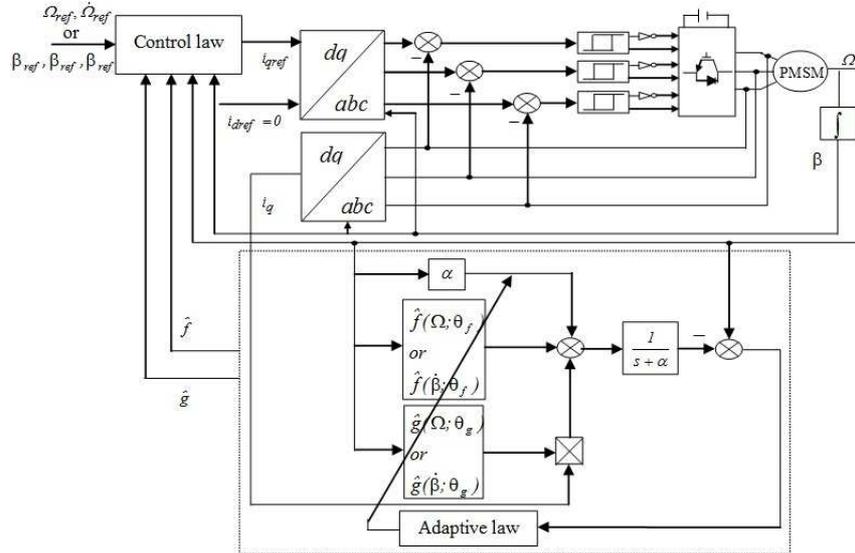
Figure 2: Indirect adaptive fuzzy control scheme of permanent magnet synchronous motor.

For this application, the fuzzy system has one variable at input and this variable is described by 3 membership functions.

The identification error is given as follows:

$$e(t) = \Omega - \hat{\Omega}. \tag{24}$$

The parameter adaptive law is given by:

$$\dot{\hat{\theta}}_f = \eta_{f\Omega} W_f(\Omega)\, e - k_\Omega \left\| e \right\| \hat{\theta}_f,$$

$$\dot{\hat{\theta}}_g = \eta_{g\Omega} W_g(\Omega)\, e\, i_{qref} - k_\Omega \left\| e \right\| \hat{\theta}_g, \tag{25}$$

where $\eta_{f\Omega}, \eta_{f\Omega}$ and k_Ω are positive constants.

Consequently, the control law i_{qref} is given by:

$$i_{qref} = \frac{v - \hat{f}(\Omega; \hat{\theta}_f)}{\hat{g}(\Omega; \hat{\theta}_g)} \tag{26}$$

with

$$v = \dot{\Omega}_{ref} + \lambda(\Omega_{ref} - \Omega), \quad \lambda > 0.$$

5.3 Position control

The procedure used previously is renewed in the case of the tracking of trajectory position. Thus, we again consider the equation (21) where $\dot{\Omega}$ is replaced by position β, which leads to:

$$\ddot{\beta} = f(\dot{\beta}) + g(\dot{\beta}) \cdot i_q \tag{27}$$

with

$$\beta = \frac{d\Omega}{dt}.$$
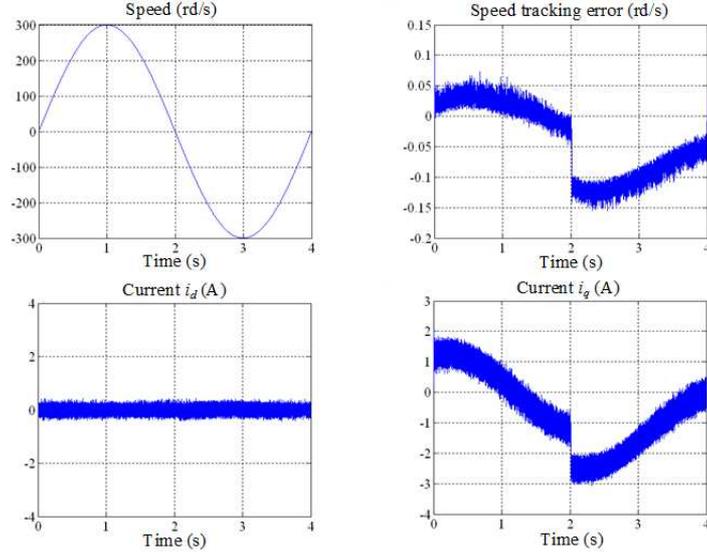
Figure 3: Speed tracking, with the nominal load torque is applied at $t = 2s$, of PMSM.

Using the equation (27), the identification model is:

$$\ddot{\hat{\beta}} = -\alpha \cdot \dot{\hat{\beta}} + \alpha \cdot \dot{\beta} + \hat{f}(\dot{\beta}; \hat{\theta}_f) + \hat{g}(\dot{\beta}; \hat{\theta}_g) \cdot i_q. \tag{28}$$

In our application we allot three membership functions to the input $\dot{\beta}$ for the two fuzzy systems. The estimated functions are given by:

$$\begin{aligned}
\hat{f}(\dot{\beta}; \hat{\theta}_f) &= W_f(\dot{\beta}) \cdot \hat{\theta}_f, \\
\hat{g}(\dot{\beta}; \hat{\theta}_g) &= W_g(\dot{\beta}) \cdot \hat{\theta}_g,
\end{aligned} \tag{29}$$

where the vector parameters $\hat{\theta}_f$ and $\hat{\theta}_g$ are updated by:

$$\begin{aligned}
\dot{\hat{\theta}}_f &= \eta_{f\beta} \cdot W_f(\dot{\theta}) \cdot e - k_\beta \cdot \|e\| \cdot \hat{\theta}_f, \\
\dot{\hat{\theta}}_g &= \eta_{g\beta} \cdot W_f(\dot{\theta}) \cdot e \cdot i_{qref} - k_\beta \cdot \|e\| \cdot \hat{\theta}_g,
\end{aligned} \tag{30}$$

with e being the identification error, it is given by:

$$e = \dot{\beta} - \dot{\hat{\beta}}, \tag{31}$$

$\eta_{f\beta}, \eta_{g\beta}$ and k_β are positive constants.

By using the functions ($\hat{f}$ and $\hat{g}$) from the fuzzy system model (29) and in accordance with the control law, the control law i_{qref} is involved by:

$$v = \ddot{\beta}_{ref} + k_{1\theta}(\dot{\beta}_{ref} - \dot{\beta}) + k_{2\theta}(\beta_{ref} - \beta).$$
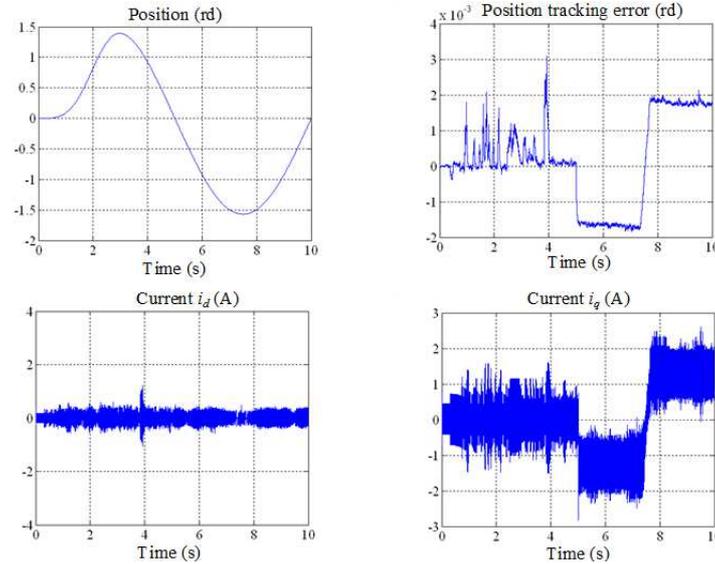
Figure 4: Position tracking, with the nominal load torque applied at $t = 5s$, of PMSM.
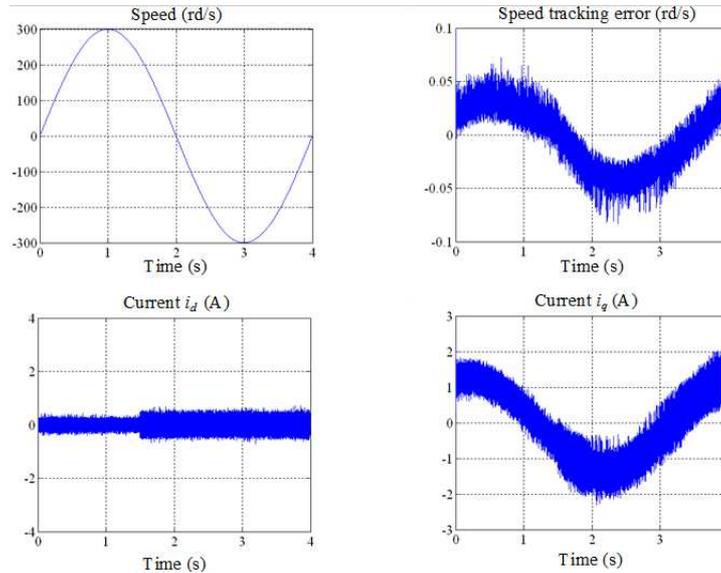

Figure 5: Speed tracking, with parameter variations at $t = 1.5s$, of PMSM.

5.3.1 Simulation results

The motor under tests is characterized by: $L_d = L_q = 0.0121\ H$, $\Phi_f = 0.013\ Wb$, $J = 0.0001\ kg \cdot m^2$, $F = 0.00005\ km^2/s$, $R_s = 3.4\ \Omega$ and $\Omega_n = 300\ rd/s$. The current-controlled inverter is fed by $70V$ continue voltage assumed constant. The proposed schema of the adaptive fuzzy controller is tested by simulation to perform the position and speed tracking of PMSM. The values of the control coefficients, which enabled us to
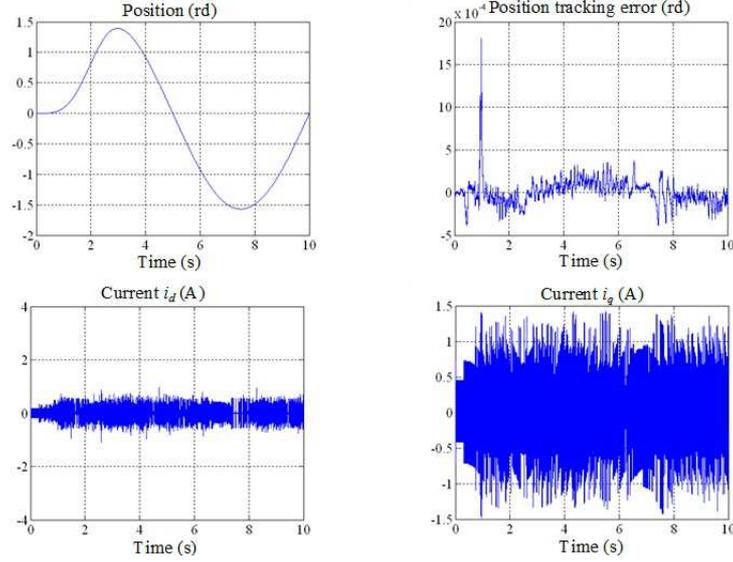
Figure 6: Position tracking, with parameter variations at $t = 1s$, of PMSM.

obtain satisfactory results, are collected in Table 1 and Table 2.

$\eta_{f\Omega}$	$\eta_{g\Omega}$	k_Ω	α	λ
50	50	0.5	5	10.85

Table 1: Speed control coefficients.

$\eta_{f\beta}$	$\eta_{g\beta}$	k_β	α	$k_{1\beta}$	$k_{2\beta}$
50	50	0.5	5	121	3694

Table 2: Position control coefficients.

The desired trajectories are imposed as:

$$\Omega_{ref} = 300 \sin\left(\frac{\pi}{2}t\right); \quad \beta_{ref} = \frac{\pi}{2}\left(1 - e^{-0.1 \cdot t^3}\right) \sin\left(\frac{\pi}{5}t\right).$$

Figure 3 and Figure 4 give respectively the responses of the speed and position control in the case where the nominal load torque is applied. It appears that, the speed and position follow respectively their reference, the disturbance rejection is fast and the stator current is aligned on the q axis (i.e. $i_d = 0$). The robustness of the trajectories tracking of speed and position is carried out in the presence of the electric parameters variations. Indeed, these variations impose an increase of 100% of the stator resistances, a reduction of 50% of stator inductances and a reduction of 10% of the inductor flux. The obtained responses are represented in Figure 5 and Figure 6.

In spite of the application of these strong parameters variations the tracking speed and position are maintained with a weak tracking error. This shows clearly that this fuzzy adaptive control has the capacity to respond quickly to the evolution of the parameters and to their variations.

6   Conclusion

In this paper, we proposed an indirect fuzzy adaptive control scheme for a class of unknown nonlinear systems. In this controller, the adaptive fuzzy control law ensures the convergence of tracking errors and boundedness of the fuzzy logic system parameters. The application of the developed method is carried out for a permanent magnet synchronous motor. The obtained simulation results show that this adaptive control fuzzy law maintains the tracking errors in an acceptable interval with feasible control inputs in the presence of hard parameters variations or external disturbances.

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Designing a Compensator Based on Extended Kalman Filter for Elimination of Noise and Delay Effect in Tracking Loop

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Abstract: This paper introduces a new control structure in tracking loop. In this new structure a position sensor has been used to eliminate noise and delay effect of tracking sensor. The EKF based Compensator estimates the desired error and provides the controller with the appropriate control signal with respect to gimbal position that is reported by the position sensor and the tracking loop error (output of tracking sensor). We have shown that in this new structure, notwithstanding the existence of noise and delay, designing of the tracking loop controller can be done and EKF based Compensator is practical for compensating noise and delay effect of the tracking sensor. Indeed a considerable feature of the presented control structure is that designing method of the controller is simple and utilizing of the delay and noise compensator has significantly reduced complication of the design.

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1   Introduction

Tracking loop is designed in order to locate the position of mobile objects in space. Indeed the main issue is designing and developing a system that is able to report position of the target object in relation to a certain reference.

Sensors with radar or optic nature are applicable for reporting the position of the (target) object. Such sensors usually have a limited field of view with regard to the technology that is used in designing tracking sensors [1-2].

Hence for reporting position of an object that is moving in space, a servomechanism system is needed to align field of view of the tracking sensor with the target position so that the target is always inside of the field of view. In other words, if the tracking sensor (with limited field of view) is set over a two degree of freedom gimbal that is tracking the target, we can be aware of precise position of the target.

The loop that is closed for target tracking and extraction of its position data is known as tracking loop. Usually it is desirable that tracking error (measured amounts by the tracking sensor) tends toward zero [3]. A simple diagram of tracking loop can be supposed as below figure [4-5].
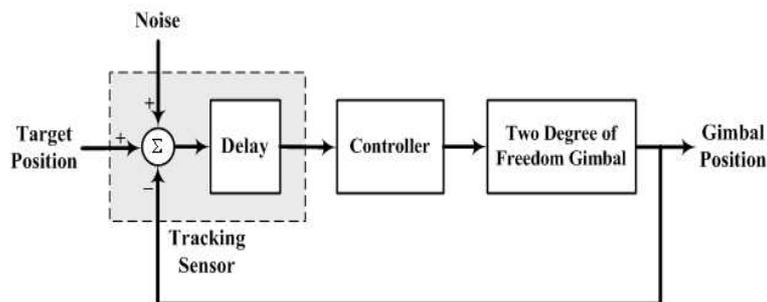
Figure 1: A simple diagram of tracking loop.

As it is shown in the above figure, it has been tried to simply model the issues of tracking precision and delay due to the process of the tracking sensor that are fundamental characteristics influencing on the tracking loop performance.

Tracking error will be different dependent on the nature of the sensor quantitatively or qualitatively. Radar sensors often have errors of stochastic nature that arise from thermal noise, glint noise and so on and impact on tracking precision [3-11]. Also Angle measurement delay causes stability margin of the tracking loop to be reduced and so the performance band width of the tracking system. Radar sensors usually have a time varying process delay.

Many texts have been written on designing an appropriate controller for reducing time delay. Most of the authors have dealt with the control problem of the time-delay system via designing robust-based controllers and predictor-based controllers. In systems that use the robust controller the desired performance with respect to time-delay is guaranteed in specified changes [7-8]. In predictor-based controllers it has been tried to remove the time delay effect from the closed-loop system [9-11].

In this paper, a structure like Figure 2 is offered for compensating the noise and delay effect of the tracking sensor. In this new structure we have used the output of position
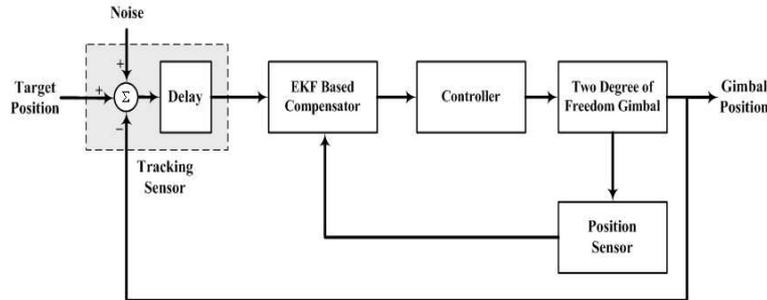
Figure 2: Proposed tracking loop structure.

sensor that ordinarily is used in gimbal frame, for eliminating the delay effect. In the structure, an EKF based compensator not only eliminates effect of the input noise, but also it is useful in eliminating the effect of delay by processing the reported data of the position sensor.

It should be noted that in designing of the mentioned compensator two main assumptions related to the position sensor have been regarded. First, the precision of the position sensor is equal to that of the tracking sensor, and the second is that the mentioned sensor is free delay (that is not far away from the reality).

The tracking loop controller is also a usual controller that can be designed despite the quantity of the delay and noise of the tracking sensor. Indeed first we can design the controller based on the desirable conditions regardless of the delay and noise that enter into the tracking loop and then with the aid of the EKF based compensator, compensate bad effects of the noise and delay inside of the tracking loop. Thus designing of the controller accomplishes with less complication that is regarded as an excellent advantage.

In the following, we are going to provide the necessary arrangements for utilizing the position sensor beside the tracking sensor in the tracking loop by deriving the kinematic relationship between them. Then we will describe the structure of the EKF based compensator by reviewing the mechanism of EKF and rewriting the related formulas. After that we will analyze the obtained results in order to evaluate the performance of this new control structure, in two cases of constant delay and variable time delay.

2 Deriving Kinematic Relationship Between Tracking Sensor and Position Sensor

Tracking sensors usually report angular position of the target in relation to their bore sight. Likewise, position sensor is able to measure angular position of the bore sight of the tracking sensor in relation to a definite reference. The purpose of this section is to determine the relationship between the reported angles by these sensors.

For this purpose, first consider the kinematic related to the tracking of a substance in space as shown in Figure 3.

In this figure, three coordinate systems are defined that are target frame (T), tracking sensor frame (S) and a frame that is attached to the ground (I). Indeed if tracking gimbal has been fixed on the ground, this ground attached frame can be considered as a reference for tracking gimbal position.
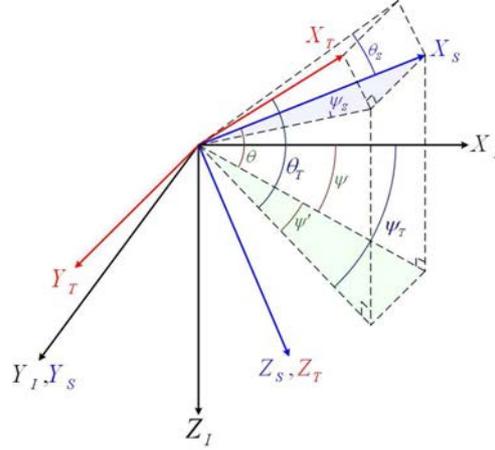
Figure 3: Target kinematic related to definite coordinate systems.

According to the figure, angles θ_T and ψ_T are elevation and azimuth position of target in relation to position sensor. Angles θ_S and ψ_S also are measured quantities of the tracking sensor. Now if we show elevation and azimuth angles of the bore sight of the tracking sensor in relation to gimbal position sensor with θ and ψ, the purpose is determining ψ' (image of ψ_R in the $X_I Y_I$ plane) as an explanation of ψ_S in relation to gimbal position sensor.

Without reducing anything from the issue, we will assume ψ and θ_S is zero. In this condition the transform matrix of I frame to T frame is obtained with rotation about Y_I by an amount of θ and then rotation about Z_S by an amount of ψ_S. This process can be shown mathematically according to formula (1)

$$\{I\} \xrightarrow{C_{Y_I}(\theta)} \{S\} \xrightarrow{C_{Z_S}(\psi_S)} \{T\}. \tag{1}$$

Now with the calculation of the rotation matrix we can determine target position relative to I frame.

$$
{}_T^I C = \begin{bmatrix} C\psi_S\,C\theta & -S\psi_S\,C\theta & S\theta \\ S\psi_S & C\psi_S & 0 \\ -C\psi_S\,S\theta & S\psi_S\,S\theta & 0 \end{bmatrix}, \; {}^T R_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
$$
$$
\Rightarrow \; {}^I R_T = \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} = {}_T^I C\,{}^T R_T = \begin{bmatrix} C\psi_S\,C\theta \\ S\psi_S \\ -C\psi_S\,S\theta \end{bmatrix}. \tag{2}
$$

On the other hand according to Figure 3 and above formula we have:

$$tg\psi' = \frac{y_I}{x_I} = \frac{1}{C\theta}\,tg\psi_S. \tag{3}$$

Now whereas ψ_S is a small value, ψ' and thereupon the relationship between the measured values of the tracking and position sensors can be approximated as follows

$$\psi' \approx \frac{1}{C\theta}\psi_S \; \Rightarrow \; \begin{cases} \psi_T = \psi + \frac{1}{C\theta}\psi_S, \\ \theta_T = \theta + \theta_S. \end{cases} \tag{4}$$

According to above formula, the values measured by the position sensor and tracking sensor relate to each other via $C\theta$ factor.

3   Explanation of Compensator Structure

This section allocated to explanation of structure of EKF based compensator. Before anything else first, we will introduce EKF and then we will describe performance of the compensator with mathematical formulas with regard to equations governing the issue.

3.1   A review of EKF equations

EKF is used as a recursive estimate algorithm for nonlinear systems. Indeed because of the ability of this algorithm in Gaussian-nonlinear filtering, EKF is used extensively. This filter is based on linearization of measurements and alteration of models by using Taylor series expansion. In other words, EKF is an estimation algorithm in which the nonlinear system is linearized firstly, and then recursive Kalman filter equations are used for time update [6].

In order to illustrate the issue, consider related equations in a nonlinear system as follows

$$
\begin{aligned}
&x(k) = f_{k-1}\left(x(k-1), u(k-1), w(k-1)\right),\\
&y(k) = h_k\left(x(k), v(k)\right),\\
&w(k) \sim (0, Q(k)),\\
&v(k) \sim (0, R(k)),
\end{aligned}
\tag{5}
$$

where $x(k)$ is state variable, $u(k)$ is process input, $y(k)$ is process output and $v(k)$ is output noise. Meanwhile process and output noise are assumed Gaussian with zero mean and $Q(k)$ and $R(k)$ covariance.

Since Kalman filter is going to minimize the expected value of the estimated square error, the initial quantification of the state estimation $\hat{x}^+(0)$ and error covariance $P^+(0)$ is as follows [6]

$$
\begin{aligned}
&\hat{x}^+(0) = E\left(x(0)\right),\\
&P^+(0) = E\left[\left(x(0) - \hat{x}^+(0)\right)\left(x(0) - \hat{x}^+(0)\right)^T\right].
\end{aligned}
\tag{6}
$$

With this initial quantification, the first step that is estimation based on premeasured values is done. The equations related to this step are as follows [6]

$$
\begin{aligned}
&F(k-1) = \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}^+(k-1)},\\
&L(k-1) = \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}^+(k-1)},\\
&P^-(k) = F_{k-1}P^+(k-1)F_{k-1}^T + L_{k-1}Q(k-1)L_{k-1}^T,\\
&\hat{x}^-(k) = f_{k-1}(\hat{x}^+(k-1), u(k-1), 0).
\end{aligned}
\tag{7}
$$

In this formula $F(k-1)$ and $L(k-1)$ are Jacobians of the process model and $P(k)$ is estimation error covariance. Then in the second step, update is done based on the present

output of measured samples according to the following equations [6]

$$
\begin{aligned}
H(k) &= \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}^-(k)}, \\
M(k) &= \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}^-(k)}, \\
K(k) &= P^-(k)H^T(k)\left(H(k)P^-(k)H^T(k) + M(k)R(k)M^T(k)\right)^{-1}, \\
\hat{x}^+(k) &= \hat{x}^-(k) + K(k)\left[y(k) - h_k\left(\hat{x}^-(k), 0\right)\right], \\
P^+(k) &= (I - K(k)H(k))P^-(k).
\end{aligned}
\tag{8}
$$

In this equation $H(k)$ and $M(k)$ are Jacobians of the output model and $K(k)$ is filter gain.

3.2   Description of compensator performance with the aid of mathematical equations

Due to the fact that the considered issue is tracking a substance in three dimension space, related equations and state variables are as follows

$$
\begin{cases}
\ddot{x} = a_x, \\
\ddot{y} = a_y, \\
\ddot{z} = a_z,
\end{cases}
\Rightarrow
\begin{cases}
x_1 = x, \\
x_2 = y, \\
x_3 = z, \\
x_4 = \dot{x}, \\
x_5 = \dot{y}, \\
x_6 = \dot{z}.
\end{cases}
\tag{9}
$$

That a_x, a_y and a_z are components of target acceleration in direction of Cartesian coordination axes.

Thus state space model is as formula (10)

$$
\begin{bmatrix}
\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6
\end{bmatrix}
=
\begin{bmatrix}
 & & & 1 & 0 & 0 \\
 & 0 & & 0 & 1 & 0 \\
 & & & 0 & 0 & 1 \\
0 & 0 & 0 & & & \\
0 & 0 & 0 & & 0 & \\
0 & 0 & 0 & & &
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6
\end{bmatrix}
+
\begin{bmatrix}
0 \\ 0 \\ 0 \\ a_x \\ a_y \\ a_z
\end{bmatrix}.
\tag{10}
$$

For digitalizing based on sampling time T, we have:

$$
\begin{aligned}
\dot{x}_i &= \frac{x_i(k+1) - x_i(k)}{T} \quad (i = 1, ..., 6) \\
&= x_{i+3}(k) \quad\quad\quad\ (i = 1, 2, 3).
\end{aligned}
\tag{11}
$$

Thus with regard to the above equation, discrete state space equations are obtained as formula (12).

$$
\underbrace{\begin{bmatrix}
x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1)
\end{bmatrix}}_{x_{k+1}}
=
\underbrace{\begin{bmatrix}
1 & 0 & 0 & T & 0 & 0 \\
0 & 1 & 0 & 0 & T & 0 \\
0 & 0 & 1 & 0 & 0 & T \\
 & & & 1 & 0 & 0 \\
 & 0 & & 0 & 1 & 0 \\
 & & & 0 & 0 & 1
\end{bmatrix}}_{A}
\underbrace{\begin{bmatrix}
x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k)
\end{bmatrix}}_{x_k}
+
\underbrace{\begin{bmatrix}
0 \\ 0 \\ 0 \\ T\,a_x \\ T\,a_y \\ T\,a_z
\end{bmatrix}}_{B\,u_k}.
\tag{12}
$$

The output of the system in Cartesian coordination based on the state variables is rewritable as follows

$$
y_k = \begin{bmatrix} R \\ \psi \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2(k) + x_2^2(k) + x_3^2(k)} \\ tg^{-1}\left(\frac{x_2(k)}{x_1(k)}\right) \\ tg^{-1}\left(\frac{x_3(k)}{\sqrt{x_1^2(k)+x_2^2(k)}}\right) \end{bmatrix} = h(x_k). \tag{13}
$$

Due to the fact that control structure of the tracking loop in both channels of elevation and azimuth is similar and for simplifying the equations, from now on we just regard azimuth channel equations. Azimuth channel related to the control structure has been shown in Figure 4. In this figure $C(z)$ and $G(z)$ are discrete model of controller and process sequently. Meanwhile Δ is function of delay. Also error of the tracking sensor modeled with $N(0,\sigma)$ that is Gaussian noise with zero mean and variance σ. $C\theta$ is the relationship factor between position sensor and tracking sensor that is obvious in formula (4). Also $1/C\theta$ is considered to compensate $C\theta$ effect. It is very clear that the only difference in elevation channel is absence of these factors.
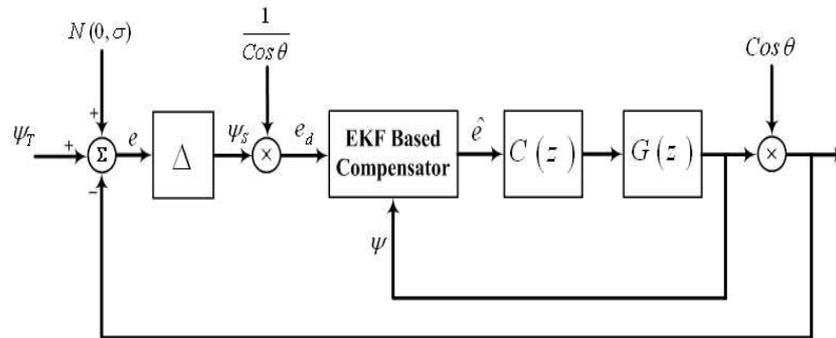
Figure 4: Mathematical model of tracking loop structure.

Thus consider output of the tracking loop in azimuth channel as formula (14).

$$
\psi(k) = tg^{-1}\left(\frac{x_2(k)}{x_1(k)}\right) = h_\psi\left(x(k)\right). \tag{14}
$$

According to this equation, the output Jacobian matrix follows as

$$
H_\psi(k) = \left.\frac{\partial h_\psi}{\partial x}\right|_{x(k)} = \begin{bmatrix} -\frac{x_2(k)}{x_1^2(k)+x_2^2(k)} & \frac{x_1(k)}{x_1^2(k)+x_2^2(k)} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{15}
$$

Due to the fact that designing of the tracking loop often is done to reset to zero the steady state error for ramp input [3], in the following we will assume that the tracking target flies with constant speed (acceleration equals to zero).

Now for implementing EKF based compensator, *F(k-1)* and *H(k)* matrices in equation sets of (7) and (8) are placed as follows.

$$
H(k) = H_\psi(k), \quad F(k-1) = A. \tag{16}
$$

Also relative to the nature and the value of the loop input noise, appropriate values for $Q(k), R(k), M(k)$ and $L(k-1)$ can be selected.

But with regard to the mathematical model of the control loop of the azimuth channel and according to Figure 4 and formula (8), $y(k)$ is assumed as below in which n states number of the delay samples that is considered in compensator structure

$$y(k) = e_d(k) + \psi(k - n). \tag{17}$$

Indeed n is a designing parameter of compensator that depends on the nature of the delay of the tracking sensor, appropriate value should be chosen for it. In the next section we will see how the selection of this parameter is done.

On the other hand, according to discrete state space model we have:

$$\begin{aligned} x(k+1) &= A\,x(k) \\ \Rightarrow x(k+n) &= A^n\,x(k). \end{aligned} \tag{18}$$

Thus in update stage, we will define $\tilde{\psi}$ as follows.

$$\begin{aligned} \tilde{X} &= A^n\,\hat{x}^+(k) \\ \Rightarrow \tilde{\psi} &= tg^{-1}\left(\frac{\tilde{X}_2}{\tilde{X}_1}\right). \end{aligned} \tag{19}$$

Therefore estimated error follows formula (20)

$$\hat{e} = \tilde{\psi} - \psi. \tag{20}$$

With applying this estimation error to controller we expect that tracking loop delay and noise effect decrease prominently.

4   Performance Evaluation of the New Control Structure in Tracking Loop

In this section we are going to evaluate the performance of the control structure that was introduced in the last section. In first stage we assume that the value of tracking sensor delay is constant and definite and we will evaluate the ability of the EKF based compensator in compensating the given delay and noise according to this assumption. In the following for creating the condition closer to the reality, we will assume that sensor delay is a time variable and then we will evaluate the performance of the control structure with this assumption.

The evaluations are done with the aid of the simulation in which certain values are allocated to parameters and components of the tracking loop. In the case of discrete model $C(z)$ and $G(z)$ with $T=0.025$ sec we have the following function

$$C(z)\,G(z) = \frac{0.055z^2 + 0.006z - 0.05}{z^3 - 2.58z^2 + 2.17z - 0.58}. \tag{21}$$

Also regarding the tracking loop input noise we assume that

$$\begin{aligned} Q(k) &= (60T_c)^2 \left[\begin{array}{cc} \frac{1}{3}T_c^2\,I_{3\times3} & \frac{1}{2}T_c\,I_{3\times3} \\ \frac{1}{2}T_c\,I_{3\times3} & I_{3\times3} \end{array} \right], \\ R(k) &= 2\ ,\ L(k-1) = 1\ ,\ M(k) = 0. \end{aligned} \tag{22}$$

In this formula T_c is regarded equal to $5T$ ($T_c = 5T$).

4.1 Study for constant delay

If constant delay value is supposed d sample, according to Figure 4, delay model Δ can be considered as formula (23)

$$\Delta = T\,d. \tag{23}$$

The above equation shows that in formulas (17) and (19) parameter n must be set equal to d and EKF based compensator equation is updated each T second. For *d=4* and *T=0.025 sec*, tracking error is estimated according to Figure 5 by means of compensator. As you can see in this figure, EKF based compensator is carefully accomplished compensating for noise and delay effect.
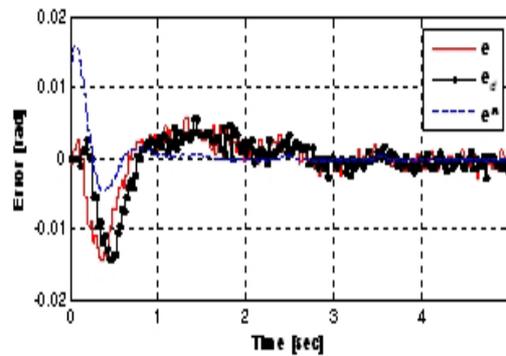
Figure 5: Comparing error signal of tracking loop with compensator estimation error.

Therefore system response to ramp input (target with constant speed) will be according to Figure 6. In this figure, three cases are supposed to show the value of compensator power. First, to the to tracking loop, that a controller is designed for *0.1sec.* delay is applied, that according to Figure 6-(a) means the loop is unstable. In Figure 6-(b) and Figure 6-(c) ideal loop (delay free) behavior is compared with delay and compensator applied loop. Figure 6-(b) shows response to ramp input in two cases and in comparison with another.

Also in Figure 6-(c) output error (difference between input and output of loop) is shown. According to the presented results you see that the designed compensator, besides catering proper stability margin, could eliminate noise effect truly.

4.2 Study for time variable delay

Now assume the considered delay is variable and the following function

$$\Delta\left(r\right) = r\left(k - \delta\left(k\right)\right) \ \ , \ \delta\left(k\right) \in \left\{D_{\min}, ..., D_{\max}\right\}. \tag{24}$$

According to this formula the upper and lower bound of variable delay is assumed definite and is shown with D_{max} and D_{min} .

In this situation a conservative selection for parameter n is the upper bound of delay. Thus, in compensator considered equation (25)

$$n = D_{\max}. \tag{25}$$
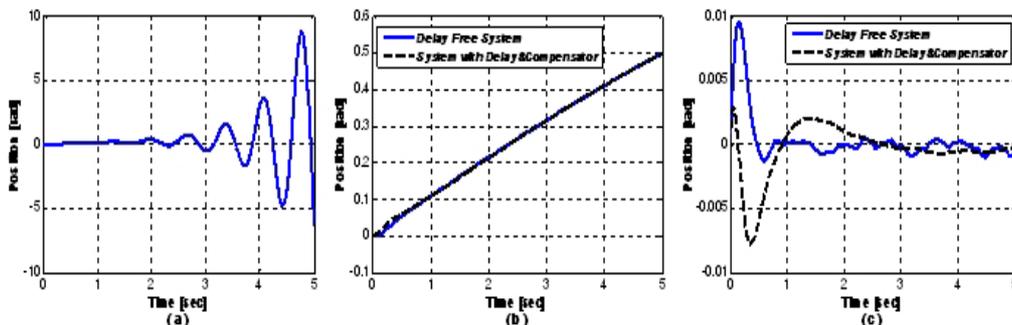
Figure 6: Simulation results for constant delay.

Another offer for selection of n in compensator related equations is as following.

$$n = \left[\frac{D_{\max} + D_{\min}}{2} \right], \tag{26}$$

where $[]$ is integer fraction. With this choice, it is adequate to consider integer fraction of mean of delay bounds in designing of compensator. Then, the estimated error will be going to zero and tracking loop output is going to desired value in steady state.

Studies have shown that the second method is better for selection of n and obtains better response. In Figure 7 the estimated error signal is compared in both of the above methods. According to this figure in the second method the estimated error has a lower oscillation magnitude.
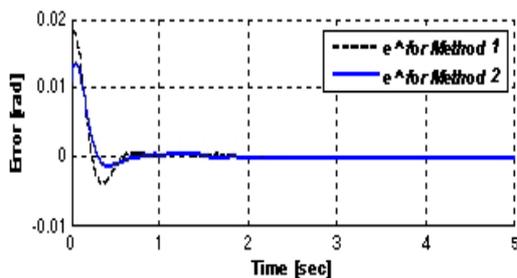
Figure 7: Comparison of the estimated error signal for n selection in both methods.

In Figure 8 tracking loop simulation results in variable delay case and for $D_{min}=2$, $D_{max}=5$ and thus $n=3$ according to formula (26) have been shown. In Figure 8-(a) you see that for applied delay, output is unstable whereas Figures 8-(b) and 8-(c) show that with the aid of EKF based compensator a near to ideal response is obtained. However the difference between input and output of tracking loop in compensator applying state

depreciates in longer time, but in less than *3 sec* with well precision goes to zero. This subject can be seen clearly in Figure 8-(c).
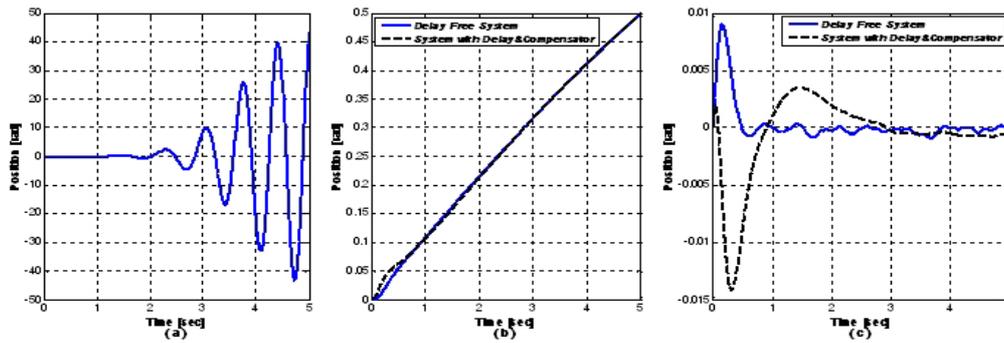
Figure 8: Simulation results for variable time delay.

Another remarkable note is that designed control structure is responsible for acceleration input tracking. In fact, however usually tracking loop is designed for constant speed input, but ability of acceleration input tracking is a significant and notable subject.
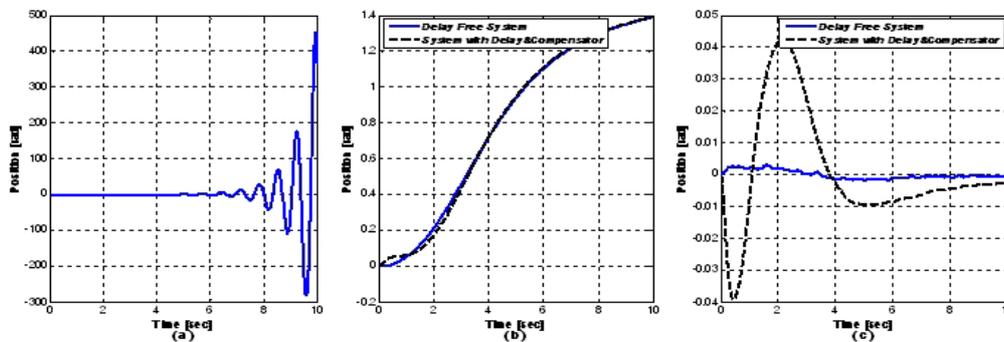
Figure 9: Simulation results for acceleration input.

Figure 9 depicts results for applying an acceleration input to tracking loop. Likewise Figure 9-(a) shows that with existence of tracking sensor delay, loop is unstable. But Figure 9-(b) shows that existence of EKF based compensator does not reduce ability of controller in acceleration input tracking. In Figure 9-(c), with comparison of error signal in ideal state (delay free) and the state when new control structure is applied in addition to ordinary controller, this fact is illustrated clearly. According to this figure, we can say that existence of compensator due to longer transient state response is practical for eliminating noise and delay effect of tracking sensor.

5   Conclusion

In this paper a new control structure is presented to compensate destructive effect of delay and noise on performance of tracking loop. We show that in the introduced structure an EKF based compensator with access to position of two degree of freedom gimbal via a position sensor, is able to estimate tracking loop error as effect of tracking sensor delay and noise reduce on output.

A significant feature of the presented structure is that notwithstanding the existence of noise and delay, and with assumption of ideal condition, designing of the tracking loop controller can be done. Then we utilize EKF based compensator to eliminate noise and delay effect without reduction in performance of tracking loop design.

In two sections the performance of the mentioned control structure for constant and time varying delay is evaluated. The obtained results confirmed that EKF based compensator, beside providing appropriate stability margin, has a good performance in reduction of delay and noise effect.

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The Structure of the Solution of Delay Differential Equations with One Unstable Positive Equilibrium

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Abstract: This paper studies the equation $\dot{x}(t) = -g(x(t)) + f(x(t - \tau))$ with one trivial equilibrium and only one unstable positive equilibrium. For a class of linear initial values, two sufficient conditions are established to guarantee that the corresponding solutions converge to the trivial equilibrium and the positive equilibrium respectively. All solutions, with the exception of two equilibria, are divided into three classes according to their eventual tendency. The first class solutions are strictly greater than 0 ultimately and converge to it; the second class ones are strictly greater than the positive equilibrium ultimately and converge to it; the third class solutions oscillate about the positive equilibrium up and down and converge to it. Furthermore, the existence of the third class of solutions is determined. Numerical simulations are given to illustrate the main results.

Keywords: *delay differential equations; convergence; oscillatory solution; attractive region; equilibrium.*

Mathematics Subject Classification (2010): 34K05; 34K60; 92B05.

1 Introduction

Delay differential equations are always the research focus of mathematicians dealing with theory of functional differential equations and scientists applying the theory to practical problems. It is not difficult to found a variety of application of delay differential equation in several fields of natural science such as viscoelasticity, mechanics, models for nuclear reactors, distributed networks, heat flow, neural networks, combustion theory, interaction of species, microbiology, learning models, epidemiology, physiology see e.g. [9, 11, 15, 22].

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The introduction of delays makes a much richer range of phenomena possible, however, it also causes sever mathematical complications.

Even with consideration of the simplest-looking equation

$$\dot{x}(t) = -\mu x(t) + f(x(t-1)), \quad \mu > 0, \tag{1}$$

where $\mu > 0$, just as pointed out by T. Krisztin in paper [17], the dynamics of equation (1) can be very rich. In the monotone feedback case, the properties of equation (1) have been explored comprehensively, including the local and global dynamics, structure of the global attractor, existence and properties of periodic orbit (see [1,7,16,19–21,29]). In the case of a non-monotone feedback function $f(x)$, the dynamics can be very complicated. Though a majority of literatures study the property of equation (1) with nonmonotone feedback (see [2,4,5,8,12]). In general there are still much unknown. One important result comes from paper [26], in which G. Röst and J. Wu showed the existence of the global attractor and gave the bounds of the attractor in the case when $f(x)$ is a general unimodal function, which is the situation for the well-known Nicholson's blowflies equation [10] and the Mackey-Glass equation [25].

Recently C. Huang, Z. Yang, T. Yi and X. Zou [14] investigated the following model

$$\dot{x}(t) = -g(x(t)) + f(x(t-\tau)), \tag{2}$$

where g and f are continuous on $\mathbb{R}^+$ with the values in $\mathbb{R}^+$, and satisfy (F_1) and (F_2).

(F_1) $g(x)$ is strictly increasing on $\mathbb{R}^+$, $\dot{g}(x) > 0$, $g(0) = 0$ and $\lim\limits_{x \to +\infty} g(x) = +\infty$.

(F_2) $f(\xi) > 0$ for all $\xi > 0, f(0) = 0$, and there exists a unique $\xi_0 > 0$, such that $f'(\xi) > 0$ if $0 < \xi < \xi_0$, $f'(\xi_0) = 0 = f'(0)$ and $f'(\xi) < 0$ if $\xi > \xi_0$, furthermore, there also exists a unique $0 < \xi_1 < \xi_0$ such that $f''(\xi) > 0$ if $0 < \xi < \xi_1$, $f''(\xi_1) = 0$ and $f''(\xi) < 0$ if $\xi_1 < \xi < \xi_0$, and $\lim\limits_{\xi \to +\infty} f(\xi) = 0$.

Evidently, the famous Allee-type model with $f(x) = ax^n e^{-x}$ in [23] satisfies conditions (F_1) and (F_2) when $n > 1$. The distinction between the models in [14] and [26] is whether $f'(0) = 0$, it is this property that makes equation (2) have different properties such as multiple equilibria or one unstable positive equilibrium. For equation (2), Huang et.al determined the invariant intervals and the multistability properties of equilibria of equation (2). When the system has only one positive equilibrium, their results imply that the positive equilibrium is unstable, but the equilibria 0 and x_1 have their own local attractive region.

The dynamics of delay differential equations can be affected by many factors. For example, delays can cause the loss of stability and induce oscillations, periodic solutions and the occurrence of Hopf bifurcations [28, 30]. Many papers consider the effect of increasing mortality and harvesting on equation (2) see e.g. [3,6,8,18,28]. E. Liz and G. Gost [24] obtained some new results for equation (2) with negative Schwarzian derivative. However, the role of initial condition on the property of solutions is not considered. In finite dimensional systems, it is direct to judge the property of orbits by initial value. As we known, systems generated by delay differential equations are infinite dimensional, the previous results can not be applied here. Thus we pay attention to the role of the initial value.

Motivated by the above discussion, we mainly explore the property of the solutions of equation (2) with a class of initial value. Throughout the paper we assume that equation (2) fulfills conditions (F_1'), (F_2) and (F_3).

(F_1') $g(x)$ is strictly increasing on $\mathbb{R}^+$, $\dot{g}(x) > 0$, $\ddot{g}(x) \leq 0$, $g(0) = 0$ and $\lim\limits_{x \to +\infty} g(x) = +\infty$.

(F_3) $f(x)$ and $g(x)$ have only one positive intersection point denoted by x_1.

Obviously, Losota's model fulfills (F_1'), (F_2) and (F_3) if $\mu = a(\frac{n-1}{e})^{n-1}$ and $n > 1$. For equation (2) with $\tau = 1$, considering the wide variety of the initial value, we mainly investigate the convergence of the solution with linear initial value $\phi(s) = ks + x_1 + h$ for $-1 \leq s \leq 0$ and $0 < h \leq x_1$. Since $\phi(s)$ is not in the attractive region of 0 or x_1 for some k and h, the results in [14] can not be directly applied to deduce the convergence of the corresponding solutions. Here, we establish two sufficient conditions to ensure that the corresponding solutions converge to 0 and x_1 respectively. Furthermore, we give more detailed description and classification of the solutions of (2). The paper divides all solutions of (2) with the exception of two equilibria into three categories according to their way of convergence. The first class solutions are strictly greater than 0 ultimately and converge to it; the second class ones are strictly greater than x_1 ultimately and converge to it; the third class solutions oscillate about x_1 up and down and converge to it. Moreover, we show the existence of the third class of solutions.

Consider one example of (2) in the form

$$\dot{x}(t) = -\mu x(t) + a_1 x(t-1)^2 e^{-a_2 x(t-1)}, \tag{3}$$

where parameters satisfy $\mu = \frac{a_1}{a_2 e}$ and the two equilibria are 0 and $\frac{1}{a_2}$. We further explore the convergence of the solution with linear initial value $\phi(s) = \frac{1}{a_2}(s+1+h)$ for $-1 \leq s \leq 0$ and $0 < h < 1$, which is across the attractive region of the two equilibria. When the information about g and f is more specific, the wider range of h can be obtained to guarantee the same convergence.

The rest of the paper is organized as follows. Section 2 mainly presents the basic definitions and introduces some relevant results. Section 3 explores the convergence of the solution with a class of linear initial value. Section 4 divides all the solutions into three classes according to their eventual tendency and shows the existence of the oscillatory solution. In Section 5 an example is given, for a class of linear initial value, more specific relationships are put forward between the location of the line and the eventual tendency of the corresponding solution. In Section 6 numerical simulations are given to illustrate the main results in Sections 4 and 5. In the final section we make a conclusion and present some unsolved issues.

2 Preliminary

Let $C = C([-\tau, 0], \mathbb{R})$ be the Banach space of continuous functions with the norm given by

$$\|\phi\| = \max_{-\tau \leq s \leq 0} |\phi(s)| \quad \text{for any} \quad \phi \in C.$$

The Banach space C contains the cone as follows,

$$C^+ = \{\phi \in C : \phi(s) \geq 0, -\tau \leq s \leq 0\}.$$

The usual notations $<, \leq$ and $\ll$ can be used to denote the various relations on C generated by the positive cone C^+. In particular, $\phi \leq \psi$ holds if $\phi(s) \leq \psi(s)$ for $-\tau \leq s \leq 0$; $\phi < \psi$ holds if $\phi(s) \leq \psi(s)$ and $\phi(s) \neq \psi(s)$ for $-\tau \leq s \leq 0$; $\phi \ll \psi$ holds if $\phi(s) < \psi(s)$ for $-\tau \leq s \leq 0$. Likewise, there are order relations $>, \geq$ and $\gg$.

Therefore, we can define the order intervals $[\phi, \psi] := \{\xi \in C : \phi \le \xi \le \psi\}$ if $\phi \le \psi$ and $(\phi, \psi) := \{\xi \in C : \phi \ll \xi \ll \psi\}$ if $\phi \ll \psi$.

Solutions of equation (2) are determined by the initial value $x(\theta) = \phi(\theta)$, where $-\tau \le \theta \le 0, \phi \in C$, and we use the universal symbol x_t to denote the state of the system at time t, where $x_t(\theta) = x(t + \theta)$ for $-\tau \le \theta \le 0$. Then $x_0(\theta) = \phi(\theta)$ and $x_t(0) = x(t)$. In order to emphasize the dependence of a solution on the initial value ϕ, we write $x_t(\phi)$ or $x(t, \phi)$. Equation (2) generates a semiflow Φ on C given by

$$\begin{aligned} \Phi : \mathbb{R}^+ \times C &\to C, \\ (t, \phi) &\mapsto x_t(\phi) := \Phi_t(\phi). \end{aligned}$$

We also define the functional $\lambda : C \to \mathbb{R}$ by

$$\lambda(\phi) := -g(\phi(0)) + f(\phi(-\tau)), \forall \phi \in C.$$

So equation (2) can be written as $\dot{x}(t) = \lambda(x_t)$. If $x \in \mathbb{R}$ we denote by x^* the element of C which takes the value x on $[-\tau, 0]$. The set of equilibria for (2) is then given by $E = \{\phi \in C | \phi \equiv x, \lambda(x^*) = 0\}$.

The positive orbit of ϕ is denoted by $O^+(\phi) = \{\Phi_t(\phi) : t \ge 0\}$. The $\omega(\phi)$ of $\phi \in C^+$ is defined by

$$\omega(\phi) = \cap_{t \ge 0} \overline{\cup_{s \ge t} \Phi_s(\phi)}.$$

i.e, whenever $\psi \in \omega(\phi)$ there exists an infinite sequence t_n such that $\lim_{t_n \to \infty} \Phi_{t_n}(\phi) = \psi$.

The semiflow Φ is said to be monotone provided $\Phi_t(\phi) \le \Phi_t(\psi)$ whenever $\phi \le \psi$ and $t \ge 0$. Φ is called strongly monotone on C^+ if it is monotone and $\Phi_t(\phi) \ll \Phi_t(\psi)$ whenever $\phi < \psi$ and $t > 0$. Φ is said to be eventually strong monotone if it is monotone and whenever $\phi < \psi$ there exists $t_0 > 0$ such that $\Phi_{t_0}(\phi) \ll \Phi_{t_0}(\psi)$. Φ is said to be strongly order-preserving on C^+ if it is monotone and whenever $\phi < \psi$ there exists open subsets $U, V \subset C^+$ and $t_0 > 0$ such that $\phi \in U, \psi \in V$ and $\Phi_{t_0}(U) \le \Phi_{t_0}(V)$. For more knowledge related to functional equations, please refer to [13] and [27].

Proposition 2.1 *[27] If Φ is eventually strongly monotone, then it is strongly order-preserving.*

Here, one main result from Huang et.al [14] about system (2) is as follows.

Theorem 2.1 *[14] For the system (2) fulfilling (F_1'), (F_2) and (F_3) (see Figure 1), $x_0^* = 0^*$ is asymptotically stable and x_1^* is unstable, there exists a heteroclinic orbit $x(t)$, which connects x_0^* and x_1^*. Furthermore, the following results hold:*
(1) $\lim_{t \to \infty} x(t, \phi) = 0$ for $\phi \in [0^*, x_1^*]/\{x_1^*\}$;
(2) $\lim_{t \longrightarrow \infty} x(t, \psi) = x_1$ for $\psi \in [x_1^*, \eta^*]$, where $\eta = \hat{f}^{-1}(f(x_1))$, $\hat{f}$ denotes the restriction of f to the interval $[\xi_0, \infty)$.
(3) The order interval $[0^*, (g^{-1}f(\xi_0))^*]$ is invariant and globally attractive on C^+.

Based on the results of Theorem 2.1, it is clear that $[0^*, \eta_0^*]$ is also invariant and globally attractive on C^+ if $\eta_0 \in [g^{-1}f(\xi_0), \eta]$. If we denote by $\tilde{f}$ the restriction of f to the interval $[0, \xi_0]$, then $\tilde{f}$ is non-decreasing on this interval. The invariance of $[0^*, \xi_0^*]$ and the monotonicity of $\tilde{f}$ guarantee that the semiflow generated by (2) is monotone on $[0^*, \xi_0^*]$ [27].
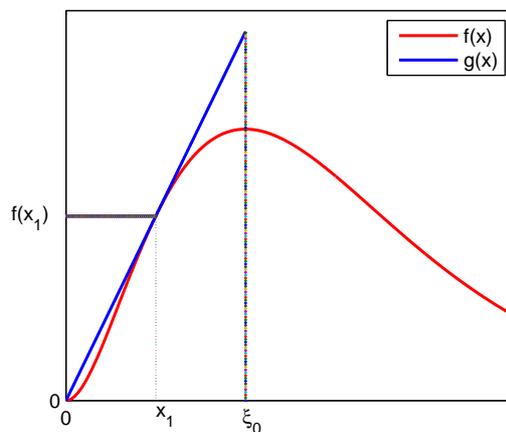
Figure 1: Schematic of equation (2) with one positive equilibrium.

3   The Convergence of the Solution with a Class of Linear Initial Value

This section mainly explores the convergence of the solution of (2) with $\tau = 1$ and linear initial value $\phi(s) = ks + x_1 + h$ for $-1 \leq s \leq 0$, where $0 < k \leq \xi_0$ and $0 < h \leq \xi_0 - x_1$. Evidently, ϕ does not completely locate in the attractive region $[0^*, x_1^*]/\{x_1^*\}$ or $[x_1^*, (g^{-1}f(\xi_0))^*]$ for some k and h. Given k, h will determine the convergence of the solution $x(t, \phi)$. Before presenting the principal results, we need to introduce some definitions and explanations. First define a new function $G(x)$, $G(x) = \frac{g(x)}{x}$ if $x > 0$ and $G(0) = g'(0)$. It is easy to check that $G(x)$ is continuous, non-increasing and $G(x) > 0$ by (F_1'). The fact that $\dot{g}(x) > 0$ and $\ddot{g}(x) \leq 0$ implies the following definition is meaningful.

$$\delta_1 = \min_{0 \leq x \leq 2x_1 + \xi_0} g'(x) = g'(2x_1 + \xi_0) > 0 \qquad \text{and} \qquad \delta_2 = \max_{0 \leq x \leq 2x_1 + \xi_0} g'(x) = g(0) > 0.$$

Therefore, $0 < \delta_1 \leq G(x) = \frac{g(x)}{x} \leq \delta_2$ for $0 \leq x \leq 2x_1 + \xi_0$.

From (2) it follows that

$$\dot{x}(t) + x(t)G(x(t)) = f(x(t - \tau)). \tag{4}$$

By multiplying both sides of (4) by $e^{\int_0^t G(x(s))ds}$ and then by integrating from $n\tau$ to t, the solutions of (2) can be obtained for ordinary differential equations on successive intervals of length τ.

$$x(t) = x(n\tau)e^{-\int_{n\tau}^t G(x(s))ds} + \int_{n\tau}^t e^{\int_t^s G(x(\omega))d\omega} f(x(s - \tau))ds \tag{5}$$

with $n \in \mathbb{N}, n\tau \leq t \leq (n+1)\tau$.

For the initial value $\phi(s) = ks + x_1 + h$, in order to ensure that $\lim_{t \to \infty} x(t, \phi) = x_1$, we give the following hypothesis denoted by (H_0).
Suppose that

$$(H_0) \qquad \xi_0 - x_1 \geq h \geq h_{up} = \frac{-(k - \delta_2 k + \delta_2 x_1) + \sqrt{(k - \delta_2 k + \delta_2 x_1)^2 + 4\delta_2^2 x_1 k}}{2\delta_2}.$$

Theorem 3.1 *Given $\phi(s) = ks + x_1 + h$ for $-1 \leq s \leq 0$, where $k \leq \xi_0$. If h fulfills* (H_0), *then* $\lim\limits_{t \to \infty} x(t, \phi) = x_1$.

Proof. If $k \leq h \leq \xi_0 - x_1$, then $\phi(s) \in [x_1^*, \xi_0^*]$, it is clear that $\lim\limits_{t \to \infty} x(t, \phi) = x_1$. If $h \leq k$, set $s_1 = -\frac{h}{k}$ and $t_1 = 1 + s_1$. When $0 \leq t \leq t_1$, from (5) it follows that

$$
\begin{aligned}
x(t, \phi) &= e^{-\int_0^t G(x(s,\phi))ds} x(0, \phi) + \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} f(x(s-1, \phi))ds \\
&\geq e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + f(\phi(-1)) \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} ds \\
&= e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + f(\phi(-1)) \int_0^t \frac{1}{G(x(s,\phi))} de^{\int_t^s G(x(\omega,\phi))d\omega} \\
&\geq e^{-\int_0^t G(x(s,\phi))ds}(x_1 + h) + \frac{f(\phi(-1))}{\delta_2}(1 - e^{-\int_0^t G(x(s,\phi))ds}) \\
&= (x_1 + h - \frac{f(\phi(-1))}{\delta_2})e^{-\int_0^t G(x(s,\phi))ds} + \frac{f(\phi(-1))}{\delta_2} \\
&\geq (x_1 + h - \frac{f(\phi(-1))}{\delta_2})(1 - \delta_2 t_1) + \frac{f(\phi(-1))}{\delta_2} \\
&= (1 - \delta_2 t_1)(x_1 + h) + f(\phi(-1))t_1 \\
&\geq (1 - \delta_2 t_1)(x_1 + h).
\end{aligned}
$$

If h satisfies $(1 - \delta_2 t_1)(x_1 + h) \geq x_1$, then $\xi_0 \geq x(t, \phi) \geq x_1$ for $s_1 \leq t \leq t_1 = s_1 + 1$. By Theorem 2.1, there holds $\lim\limits_{t \to \infty} x(t, \phi) = x_1$.

Therefore, it suffices to show that

$$(1 - \delta_2 t_1)(x_1 + h) \geq x_1, \tag{6}$$

$$i.e. \qquad \frac{\delta_2}{k}h^2 + (1 - \delta_2 + \frac{\delta_2 x_1}{k})h - \delta_2 x_1 \geq 0.$$

It is easy to check that (6) holds if h fulfills (H_0). By the fact that $k \leq \xi_0 - x_1$ and Theorem 2.1, there holds $\lim\limits_{t \to \infty} x(t, \phi) = x_1$.

In the following we consider the convergence of the solution of (2) with initial value $\psi(s) = x_1 s + x_1 + h$. First, we introduce some hypotheses as follows.

Suppose that

(H_1) $1 \leq \dfrac{\delta_2}{\delta_1} = \alpha < 2$

and

(H_2) $(\alpha - 1)(e^{\delta_2} + \dfrac{\delta_2}{4} - 1) - (1 - e^{-\frac{\delta_2}{2}})(1 - \dfrac{\alpha}{2}) < 0.$

Set

$$\underline{h_1} = \frac{x_1(\sqrt{1 + (2-\alpha)^2(1 - e^{-\frac{\delta_1}{2}})^2} - 1)}{(2 - \alpha)(1 - e^{-\frac{\delta_1}{2}})}$$

and

$$\underline{h_2} = \frac{2x_1(\sqrt{\triangle_2} - (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1))}{3\delta_2},$$

where

$$\triangle_2 = (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)^2 - 3\delta_2((\alpha - 1)(e^{\delta_2} - 1) - \frac{\delta_2}{4}).$$

Set

$$\underline{h_3} = \frac{x_1(\sqrt{\triangle_3} - (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1))}{2((1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1))},$$

where

$$\begin{aligned}
\triangle_3 &= (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)^2 - 4((1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1)) \\
&\quad ((\alpha - 1)(e^{\delta_2} + \frac{\delta_2}{4} - 1) - (1 - e^{-\frac{\delta_2}{2}})(1 - \frac{\alpha}{2})).
\end{aligned}$$

Another hypothesis is as follows.

(H_3) $\qquad\qquad\qquad\qquad h \leq h_{down} \triangleq \min\{\underline{h_1}, \underline{h_2}, \underline{h_3}\}.$

Theorem 3.2 *Given $\psi(s) = x_1 s + x_1 + h$ for $-1 \leq s \leq 0$. If (H_1)–(H_3) hold, then $\lim\limits_{t \to \infty} x(t, \psi) = 0$.*

Proof. Set $s_1 = -\frac{h}{x_1}$ and $s_0 = \frac{s_1 - 1}{2}$, $t_0 = 1 + s_0$ and $t_1 = 1 + s_1$. The aim of the following part is to show $x(t, \psi) \leq x_1$ for $t_0 \leq t \leq 1 + t_0$. Here we divide the proof into four points.

(1) When $0 \leq t \leq t_0$, from (5) it follows that

$$\begin{aligned}
x(t, \psi) &= e^{-\int_0^t G(x(s,\psi))ds} x(0, \psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1, \psi))ds \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + f(\psi(s_0)) \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} ds \\
&= e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + f(\psi(s_0)) \int_0^t \frac{1}{G(x(s,\psi))} de^{\int_t^s G(x(\omega,\psi))d\omega} \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + \frac{\delta_2 \psi(s_0)}{\delta_1}(1 - e^{-\int_0^t G(x(s,\psi))ds}) \\
&\leq (x_1 + h)(1 - \frac{\alpha}{2})e^{-\delta_1 t} + \frac{\alpha}{2}(x_1 + h).
\end{aligned}$$

Therefore,

$$\begin{aligned}
x(t_0, \psi) &\leq (x_1 + h)(1 - \frac{\alpha}{2})e^{-\delta_1 t_0} + \frac{\alpha}{2}(x_1 + h) \qquad\qquad\qquad (7) \\
&\leq (x_1 + h)(1 - \frac{\alpha}{2})(2(e^{-\frac{\delta_1}{2}} - 1)t_0 + 1) + \frac{\alpha}{2}(x_1 + h).
\end{aligned}$$

Let

$$(x_1 + h)(1 - \frac{\alpha}{2})(2(e^{-\frac{\delta_1}{2}} - 1)t_0 + 1) + \frac{\alpha}{2}(x_1 + h) \leq x_1, \qquad\qquad (8)$$

i.e. $\quad (1 - \frac{\alpha}{2})(1 - e^{-\frac{\delta_1}{2}})\frac{h^2}{x_1} + h - (1 - \frac{\alpha}{2})(1 - e^{-\frac{\delta_1}{2}})x_1 \leq 0.$

Since $h \leq h_{down} \leq \underline{h_1}$, it is easy to check that (8) holds. Therefore, $x(t_0, \psi) \leq x_1$.

(2) When $t_1 \leq t \leq 1$, from (5) it follows that

$$
\begin{aligned}
x(t, \psi) &= e^{-\int_0^t G(x(s,\psi))ds} x(0, \psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1, \psi))ds \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + f(\psi(s_0)) \int_0^{t_0} e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\
&\quad + f(\psi(s_1)) \int_{t_0}^{t_1} e^{\int_t^s G(x(\omega,\psi))d\omega}ds + f(\psi(0)) \int_{t_1}^t e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1 + h) + \frac{\alpha}{2}(x_1 + h)(e^{-\int_{t_0}^t G(x(s,\psi))ds} - e^{-\int_0^t G(x(s,\psi))ds}) \\
&\quad + \alpha x_1(e^{-\int_{t_1}^t G(x(s,\psi))ds} - e^{-\int_{t_0}^t G(x(s,\psi))ds}) + \alpha(x_1 + h)(1 - e^{-\int_{t_1}^t G(x(s,\psi))ds}) \\
&= e^{-\int_0^t G(x(s,\psi))ds}((1 - \frac{\alpha}{2})(x_1 + h) + \frac{\alpha}{2}(h - x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \\
&\quad - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds}) + \alpha(x_1 + h).
\end{aligned}
$$

Since

$$
\begin{aligned}
&(1 - \frac{\alpha}{2})(x_1 + h) + \frac{\alpha}{2}(h - x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds} \\
&= x_1(1 - \frac{\alpha}{2} - \frac{\alpha}{2}e^{\int_0^{t_0} G(x(s,\psi))ds}) + h(1 - \frac{\alpha}{2} - \frac{\alpha}{2}e^{\int_0^{t_1} G(x(s,\psi))ds}) \\
&\quad + \frac{\alpha h}{2}(e^{\int_0^{t_0} G(x(s,\psi))ds} - e^{\int_0^{t_1} G(x(s,\psi))ds}) \\
&< 0,
\end{aligned}
$$

there holds

$$
\begin{aligned}
x(t, \psi) &\leq e^{-\delta_2}((1 - \frac{\alpha}{2})(x_1 + h) + \frac{\alpha}{2}(h - x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} & (9) \\
&\quad - \alpha h e^{\int_0^{t_1} G(x(s,\psi))ds}) + \alpha(x_1 + h) \\
&\leq e^{-\delta_2}((1 - \frac{\alpha}{2})(x_1 + h) + \frac{\alpha}{2}(h - x_1)e^{\delta_1 t_0} - \alpha h e^{\delta_1 t_1}) + \alpha(x_1 + h) & (10) \\
&\leq e^{-\delta_2}((1 - \frac{\alpha}{2})(x_1 + h) + \frac{\alpha}{2}(h - x_1)(1 + \delta_1 t_0) - \alpha h(1 + \delta_1 t_1)) + \alpha(x_1 + h) \\
&= e^{-\delta_2}(\frac{3\delta_2}{4x_1}h^2 - (\alpha + \frac{\delta_2}{2} - 1)h - (\alpha + \frac{\delta_2}{4} - 1)x_1) + \alpha(x_1 + h).
\end{aligned}
$$

Let

$$
e^{-\delta_2}(\frac{3\delta_2}{4x_1}h^2 - (\alpha + \frac{\delta_2}{2} - 1)h - (\alpha + \frac{\delta_2}{4} - 1)x_1) + \alpha(x_1 + h) \leq x_1, \qquad (11)
$$

$$
\text{i.e.} \quad \frac{3\delta_2}{4x_1}h^2 + (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)h + ((\alpha - 1)(e^{\delta_2} - 1) - \frac{\delta_2}{4})x_1 \leq 0.
$$

By (H_2), there holds $(\alpha - 1)(e^{\delta_2} - 1) - \frac{\delta_2}{4} < 0$. Since $h \leq h_{down} \leq \underline{h_2}$, (11) holds, i.e. $x(t, \psi) \leq x_1$ for $t_1 \leq t \leq 1$.

(3) When $t_0 \leq t \leq t_1$, from (5) it follows that

$$
\begin{aligned}
x(t,\psi) &= e^{-\int_0^t G(x(s,\psi))ds} x(0,\psi) + \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1+h) + f(\psi(s_0))\int_0^{t_0} e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\
&\quad + f(\psi(s_1))\int_{t_0}^t e^{\int_t^s G(x(\omega,\phi))d\omega}ds \\
&\leq e^{-\int_0^t G(x(s,\psi))ds}(x_1+h+\frac{\alpha}{2}(x_1+h)(e^{\int_0^{t_0} G(x(s,\psi))ds}-1) \\
&\quad + \alpha x_1 e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1 \\
&= e^{-\int_0^t G(x(s,\psi))ds}((x_1+h)(1-\frac{\alpha}{2})+\frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1.
\end{aligned}
$$

If $(x_1+h)(1-\frac{\alpha}{2})+\frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \geq 0$, there holds

$$
\begin{aligned}
x(t,\psi) &\leq e^{-\int_0^{t_0} G(x(s,\psi))ds}((x_1+h)(1-\frac{\alpha}{2})+\frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1 \\
&\leq (x_1+h)(1-\frac{\alpha}{2})e^{-\delta_1 t_0}+\frac{\alpha}{2}(h+x_1). \quad\quad (12)
\end{aligned}
$$

As we have proved that the right-hand part of (12) (i.e.inequaltiy (7)) is less than x_1, which means that $x(t,\psi) \leq x_1$ for $t_0 \leq t \leq t_1$.

If $(x_1+h)(1-\frac{\alpha}{2})+\frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds} \leq 0$, there holds

$$
x(t,\psi) \leq e^{-\delta_2}((x_1+h)(1-\frac{\alpha}{2})+\frac{\alpha}{2}(h-x_1)e^{\int_0^{t_0} G(x(s,\psi))ds}) + \alpha x_1. \quad\quad (13)
$$

Subtracting the right-hand part of inequalities (9) from that of (13) gives

$$
\alpha h e^{-\delta_2+\int_0^{t_1} G(x(s,\psi))ds} - \alpha h \leq 0.
$$

Since the right-hand part of (9) is less than x_1, then $x(t,\psi) \leq x_1$ for $t_0 \leq t \leq t_1$.

(4) When $1 \leq t \leq 1+t_0$, from (5) it follows that

$$
\begin{aligned}
x(t,\psi) &= e^{-\int_1^t G(x(s,\psi))ds} x(1,\psi) + \int_1^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-1,\psi))ds \\
&\leq e^{-\int_1^t G(x(s,\psi))ds} x(1,\psi) + f(\psi(0))\int_1^t e^{\int_t^s G(x(\omega,\psi))d\omega}ds \\
&\leq e^{-\int_1^t G(x(s,\psi))ds} x(1,\psi) + \alpha(x_1+h)(1-e^{-\int_1^t G(x(s,\psi))ds}) \\
&\leq (x(1,\psi)-\alpha(x_1+h))e^{-\delta_2 t_0} + \alpha(x_1+h).
\end{aligned}
$$

Since (10) implies that

$$
x(1,\psi) \leq e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h)+\frac{\alpha}{2}(h-x_1)e^{\delta_1 t_0}-\alpha h e^{\delta_1 t_1}) + \alpha(x_1+h),
$$

there holds

$$
\begin{aligned}
x(t,\psi) \ \leq \ & e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h)e^{-\delta_2 t_0} + \frac{\alpha}{2}(h-x_1)e^{\delta_1 t_0 - \delta_2 t_0} - \alpha h e^{\delta_1 t_1 - \delta_2 t_0}) + \alpha(x_1+h) \\
\leq \ & e^{-\delta_2}((1-\frac{\alpha}{2})(x_1+h)(2(e^{-\frac{\delta_2}{2}}-1)t_0+1) + \frac{\alpha}{2}(h-x_1)(1+\delta_1 t_0 - \delta_2 t_0) \\
& -\alpha h(1+\delta_1 t_1 - \delta_2 t_0)) + \alpha(x_1+h) \qquad (b = 1 - e^{-\frac{\delta_2}{2}}) \\
= \ & e^{-\delta_2}((b(1-\frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1))\frac{h^2}{x_1} - (\alpha + \frac{\delta_2}{2} - 1)h \\
& +((\alpha-1)(\frac{\delta_2}{4}-1) - b(1-\frac{\alpha}{2}))x_1) + \alpha(x_1+h).
\end{aligned}
$$

Letting the right-hand part of the above inequality be less than x_1, by equivalent transformation, we have

$$
\begin{aligned}
(b(1-\frac{\alpha}{2}) - \frac{\alpha}{4}(\delta_2 - 3\delta_1))\frac{h^2}{x_1} + (\alpha e^{\delta_2} - \alpha - \frac{\delta_2}{2} + 1)h \\
+((\alpha-1)(e^{\delta_2} + \frac{\delta_2}{4} - 1) - b(1-\frac{\alpha}{2}))x_1 \leq 0.
\end{aligned}
\tag{14}
$$

Based on (H_2) and the fact that $h \leq h_{down} \leq \underline{h_3}$, (14) holds, i.e. $x(t,\psi) \leq x_1$ for $1 \leq t \leq 1 + t_0$.

As a conclusion, $x(t,\psi) \leq x_1$ for $t_0 \leq t \leq 1 + t_0$ if (H_1)–(H_3) hold. By Theorem 2.1, there holds $\lim\limits_{t\to\infty} x(t,\psi) = 0$.

4   The Classification of Solutions and the Existence of Oscillatory Solution

This section is devoted to divide all solutions of (2) into three categories according to their eventual tendency and show the existence of oscillatory solution. First the definition of oscillatory solutions is formulated as follows.

Definition 4.1 [9, 11, 12] The solution $x(t,\phi)$ of (2) with initial value $\phi \in C^+$ is said to be oscillatory about $\bar{x}$, if there exists a sequence $\{\xi_n\} \to \infty$ as $n \to \infty$ such that $x(\xi_n,\phi) = \bar{x}$ and $x(t,\phi) - \bar{x}$ simultaneously has positive and negative values in (ξ_n, ξ_{n+1}) for $n = 1, 2, 3, \cdots$. Otherwise, $x(t,\phi)$ is said to be non-oscillatory about $\bar{x}$.

For the systems of delay differential equations, there are various ways to define oscillation. For instance, in [9, 11] the real function x is said to be oscillatory about zero if x has arbitrarily large zeros. Here the definition is stricter than those mentioned above. Consider $x(t) = \sin t + 2$, which is oscillatory about 1 according to the concept in [9, 11]. However, it is non-oscillatory about 1 according to Definition 4.1.

Theorem 4.1 If $x(t,\phi)$ is oscillatory about x_1, then $\lim\limits_{t\to\infty} x(t,\phi) = x_1$.

Proof. First we assert that the semiflow generated by (2) is eventually strongly monotone on $[0^*, \xi_0^*]$, then by Proposition 2.1, it is strongly order-preserving. For any $\phi, \psi \in [0^*, \xi_0^*]$, if $\phi < \psi$, there exists a $t_0 \in [0, \tau]$ such that $x(t_0, \phi) < x(t_0, \psi)$. Otherwise, $x(t,\phi) = x(t,\psi)$ for $0 \leq t \leq \tau$.

From (5) it follows immediately that

$$
\begin{aligned}
x(t,\phi) - x(t,\psi) &= e^{-\int_0^t G(x(s,\phi))ds} x(0,\phi) - e^{-\int_0^t G(x(s,\psi))ds} x(0,\psi) \\
&\quad + \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} f(x(s-\tau,\phi))ds \\
&\quad - \int_0^t e^{\int_t^s G(x(\omega,\psi))d\omega} f(x(s-\tau,\psi))ds,
\end{aligned}
$$

$$
i.e. \quad 0 = \int_0^t e^{\int_t^s G(x(\omega,\phi))d\omega} (f(x(s-\tau,\phi)) - f(x(s-\tau,\psi)))ds
$$

with $0 \le t \le \tau$. By the fact that $x(t,\phi) \le x(t,\psi) \le \xi_0$ and $f(x)$ is strictly increasing on $[0,\xi_0]$, there holds $x(s-\tau,\phi) = x(s-\tau,\psi)$ for $0 \le s \le \tau$, i.e. $\phi = \psi$, which contradicts the assumption.

Replacing $n\tau$ in (5) by t_0, we have

$$
x(t) = e^{-\int_{t_0}^t G(x(s))ds} x(t_0) + \int_{t_0}^t e^{\int_t^s G(x(\omega))d\omega} f(x(s-\tau))ds
$$

with $t_0 \le t \le t_0 + \tau$. By the fact that $x(t,\phi) \le x(t,\psi) \le \xi_0$, $f(x)$ is strictly increasing on $[0,\xi_0]$ and $G(x)$ is non-increasing, there holds $f(x(t,\phi)) \le f(x(t,\psi))$ and $G(x(t,\phi)) \ge G(x(t,\psi))$. Furthermore,

$$
\begin{aligned}
x(t,\phi) - x(t,\psi) &\le e^{-\int_{t_0}^t G(x(s,\phi))ds} x(t_0,\phi) - e^{-\int_{t_0}^t G(x(s,\psi))ds} x(t_0,\psi) \\
&< 0 \qquad \text{whenever} \qquad t_0 \le t \le t_0 + \tau,
\end{aligned}
$$

i.e. for $\phi < \psi$, there exists a $t_1 = t_0 + \tau$ such that $x_{t_1}(\phi) \ll x_{t_1}(\psi)$, then the semiflow generated by (2) is eventually strongly monotone. Therefore, it is strongly order-preserving on $[0^*,\xi_0^*]$.

If $x(t,\phi)$ is oscillatory about x_1 with $0^* \le \phi < \xi_0^*$, by Theorem 3.7 in [27], we have $\omega(\phi) <$ or $= \omega(\xi_0^*) = \{x_1^*\}$. If the former holds, the compactness of $\overline{O^+(\phi)}$ suggests that $\omega(\phi)$ is nonempty, compact, invariant and connected, so $0^* \in \omega(\phi)$. Obviously, $0^* \le \omega(\phi)$, Corollary 2.4 in [27] implies that $\omega(\phi) = \{0^*\}$, which contradicts the oscillation of $x(t,\phi)$. Thus $\omega(\phi) = \{x_1^*\}$, and $x_{t_k}(\phi) \to x_1^*$ if and only if $x_{t_k}(\xi_0^*) \to x_1^*$. The fact that $x(t,\xi_0^*) \to x_1$ implies $x(t,\phi) \to x_1$.

Based on the global attractivity of $[0^*,\xi_0^*]$, the solution $x(t,\phi)$ with $\phi \in C^+$ oscillating about x_1 will eventually tend to x_1.

Proposition 4.1 *Given any* $\phi \in C^+ \setminus \{0^*, x_1^*\}$*, only one of the following results holds:*

(1) $x(t,\phi)$ enters $(0,x_1)$ ultimately , thus $\lim\limits_{t\to\infty} x(t,\phi) = 0$.

(2) $x(t,\phi)$ enters $(x_1,\xi_0]$ ultimately, thus $\lim\limits_{t\to\infty} x(t,\phi) = x_1$.

(3) $x(t,\phi)$ oscillates about x_1, thus $\lim\limits_{t\to\infty} x(t,\phi) = x_1$.

Proof. Assume, by contradiction, that there exists $\phi \in C^+ \setminus \{0^*, x_1^*\}$, a T and a sequence $\{\xi_n\} \to \infty$ as $n \to \infty$ such that one of the following two cases holds.
(a) $x(\xi_n,\phi) = x_1$ for $n = 1,2,3,\cdots$, and $x(t,\phi) \ge x_1$ for $t > T$.
(b) $x(\xi_n,\phi) = 0$ for $n = 1,2,3,\cdots$, and $0 \le x(t,\phi) < x_1$ for $t > T$.

Assume that case (a) holds. Choose a sufficiently large $\xi_n > T + 2\tau$ and denote it by ξ_{n_0+2} such that $x(\xi_{n_0+2}, \phi) = x_1$. Note that $x(t, \phi)$ eventually enters $[x_1, \xi_0]$ and the derivative of $x(t, \phi)$ is continuous, then $\dot{x}(\xi_n, \phi) = 0$ for $n = 1, 2, 3, \cdots$. Therefore,

$$
\begin{aligned}
0 = \dot{x}(\xi_{n_0+2}, \phi) &= -g(x(\xi_{n_0+2}, \phi)) + f(x(\xi_{n_0+2} - \tau, \phi)), \\
\text{i.e.} \quad g(x_1) &= f(x(\xi_{n_0+2} - \tau, \phi)),
\end{aligned}
$$

which implies that $x(\xi_{n_0+2} - \tau, \phi) = x_1$. Here, denote $\xi_{n_0+2} - \tau$ by ξ_{n_0+1} and $\xi_{n_0+2} - 2\tau$ by ξ_{n_0} for brevity. Then they satisfy the following coditions.
(a1) $x(\xi_{n_0+i}, \phi) = x_1$ where $i = 0, 1, 2$.
(a2) $\dot{x}(\xi_{n_0+i}, \phi) = 0$ where $i = 0, 1, 2$.

Let ξ_{n_0+1} be an initial point of integration in (5), then

$$
x(t, \phi) = e^{-\int_{\xi_{n_0+1}}^{t} G(x(s,\phi))ds} x(\xi_{n_0+1}, \phi) + \int_{\xi_{n_0+1}}^{t} e^{\int_{t}^{s} G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds \quad (15)
$$

with $\xi_{n_0+1} \le t \le \xi_{n_0+1} + \tau = \xi_{n_0+2}$.
Replacing t by ξ_{n_0+2} in (15) gives

$$
x(\xi_{n_0+2}, \phi) = x_1 e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds} + \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds, \quad (16)
$$

i.e.

$$
\begin{aligned}
x_1\left(1 - e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds}\right) &= \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} f(x(s - \tau, \phi))ds \\
&= \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} \frac{f(x(s - \tau, \phi))}{G(x(s, \phi))} G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds.
\end{aligned}
$$

Note that

$$
\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds = 1 - e^{-\int_{\xi_{n_0+1}}^{\xi_{n_0+2}} G(x(s,\phi))ds},
$$

by equivalent transformation, (16) becomes

$$
0 = \int_{\xi_{n_0+1}}^{\xi_{n_0+2}} \left(\frac{f(x(s - \tau, \phi))}{x_1 G(x(s, \phi))} - 1\right) G(x(s, \phi)) e^{\int_{\xi_{n_0+2}}^{s} G(x(\omega,\phi))d\omega} ds. \quad (17)
$$

By the fact that $\xi_0 \ge x(t, \phi) \ge x_1$ for $t > T$, $f(x)$ increases on $[x_1, \xi_0]$ and $G(x)$ is non-increasing, there holds

$$
\frac{f(x(s - \tau, \phi))}{x_1 G(x(s, \phi))} \ge \frac{f(x_1)}{x_1 G(x(s, \phi))} = \frac{G(x_1)}{G(x(s, \phi))} \ge 1.
$$

Equality in (17) holds if and only if $x(s - \tau, \phi) = x_1$ and $G(x(s, \phi)) = G(x_1)$ for $\xi_{n_0+1} \le s \le \xi_{n_0+2}$. Induction implies $\phi = x_1$, which contradicts the assumption. Similarly, case (b) does not hold. So far the proof is completed.

In the following part, attention will be paid to show the existence of the oscillatory solution. Here consider the initial value $\phi(s) = ks + b$ for $-\tau \le s \le 0$, where $0 < k \le \min\{\frac{x_1}{\tau}, \frac{\xi_0 - x_1}{\tau}\}$ and $x_1 \le b \le \xi_0$. Given k, the parameter b will determine the eventual tendency of the solution $x(t, \phi)$. In order to stress the dependence of the eventual tendency of $x(t, \phi)$ on the parameter b, we abbreviate $\phi(s)$ to ϕ^b.

Proposition 4.2 *Given $\phi \in C^+$, if $\lim\limits_{t\to\infty} x(t,\phi) = 0$, then there exists a $\delta > 0$ such that $\lim\limits_{t\to\infty} x(t,\psi) = 0$ for any $\psi \in O(\phi,\delta)$.*

Proof. If $\lim\limits_{t\to\infty} x(t,\phi) = 0$, then there exists a $T_0 > 0$ such that $x(t,\phi) < x_1$ for $t \in [T_0, T_0 + 2\tau]$. Set $l = \max\limits_{T_0 \le t \le T_0 + 2\tau} x(t,\phi)$, $\epsilon = (x_1 - l)/3$ and $T = T_0 + 2\tau$, by the continuous dependence of solutions on the initial value [13, 15, 27], there exists a $\delta(\epsilon, T) > 0$ such that $|x(t,\phi) - x(t,\psi)| < \epsilon$ for $0 \le t \le T$ and any $\psi \in O(\phi,\delta)$. This means that $x(t,\psi) < x_1$ for $T_0 \le t \le T_0 + 2\tau$. Therefore $\lim\limits_{t\to\infty} x(t,\psi) = 0$ by Theorem 2.1.

Remark 4.1 From the above proposition it is easy to get the following conclusion. If $b = b_0$, i.e. the initial value $\phi(s) = ks + b_0$ for $-\tau \le s \le 0$, and $\lim\limits_{t\to\infty} x(t,\phi^{b_0}) = 0$, then there exists a $\delta > 0$ such that $\lim\limits_{t\to\infty} x(t,\phi^b) = 0$ for any $b \in O(b_0,\delta) \cap [x_1, \xi_0]$.

Remark 4.2 The above proposition can not be generalized to $\lim\limits_{t\to\infty} x(t,\phi) = x_1$, i.e. if $\lim\limits_{t\to\infty} x(t,\phi) = x_1$, it does not provide that there exists a $\delta > 0$, such that $\lim\limits_{t\to\infty} x(t,\psi) = x_1$ for any $\psi \in O(\phi,\delta)$. This case can be confirmed in the following part. The following proposition is a special case.

Proposition 4.3 *If $b = \xi_0$, i.e, the initial value $\phi(s) = ks + \xi_0$ for $-\tau \le s \le 0$, then there exists a $\delta > 0$ such that $\lim\limits_{t\to\infty} x(t,\phi^b) = x_1$ for any $b \in [\xi_0 - \delta, \xi_0]$.*

Proof. Note that $\phi^{\xi_0} \in [x_1^*, \xi_0^*]$, the argument of Theorem 4.1 implies that there exists a T_1 such that $x(t,\phi) > x_1$ for $t \ge 0$. Let $T_2 = T_1 + 2\tau$, $l = \min\limits_{0 \le t \le 2\tau} x(t,\phi^{\xi_0})$ and $\epsilon = (l - x_1)/3$, by the continuous dependence of solutions on the initial value [13,15,27], there exists a $\delta(\epsilon, T_2) > 0$, when $b \in [\xi_0 - \delta, \xi_0]$, $0 \le x(t,\phi^{\xi_0}) - x(t,\phi^b) < \epsilon$ for $0 \le t \le T_2$. This means that $x(t,\phi^b) > x_1$ for $T_1 \le t \le T_2$, so $\lim\limits_{t\to\infty} x(t,\phi^b) = x_1$ by Theorem 2.1.

Theorem 4.2 *There exists an initial value ϕ such that $x(t,\phi)$ oscillates about x_1.*

Proof. Consider the linear initial value $\phi(s) = ks + b$ for $-\tau \le s \le 0$. We restrict b to $[x_1, \xi_0]$. Then there must exist a $b_0 \in (x_1, \xi_0)$ such that $x(t,\phi^b)$ oscillates about x_1. Otherwise, given $b \in [x_1, \xi_0]$, by Theorem 4.1, Propositions 4.2 and 4.3, there exists a δ such that $\lim\limits_{t\to\infty} x(t,\phi^{b'}) = \lim\limits_{t\to\infty} x(t,\phi^b) = 0$ or x_1 for any $b' \in O(b,\delta)$. This contradicts the finiteness of b, which is restricted to $[x_1, \xi_0]$. Thus such a b_0 exists, i.e. the oscillatory solution exists.

In the following section, denote

$$B := \{ \quad b \quad | \quad x_1 \le b \le \xi_0, \quad \lim\limits_{t\to\infty} x(t,\phi^b) = x_1\}, \quad \beta = \inf B,$$

$$A := \{ \quad b \quad | \quad x_1 \le b \le \xi_0, \quad \lim\limits_{t\to\infty} x(t,\phi^b) = 0\}, \quad \alpha = \sup A.$$

Proposition 4.4 *The solution $x(t,\phi^\alpha)$ oscillates about x_1, $\lim\limits_{t\to\infty} x(t,\phi) = 0$ for $\phi \in [0^*, \phi^\alpha)$ and $\lim\limits_{t\to\infty} x(t,\phi) = x_1$ for $\phi \in [\phi^\alpha, \xi_0^*]$.*

Proof. If $x(t, \phi^\alpha)$ does not oscillate about x_1, then it will eventually enter the domain $(x_1, \xi_0]$ or $(0, x_1)$. If it enters $(0, x_1)$, by Proposition 4.2, there exists a δ such that $\lim_{t \to \infty} x(t, \phi^b) = 0$ for $b \in O(\alpha, \delta)$, which contradicts the definition of α. Similarly, it will not eventually enter the domain $(x_1, \xi_0]$. Therefore $x(t, \phi^\alpha)$ oscillates about x_1. The second part is clear by the monotonicity of the semiflow generated by (2).

In the same way, we can immediately get the following result.

Corollary 4.1 *The solution $x(t, \phi^\beta)$ oscillates about x_1 and $\alpha = \beta$.*

Remark 4.3 For system (2) with $\tau = 1$ and the initial value $\phi(s) = x_1(s + 1 + h)$ in Section 3, according to Theorem 4.2, there exists a h_0 such that $x(t, \phi)$ oscillates about x_1 and then converges to it if $h = h_0$, $\lim_{t \to \infty} x(t, \phi) = 0$ if $0 \le h < h_0$ and $\lim_{t \to \infty} x(t, \phi) = x_1$ if $h_0 \le h \le \xi_0 - x_1$.

5   Example

This section mainly investigates model (3)

$$\dot{x}(t) = -\mu x(t) + a_1 x(t-1)^2 e^{-a_2 x(t-1)},$$

where parameters satisfy $\mu = \frac{a_1}{a_2 e}$ and the two equilibria are 0 and $\frac{1}{a_2}$. Their attractive regions are $[0^*, (\frac{1}{a_2})^*] \setminus \{(\frac{1}{a_2})^*\}$ and $[(\frac{1}{a_2})^*, (\hat{f}^{-1}(f(\frac{1}{a_2})))^*]$ respectively [14]. Let us set the linear initial value $\phi(s) = \frac{1}{a_2}(s + 1 + h)$ for $-1 \le s \le 0$ and $0 < h < 1$. Obviously, ϕ does not completely locate in any attractive region. The parameter function h will determine the convergence of the solution $x(t, \phi)$.

The following two theorems describe the relationship between the eventual tendency of the solution $x(t, \phi)$ and the parameter μ (i.e. a_1 and a_2).

Theorem 5.1 *Set $h_1(\mu) = \frac{\mu}{\mu+1}$ for $0 < \mu < \infty$ and $\phi(s) = \frac{1}{a_2}(s + 1 + h)$ for $-1 \le s \le 0$, if $h_1 \le h \le 1$, then $\lim_{t \to \infty} x(t, \phi) = \frac{1}{a_2}$.*

The proof of this theorem is given in Appendix A. For system (3) and the initial value with slope $\frac{1}{a_2}$, if μ increases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be increased appropriately so that the corresponding solution converges to $\frac{1}{a_2}$. If μ decreases, appropriate reduction in the ratio can still guarantee that the corresponding solution converges to $\frac{1}{a_2}$.

According to Theorem 3.1, $h_{up} = \frac{-1 + \sqrt{1 + 4\mu^2}}{2\mu} \frac{1}{a_2}$, i.e. $\lim_{t \to \infty} x(t, \phi) = \frac{1}{a_2}$ if $\phi(s) = \frac{1}{a_2}(s + 1) + h$ for $h_{up} \le h \le \frac{1}{a_2}$. Note that $h_{up} \ge \frac{1}{a_2} h_1$, it implies that Theorem 5.1 gives wider range of linear initial value, the corresponding solutions of which converge to the positive equilibrium of system (3).

Theorem 5.2 *Set*

$$h_2(\mu) = \begin{cases} \dfrac{\mu}{3(\mu+1)}, & 0 < \mu \le 1, \\[2mm] \dfrac{1}{6\mu}, & 1 < \mu < \infty, \end{cases}$$

and $\psi(s) = \frac{1}{a_2}(s + 1 + h)$ for $-1 \le s \le 0$, if $0 \le h \le h_2(\mu)$, then $\lim_{t \to \infty} x(t, \psi) = 0$.

The proof of this theorem is given in Appendix A. Note that in the case $0 < \mu \leq 1$, for system (3) and the initial value with slope $\frac{1}{a_2}$, the ratio of the intercept to $\frac{1}{a_2}$ needs to be decreased appropriately so that the corresponding solution converges to 0 if μ decreases. Appropriate increase in the ratio still can guarantee that the corresponding solution converges to 0 if μ increases.

According to Theorem 3.2, $\underline{h_2} = \frac{\frac{\mu}{2} - e^\mu + \sqrt{\triangle_2}}{1.5a_2\mu}$ where $\triangle_2 = e^{2\mu} - \mu e^\mu + \mu^2$. Note that $h_{down} \leq \underline{h_2} \leq \frac{1}{a_2}h_2$, it implies that Theorem 5.2 gives wider range of linear initial value, the corresponding solutions of which converge to the trivial equilibrium of system (3).

Remark 5.1 The above two parameter functions indeed guarantee that the corresponding solution belongs to the first class and the second class mentioned in Proposition 4.1. However, they are just sufficient conditions. For system (3) with the initial $\phi(s) = \frac{1}{a_2}(s + 1 + h)$, according to Theorem 4.2, there exists a h_0 such that $x(t, \phi)$ oscillates about $\frac{1}{a_2}$ and then converges to it if $h = h_0$, $\lim_{t \to \infty} x(t, \phi) = 0$ if $0 \leq h < h_0$ and $\lim_{t \to \infty} x(t, \phi) = \frac{1}{a_2}$ if $h_0 \leq h \leq \frac{2}{a_2}$.

6   Simulations

In this section, numerical simulations are given to illustrate some results in Sections 4 and 5.

Consider the model from Section 5

$$\dot{x}(t) = -\mu x(t) + a_1 x(t-1)^2 e^{-a_2 x(t-1)}$$

and the initial value $\phi(s) = \frac{1}{a_2}(s + 1 + h(\mu))$ for $-1 \leq s \leq 0$.

Simulation 1: Let $h(\mu) = h_1(\mu) = \frac{\mu}{\mu+1}$ for $0 < \mu < \infty$.

Case A: Fix $a_1 = e$.

(1) Choose $a_2 = 10$, then $\mu = \frac{1}{10}$ and $\phi_1(s) = \frac{1}{10}(s + \frac{12}{11})$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \phi_1) = \frac{1}{10}$ (see Figure 2).

(2) Choose $a_2 = 4$, then $\mu = \frac{1}{4}$ and $\phi_2(s) = \frac{1}{4}(s + \frac{6}{5})$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \phi_2) = \frac{1}{4}$. However, if set $\phi_3(s) = \frac{1}{4}(s + \frac{12}{11})$, simulation implies $\lim_{t \to \infty} x(t, \phi_3) = 0$ (see Figure 4).

(3) Choose $a_2 = 1$, then $\mu = 1$ and $\phi_4(s) = s + \frac{3}{2}$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \phi_4) = 1$. However, if set $\phi_5(s) = s + \frac{6}{5}$, simulation implies $\lim_{t \to \infty} x(t, \phi_5) = 0$ (see Figure 6).

Case B: Fix $a_2 = 5$.

(1) Choose $a_1 = e$, then $\mu = \frac{1}{5}$ and $\psi_1(s) = \frac{1}{5}(s + \frac{7}{6})$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \psi_1) = \frac{1}{5}$ (see Figure 3).

(2) Choose $a_1 = 3e$, then $\mu = \frac{3}{5}$ and $\psi_2(s) = \frac{1}{5}(s + \frac{11}{8})$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \psi_2) = \frac{1}{5}$. However, if set $\psi_3(s) = \frac{1}{5}(s + \frac{7}{6})$, simulation implies $\lim_{t \to \infty} x(t, \psi_3) = 0$ (see Figure 5).

(3) Choose $a_1 = 20e$, then $\mu = 4$ and $\psi_4(s) = \frac{1}{5}(s + \frac{9}{5})$. From Theorem 5.1 it follows $\lim_{t \to \infty} x(t, \psi_4) = \frac{1}{5}$. However, if set $\psi_5(s) = \frac{1}{5}(s + \frac{11}{8})$, simulation implies $\lim_{t \to \infty} x(t, \psi_5) = 0$ (see Figure 7).
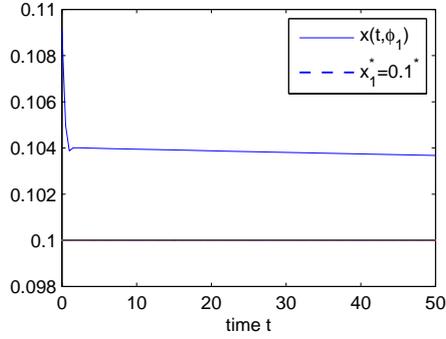
Figure 2: The numerical solution of $\dot{x}(t) = -\frac{x(t)}{10} + ex(t-1)^2 e^{-10x(t-1)}$ with the initial value ϕ_1.
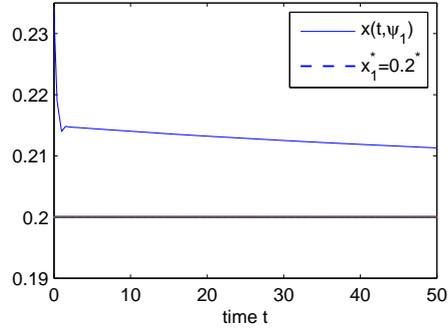
Figure 3: The numerical solution of $\dot{x}(t) = -\frac{x(t)}{5} + ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_1.
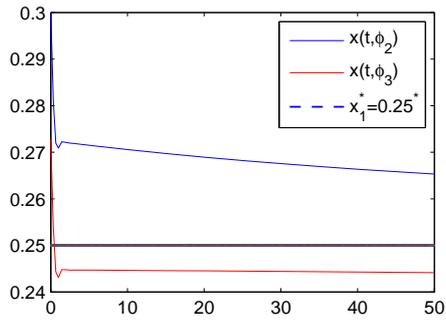
Figure 4: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{4} + ex(t-1)^2 e^{-4x(t-1)}$ with the initial value ϕ_2 and ϕ_3.
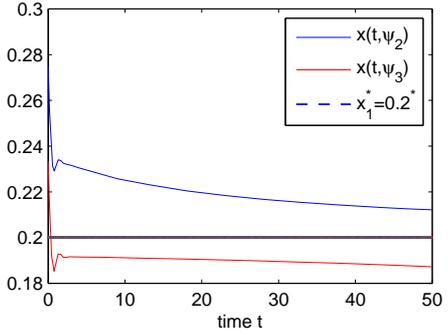

Figure 5: The numerical solutions of $\dot{x}(t) = -\frac{3x(t)}{5} + 3ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_2 and ψ_3.

Remark 6.1 For model (3) with $\tau = 1$ and the linear initial value with slope $\frac{1}{a_2}$, if μ increases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be increased appropriately to ensure the same convergence of the corresponding solution. Otherwise, it probably converges to 0. If μ decreases, appropriate reduction in the ratio can still guarantee that the corresponding solution converges to $\frac{1}{a_2}$.

Simulation 2: Let $h(\mu) = h_2(\mu) = \frac{\mu}{3(\mu+1)}$ for $0 < \mu \leq 1$.

Case A: Fix $a_1 = e$.

(1) Choose $a_2 = 1$, then $\mu = 1$ and $\phi_1(s) = s + \frac{7}{6}$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \phi_1) = 0$ (see Figure 8).

(2) Choose $a_2 = 4$, then $\mu = \frac{1}{4}$ and $\phi_2(s) = \frac{1}{4}(s + \frac{16}{15})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \phi_2) = 0$. However, if set $\phi_3(s) = \frac{1}{4}(s + \frac{7}{6})$, simulation implies $\lim_{t\to\infty} x(t, \phi_3) = \frac{1}{4}$ (see Figure 10).

(3) Choose $a_2 = 10$, then $\mu = \frac{1}{10}$ and $\phi_4(s) = \frac{1}{10}(s + \frac{34}{33})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \phi_4) = 0$. However, if set $\phi_5(s) = \frac{1}{10}(s + \frac{16}{15})$, simulation implies $\lim_{t\to\infty} x(t, \phi_5) = \frac{1}{10}$
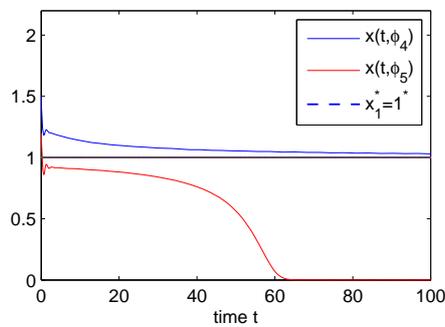
Figure 6: The numerical solutions of $\dot{x}(t) = -x(t)+ex(t-1)^2e^{-x(t-1)}$ with the initial value ϕ_4 and ϕ_5.
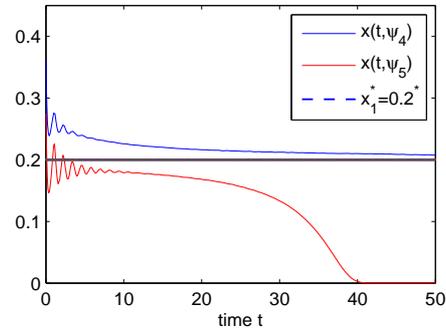
**Figure 7**: The numerical solutions of $\dot{x}(t) = -4x(t)+20ex(t-1)^2e^{-5x(t-1)}$ with the initial value ψ_4 and ψ_5.
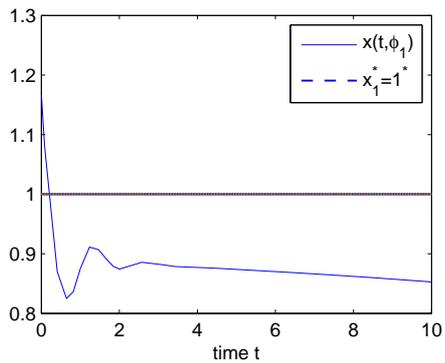
(see Figure 12).

**Figure 8**: The numerical solution of $\dot{x}(t) = -x(t)+ex(t-1)^2e^{-x(t-1)}$ with the initial value ϕ_1.
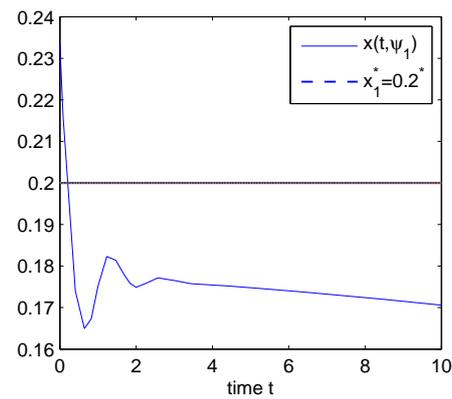
Figure 9: The numerical solution of $\dot{x}(t) = -x(t) + 5ex(t - 1)^2e^{-5x(t-1)}$ with the initial value ψ_1.

Case B: Fix $a_2 = 5$.

(1) Choose $a_1 = 5e$, then $\mu = 1$ and $\psi_1(s) = \frac{1}{5}(s + \frac{7}{6})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_1) = 0$ (see Figure 9).

(2) Choose $a_1 = 2e$, then $\mu = \frac{2}{5}$ and $\psi_2(s) = \frac{1}{5}(s + \frac{23}{21})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_2) = 0$. However, if set $\psi_3(s) = \frac{1}{5}(s + \frac{7}{6})$, simulation implies $\lim_{t\to\infty} x(t, \psi_3) = \frac{1}{5}$ (see Figure 11).

(3) Choose $a_1 = e$, then $\mu = \frac{1}{5}$ and $\psi_4(s) = \frac{1}{5}(s + \frac{19}{18})$. From Theorem 5.2 it follows $\lim_{t\to\infty} x(t, \psi_4) = 0$. However, if set $\psi_5(s) = \frac{1}{5}(s + \frac{23}{21})$, simulation implies $\lim_{t\to\infty} x(t, \psi_5) = \frac{1}{5}$ (see Figure 13).

Remark 6.2 For model (3) with $\tau = 1$ and the linear initial value with slope $\frac{1}{a_2}$, if μ decreases, the ratio of the intercept to $\frac{1}{a_2}$ needs to be decreased appropriately to ensure
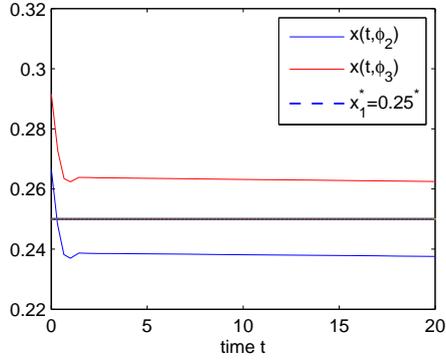
Figure 10: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{4} + ex(t-1)^2 e^{-4x(t-1)}$ with the initial value ϕ_2 and ϕ_3.
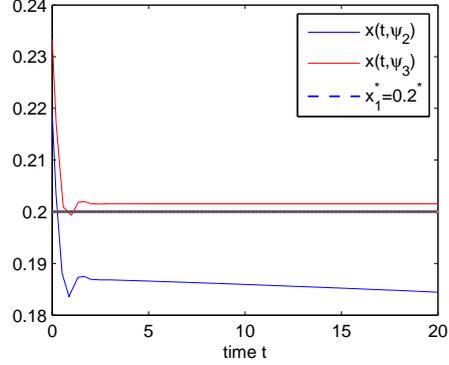
Figure 11: The numerical solutions of $\dot{x}(t) = -\frac{2x(t)}{5} + 2ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_2 and ψ_3.
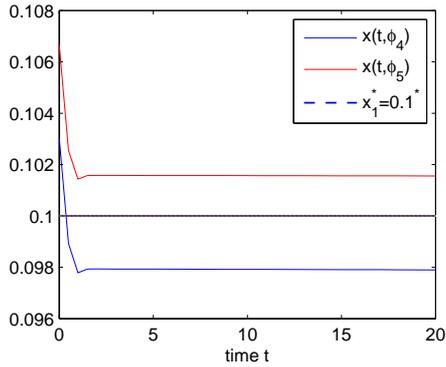
Figure 12: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{10} + ex(t-1)^2 e^{-10x(t-1)}$ with the initial value ϕ_4 and ϕ_5.
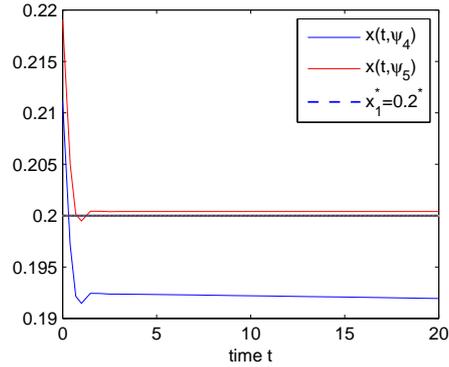
Figure 13: The numerical solutions of $\dot{x}(t) = -\frac{x(t)}{5} + ex(t-1)^2 e^{-5x(t-1)}$ with the initial value ψ_4 and ψ_5.

the same convergence of the corresponding solution. Otherwise, it probably converges to $\frac{1}{a_2}$. If μ increases, appropriate increase in the ratio can still guarantee that the corresponding solution converges to 0.

Simulation 3: Set $a_1 = e$, $a_2 = 1$ and $\tau = 1$, then $\mu = 1$. The model is:

$$\dot{x}(t) = -x(t) + ex(t-1)^2 e^{-x(t-1)}. \tag{18}$$

For the initial value $\phi(s) = s + b$, by Proposition 4.2, there must exist a special b_0 such that the solution $x(t, \phi^{b_0})$ oscillates about 1. By making use of the dichotomy, the range of b_0 is given as follows.

Step 1: Set $\phi_1(s) = s + \frac{4}{3}$ and $\phi_2(s) = s + \frac{7}{6}$, by Theorem 5.1 and 5.2, we have $\lim_{t \to \infty} x(t, \phi_1) = 1$ and $\lim_{t \to \infty} x(t, \phi_2) = 0$ (see Figure 14).

Step 2: Set $\phi_3(s) = s + \frac{5}{4}$ (i.e. $\frac{1}{2}(\phi_1 + \phi_2)$) and $\phi_4(s) = s + \frac{31}{24}$ (i.e. $\frac{1}{2}(\phi_1 + \phi_3)$), simulation implies that $\lim_{t \to \infty} x(t, \phi_3) = 0$ and $\lim_{t \to \infty} x(t, \phi_4) = 1$ (see Figure 15).
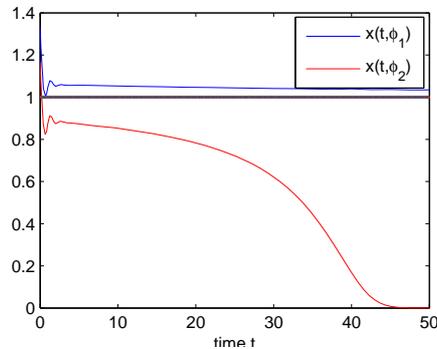
Figure 14: The numerical solutions of (18) with the initial value ϕ_1 and ϕ_2.
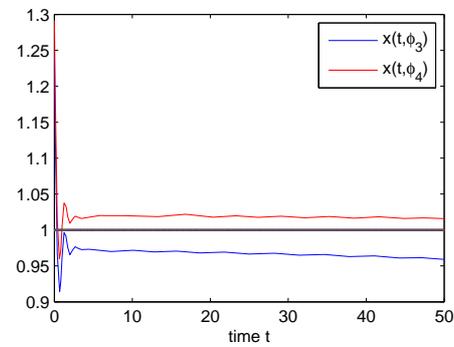
Figure 15: The numerical solutions of (18) with the initial value ϕ_3 and ϕ_4.
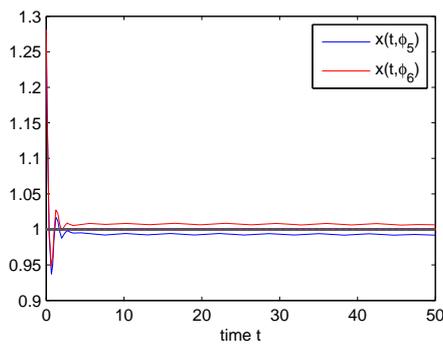
Figure 16: The numerical solutions of (18) with the initial value ϕ_5 and ϕ_6.
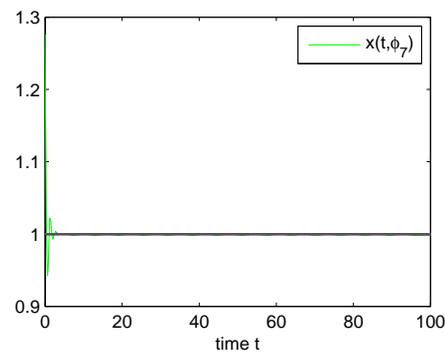
Figure 17: The numerical solutions of (18)with the initial value ϕ_7.

Step 3: Set $\phi_5(s) = s + \frac{61}{48}$ (i.e. $\frac{1}{2}(\phi_3 + \phi_4)$) and $\phi_6(s) = s + \frac{123}{96}$ (i.e. $\frac{1}{2}(\phi_4 + \phi_5)$), simulation implies that $\lim\limits_{t\to\infty} x(t, \phi_5) = 0$ and $\lim\limits_{t\to\infty} x(t, \phi_6) = 1$ (see Figure 16).

Step 4: Set $\phi_7(s) = s + \frac{245}{192}$ (i.e.$\frac{1}{2}(\phi_5 + \phi_6)$), the convergence of $x(t, \phi_7)$ is not evident (see Figure 17). Therefore the special b_0 that makes $x(t, \phi^{b_0})$ oscillate about 1 must locate in $\left[\frac{61}{48}, \frac{123}{96}\right]$.

7   Conclusions and Discussions

For equation (2) with $\tau = 1$, when the unique positive equilibrium is not globally asymptotic stable, the initial value plays an important role in practical problems. In order to ensure that the solution converges to the trivial or positive equilibrium, i.e. population size or density disappears or approximates a positive steady state, we need to fully consider the effects of the initial value. Since the form of initial value is so abundant, the paper conducts a preliminary study of the convergence of the solution with the initial value $\phi(s)$, which means that population size or density increases linearly in the initial stage. Theorem 3.1 implies that $\lim\limits_{t\to\infty} x(t, \phi) = x_1$ if (H_0) is satisfied. Theorem 3.2 im-

plies that $\lim_{t \to \infty} x(t, \phi) = 0$ if (H_1)–(H_3) hold. By the monotonicity of the flow generated by (2), we show the existence of the oscillatory solution, and prove that the solution oscillating about x_1 must converge to it. Furthermore, we give more detailed descriptions and classifications of all solutions of (2). When an example (3) is given, wider range of h compared to that of the general case is established to guarantee that the corresponding solution converges to 0 and x_1 respectively.

For equation (2) with a class of linear initial value $\phi(s)$, by the argument of Theorem 4.2, there exists a unique h_0 such that $x(t, \phi)$ oscillates about x_1 if $h = h_0$, $x(t, \phi)$ converges to 0 if $0 \le h < h_0$ and $x(t, \phi)$ converges to x_1 if $h_0 \le h < \xi_0 - x_1$. However, which h_0 should be chosen needs to be further explored.

Here we mainly investigate the convergence of the solution with the initial value that is linear and across the attractive region of 0 and x_1. However, in real-world problems, the initial value is various. When $\phi(s)$ is in other form and not in the attractive region of 0 and x_1 such as $\phi(s) = k \sin s + x_1 + h$, new method needs to be explored to found the condition which guarantees that the corresponding solution converges to 0 or x_1.

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