Nonlinear Dynamics and Systems Theory

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Volume 14   Number 1   2014

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An International Journal of Research and Surveys

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NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2011: SNIP – 0.84, RIP – 0.77

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Papers published in this journal are indexed or abstracted in: Mathematical Reviews / MathSciNet, Zentralblatt MATH / Mathematics Abstracts, PASCAL database (INIST–CNRS) and SCOPUS.

Permanence and Ultimate Boundedness for Discrete-Time Switched Models of Population Dynamics

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Abstract: The problems of permanence and ultimate boundedness for a class of discrete-time Lotka–Volterra type systems with switching of parameter values are studied. Two new approaches for the constructing of a common Lyapunov function for the family of subsystems corresponding to a switched system are suggested. Sufficient conditions in terms of linear inequalities are obtained to guarantee that the solutions of the considered system are ultimately bounded or permanent for an arbitrary switching law. An example is presented to demonstrate the effectiveness of the obtained results.

Keywords: *population dynamics; ultimate boundedness; switched system; discrete-time models; common Lyapunov function; linear inequalities.*

Mathematics Subject Classification (2010): 92D25, 39A22, 39A60.

1 Introduction

The Lotka–Volterra type differential and difference equations systems are extensively used in modeling of population dynamics [6, 7, 9, 12, 14, 15]. A very important ecological problem associated with multispecies population interactions is the following one: whether or not the densities of all species are bounded [5, 7, 9, 15]. Of particular interest is the situation when there exists a bounded region in the phase space of the system, such that every solution enters this region for finite time and remains within it thereafter. Solutions of systems possessing this property are called ultimately bounded [6, 7].

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It is worth mentioning that, in the analysis of population models, it is important not only to check the ultimate boundedness, but also to verify whether or not the considered system is permanent [5, 7, 12, 17]. The permanence property, in addition to the ultimate boundedness of densities of all species, implies that if initially all species are present, even in very small quantities, then after a certain time some sizeable amount of each of them will be present.

Conditions of ultimate boundedness and permanence are well investigated for Lotka–Volterra type models with constant parameters, see, for example, [5–7, 9] and the references cited therein. However, owing to many natural and man-made factors, such as fire, drought, raining season, changing in nutrition, deforestation, radiation, etc., the intrinsic discipline of biological species or ecological environment usually undergoes some discrete changes of relatively short duration at some fixed times. For more adequate modeling of such processes, stochastic, switched or impulsive systems are used [4, 8, 13, 17, 18]. The problem of ultimate boundedness and permanence analysis for these models is much more complicated than that one for differential and difference systems with constant parameters.

In the present paper, a discrete-time switched Lotka–Volterra type system is studied. The system consists of a family of subsystems of difference equations and a switching law determining at each time instant which subsystem is active. We will look for conditions providing the ultimate boundedness or permanence of the considered system for an arbitrary switching law. A general approach to the problem is based on the construction of a common Lyapunov function (CLF) for the family of subsystems corresponding to the switched system. This approach has been effectively used for the analysis of stability and boundedness for many classes of switched systems, see, for instance, [1–3, 10, 11, 16], and the references therein. However, the problem of the existence of a CLF has not got a constructive solution even for the case of family of linear time-invariant systems [11].

In [3], for the investigated switched system, a special form of Lyapunov function has been used. The sufficient condition in terms of linear inequalities was obtained to guarantee the existence of a CLF in the prescribed form, and thereby to ensure that solutions of the switched system are ultimately bounded or permanent for an arbitrary switching signal. In the present paper, two different approaches for the constructing of a CLF are proposed. The usage of these approaches permits to relax the ultimate boundedness and the permanence conditions found in [3].

2   Statement of the Problem

Consider the switched difference system

$$x_i(k+1) = x_i(k) \exp\left(h\left(c_i^{(\sigma)} + \sum_{j=1}^{n} p_{ij}^{(\sigma)} f_j(x_j(k)) \right) \right), \quad i = 1, \ldots, n. \qquad (1)$$

The system describes interaction of n species in a biological community. Here $x_i(k)$ is the density of population i at the kth generation; functions $f_i(z_i)$ are defined for $z_i \in [0, +\infty)$; $\sigma = \sigma(k)$, $k = 0, 1, \ldots$, with $\sigma(k) \in \{1, \ldots, N\}$ defines a switching law; $c_i^{(s)}$ and $p_{ij}^{(s)}$, $s = 1, \ldots, N$, $i, j = 1, \ldots, n$, are constant coefficients; h is a positive parameter characterizing the transient time between two consecutive generations. Thus, at each

time instant, the dynamics of (1) is described by one of the subsystems

$$x_i(k+1) = x_i(k) \exp\left(h\left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(x_j(k))\right)\right), \quad i = 1, \ldots, n, \ s = 1, \ldots, N. \quad (2)$$

Subsystems of the form (2) are discrete counterparts of the continuous generalized Lotka–Volterra ecosystem models [5–7, 12, 15]. It is known [6, 7, 12] that if the populations have non-overlapping generations, then discrete time models are more appropriate than the continuous ones. Moreover, they provide efficient schemes for the numerical simulation of continuous processes.

In (1), coefficients $c_i^{(s)}$ characterize the intrinsic growth rate of the ith population; the introduction of self-interaction terms $p_{ii}^{(s)} f_i(z_i)$ with $p_{ii}^{(s)} < 0$ is justified by the natural limitation of resources in the environment, the terms $p_{ij}^{(s)} f_j(z_j)$ for $j \neq i$ measure influence of population j on population i. It is supposed that environment fluctuations provoke switching of the system parameters.

According to standard assumptions [6, 7, 15], we assume that functions $f_i(z_i)$, $i = 1, \ldots, n$, possess the following properties:

(i) $f_i(z_i)$ are continuous for $z_i \in [0, +\infty)$;
(ii) $f_i(0) = 0$, and for $z_i > 0$ the inequality $f_i(z_i) > 0$ holds, and
(iii) $f_i(z_i) \to +\infty$ as $z_i \to +\infty$.

By R_+^n we denote the non-negative orthant of R^n; int R_+^n being the interior of R_+^n; $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ denotes the solution of (1) starting from $\mathbf{x}^{(0)}$ at $k = k_0$; $\mathbf{P}_s = \left(p_{ij}^{(s)}\right)_{i,j=1}^{n}$, $s = 1, \ldots, N$, are given matrices; and $B_Q = \{\mathbf{z} : \ \mathbf{z} \in \text{int } R_+^n, \|\mathbf{z}\| \leq Q\}$ for a given positive number Q. For biological reasons, we will consider (1) in int R_+^n which is an invariant set for this system.

Definition 2.1 System (1) is called ultimately bounded in int R_+^n with the ultimate bound $R > 0$ if, for any $\mathbf{x}^{(0)} \in \text{int } R_+^n$ and $k_0 \geq 0$, there exists $T > 0$, such that $\|\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)\| \leq R$ for $k \geq k_0 + T$.

Definition 2.2 System (1) is called uniformly ultimately bounded in int R_+^n with the ultimate bound $R > 0$ if, for any $Q > 0$, there exists $T = T(Q) > 0$, such that $\|\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)\| \leq R$ for all $k_0 \geq 0$, $\mathbf{x}^{(0)} \in B_Q$, $k \geq k_0 + T$.

Definition 2.3 System (1) is called permanent if there exists a compact set $D \subset \text{int } R_+^n$, such that, for any $\mathbf{x}^{(0)} \in \text{int } R_+^n$ and $k_0 \geq 0$, the solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) ultimately remains in D.

Definition 2.4 System (1) is called uniformly permanent if there exist numbers Δ_1 and Δ_2, $0 < \Delta_1 < \Delta_2$, such that, for any δ_1 and δ_2, $0 < \delta_1 < \delta_2$, one can choose $T > 0$ satisfying the following condition: if for the initial values of a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ the inequalities $k_0 \geq 0$, $\delta_1 \leq x_i^{(0)} \leq \delta_2$, $i = 1, \ldots, n$, hold, then $\Delta_1 \leq x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \ldots, n$, for $k \geq k_0 + T$.

Conditions of the ultimate boundedness and the permanence are well investigated for individual subsystems from (2) without switching [5–7, 12]. The goal of the present paper is the ultimate boundedness and the permanence analysis for switched system (1).

3   Ultimate Boundedness Conditions

Sufficient conditions of uniform ultimate boundedness for switched system (1) have been obtained in [3]. The case was considered when, for the functions $f_1(z_1), \ldots, f_n(z_n)$, in addition to the properties (i)–(iii), the following assumptions are fulfilled.

Assumption 3.1 Let $\int_0^1 \frac{f_i(\tau)}{\tau}\, d\tau < +\infty$, $i = 1, \ldots, n$.

Assumption 3.2 The functions $\tilde{f}_i(z_i) = f_i(\exp(z_i))$ satisfy the Lipschitz condition with constant L for all $z_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$.

For example, the properties (i)–(iii) and Assumptions 3.1 and 3.2 are fulfilled for functions $f_i(z_i) = \log(z_i + 1)$, $i = 1, \ldots, n$.

Let us introduce the auxiliary matrices $\overline{\mathbf{P}}_s = \left(\bar{p}_{ij}^{(s)} \right)_{i,j=1}^n$ whose entries are defined by the formulae $\bar{p}_{ii}^{(s)} = p_{ii}^{(s)}$, and $\bar{p}_{ij}^{(s)} = \max \left\{ p_{ij}^{(s)}; 0 \right\}$ for $j \neq i$; $i, j = 1, \ldots, n$; $s = 1, \ldots, N$. Thus, the matrices $\overline{\mathbf{P}}_1, \ldots, \overline{\mathbf{P}}_N$ are Metzler ones [9, 10].

Consider the two families of linear inequalities systems

$$\overline{\mathbf{P}}_s \theta < \mathbf{0}, \quad s = 1, \ldots, N, \tag{3}$$

$$\overline{\mathbf{P}}_s^T \mathbf{b} < \mathbf{0}, \quad s = 1, \ldots, N, \tag{4}$$

where $\theta = (\theta_1, \ldots, \theta_n)^T$, $\mathbf{b} = (b_1, \ldots, b_n)^T$. These inequalities in vector form are understood to be component-wise. That is to say, a vector is less than zero if and only if so is each component of the vector. For convenience, one can call a vector to be negative (respectively, positive) if it is less (respectively, greater) than zero.

In [3], a CLF for (2) has been chosen in the form

$$V_1(\mathbf{z}) = \sum_{i=1}^n \lambda_i \int_1^{z_i} \frac{f_i(\tau)}{\tau} d\tau, \tag{5}$$

where $\lambda_1, \ldots, \lambda_n$ are positive coefficients. By the usage of function (5), the following theorem was proved.

Theorem 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled. If systems (3) and (4) admit positive solutions, then there exists $h_0 > 0$ such that system (1) is uniformly ultimately bounded in* int R_+^n *for any $h \in (0, h_0)$ and for arbitrary switching law.*

Remark 3.1 Necessary and sufficient conditions of solvability for inequality systems of the form (3) and (4) with Metzler matrices have been found in [2, 10]. Furthermore, in [2], an effective algorithm based on a modification of Gaussian elimination procedure for the construction of positive solutions of such systems was suggested.

Remark 3.2 It is known [9] that if a matrix $\mathbf{P}$ is Metzler one, then the system of inequalities $\mathbf{P}\theta < \mathbf{0}$ possesses a positive solution if and only if the system of inequalities $\mathbf{P}^T \mathbf{b} < \mathbf{0}$ possesses a positive solution as well. However, it is not true for the families of inequalities (3), (4) [2, 3]. Generally, from the existence of a positive solution for the inequalities (3) with Metzler matrices $\overline{\mathbf{P}}_1, \ldots, \overline{\mathbf{P}}_N$, it does not follow that a positive solution also exists for the corresponding inequalities (4).

In the present section, we shall suggest another approach for the constructing of a CLF for family (2). The usage of this approach permits to relax the conditions of Theorem 3.1. In particular, we will prove that in the case when for functions $f_1(z_1), \ldots, f_n(z_n)$, instead of Assumptions 3.1 and 3.2, an additional assumption is fulfilled, the existence of a positive solution for (3) is sufficient to ensure that (1) is uniform ultimately bounded for sufficiently small values of h and for any switching law. Thus, another condition of Theorem 3.1, i.e., the condition of the existence of a positive solution for (4), can be dropped.

Assumption 3.3 The functions $\tilde{f}_i(z_i) = f_i(\exp(z_i))$ are continuously differentiable for $z_i \in (-\infty, +\infty)$, and $0 < \tilde{f}_i'(z_i) \le L$, $i = 1, \ldots, n$, where L is a positive constant.

Theorem 3.2 *Let Assumption 3.3 be fulfilled. If system* (3) *admits a positive solution, then there exists* $h_0 > 0$ *such that system* (1) *is uniformly ultimately bounded in* int R_+^n *for any* $h \in (0, h_0)$ *and for arbitrary switching law.*

Proof. Let a positive vector $\theta = (\theta_1, \ldots, \theta_n)^T$ satisfy the inequalities (3). Then there exists a number $\gamma > 0$, such that $\sum_{j=1}^n \bar{p}_{ij}^{(s)} \theta_j \le -\gamma$, $i = 1, \ldots, n$, $s = 1, \ldots, N$.
Construct a CLF for (2) in the form

$$V_2(\mathbf{z}) = \max_{i=1,\ldots,n} \frac{f_i(z_i)}{\theta_i}. \tag{6}$$

Function $V_2(\mathbf{z})$ is continuous for $\mathbf{z} \in R_+^n$, and $V_2(\mathbf{z}) \to +\infty$ as $\|\mathbf{z}\| \to \infty$.

For some s in $\{1, \ldots, N\}$, consider the difference of the function (6) with respect to the sth subsystem from (2). Let $\hat{\mathbf{x}} \in$ int R_+^n, and $\mathbf{x}(k) = (x_1(k), \ldots, x_n(k))^T$ be the solution of the sth subsystem starting from $\hat{\mathbf{x}}$ at $k = 0$. For every $k = 0, 1, \ldots$, find

$$B_k = \max_{i=1,\ldots,n} \frac{f_i(x_i(k))}{\theta_i}.$$

Denote by A_k a subset of $\{1, \ldots, n\}$ such that $f_i(x_i(k))/\theta_i = B_k$ for $i \in A_k$, and $f_i(x_i(k))/\theta_i < B_k$ for $i \notin A_k$.

Choose a nonnegative integer k. Let $r \in A_k$, $i \in A_{k+1}$. We obtain

$$\Delta V_2\big|_{(s)} = V_2(\mathbf{x}(k+1)) - V_2(\mathbf{x}(k)) = \frac{f_i(x_i(k+1))}{\theta_i} - \frac{f_r(x_r(k))}{\theta_r}$$

$$= \left(\frac{f_i(x_i(k+1))}{\theta_i} - \frac{f_i(x_i(k))}{\theta_i} \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$= \left(\frac{\tilde{f}_i(y_i(k+1))}{\theta_i} - \frac{\tilde{f}_i(y_i(k))}{\theta_i} \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$\le \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} + \sum_{j=1}^n \bar{p}_{ij}^{(s)} f_j(x_j(k)) \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$\le \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} + \bar{p}_{ii}^{(s)} f_i(x_i(k)) + \sum_{j=1}^n \bar{p}_{ij}^{(s)} \theta_j \frac{f_r(x_r(k))}{\theta_r} - \bar{p}_{ii}^{(s)} \theta_i \frac{f_r(x_r(k))}{\theta_r} \right)$$

$$- \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$\leq \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} - \gamma B_k - \bar{p}_{ii}^{(s)}\theta_i \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \right)$$

$$- \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$= \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} - \gamma B_k \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \left(1 + Lh\bar{p}_{ii}^{(s)} \right).$$

Here $y_i(k) = \log x_i(k)$, $\Delta y_i(k) = y_i(k+1) - y_i(k)$, $\xi_{ik} \in (0,1)$.

Let $D = \max\limits_{s=1,\ldots,N} \max\limits_{i=1,\ldots,n} |\bar{p}_{ii}^{(s)}|$,

$$0 < h_0 < \frac{1}{LD}, \tag{7}$$

and $h \in (0, h_0)$. Then there exists a positive number H, such that $\Delta V_2\big|_{(s)} < 0$ for $\|\mathbf{x}(k)\| > H$ and for all $s = 1, \ldots, N$.

Define the constants M and M_1 by the following formulae:

$$M = \max\limits_{\mathbf{z} \in R_+^n, \ \|\mathbf{z}\| \leq H} V_2(\mathbf{z}), \quad M_1 > M + hL \max\limits_{s=1,\ldots,N} \max\limits_{i=1,\ldots,n} \left| \frac{c_i^{(s)}}{\theta_i} \right|.$$

Consider the region $G = \{\mathbf{z} : \mathbf{z} \in \text{int } R_+^n, V_2(\mathbf{z}) \leq M_1\}$. We obtain that $V_2(\mathbf{x}(k+1)) \leq M_1$ if $\|\mathbf{x}(k)\| \leq H$, and $V_2(\mathbf{x}(k+1)) < V_2(\mathbf{x}(k))$ if $\|\mathbf{x}(k)\| > H$. Hence, once a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) enters into G at $k = k_1 \geq k_0$, it remains within the region for $k \geq k_1$.

Choose a positive number Q. We will show that there exists $T = T(Q) \geq 0$ such that $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$ for all $k_0 \geq 0$, $\mathbf{x}^{(0)} \in B_Q$ and $k \geq k_0 + T(Q)$.

Let $U = \max_{\mathbf{z} \in R_+^n, \ \|\mathbf{z}\| \leq Q} V_2(\mathbf{z})$. If $U \leq M_1$, then we can take $T(Q) = 0$.

Now consider the case when $U > M_1$. If $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) > M_1$ for $k = k_0, k_0 + 1, \ldots, \tilde{k}$, then the inequalities

$$M_1 < V_2(\mathbf{x}(\tilde{k}, \mathbf{x}^{(0)}, k_0)) \leq V_2(\mathbf{x}^{(0)}) - \rho(\tilde{k} - k_0) \leq U - \rho(\tilde{k} - k_0)$$

hold, where

$$\rho = -\max\limits_{s=1,\ldots,N} \max\limits_{\mathbf{z} \in R_+^n, \ M_1 \leq V_2(\mathbf{z}) \leq U} \Delta V_2\big|_{(s)} > 0.$$

Hence, $\tilde{k} < k_0 + (U - M_1)/\rho$. By taking $T(Q) = (U - M_1)/\rho$, one gets $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$ for $k \geq k_0 + T(Q)$. Thus, system (1) is uniformly ultimately bounded in int R_+^n.

Corollary 3.1 Let $c_i^{(s)} \leq 0$, $i = 1, \ldots, n$; $s = 1, \ldots, N$, and Assumption 3.3 be fulfilled. If system (3) admits a positive solution, then there exists $h_0 > 0$ such that the zero solution of (1) is globally asymptotically stable in int R_+^n for any $h \in (0, h_0)$ and for any switching law.

Remark 3.3 In the case when all the coefficients $c_i^{(s)}$ are negative, instead of (3), it is sufficient to consider the nonstrict inequalities

$$\overline{\mathbf{P}}_s\theta \leq \mathbf{0}, \quad s = 1, \ldots, N. \tag{8}$$

Corollary 3.2 *Let $c_i^{(s)} < 0$, $i = 1, \ldots, n$; $s = 1, \ldots, N$, and Assumption 3.3 be fulfilled. If system* (8) *admits a positive solution, then there exists $h_0 > 0$ such that the zero solution of* (1) *is globally asymptotically stable in* int R_+^n *for any $h \in (0, h_0)$ and for any switching law.*

4   Permanence Conditions

In this section, we consider the case when, in system (1), parameters $c_i^{(s)}$ and $p_{ij}^{(s)}$ satisfy an additional restriction.

Assumption 4.1 The following inequalities are valid $c_i^{(s)} > 0$, and $p_{ij}^{(s)} \geq 0$ for $j \neq i$; $i, j = 1, \ldots, n$; $s = 1, \ldots, N$.

Theorem 4.1 *Let Assumptions 3.3 and 4.1 be fulfilled. If system* (3) *admits a positive solution, then there exists $h_0 > 0$ such that system* (1) *is uniformly permanent for any $h \in (0, h_0)$ and for arbitrary switching law.*

Proof. Let for a constant h_0 the condition (7) be valid. Choose a number $h \in (0, h_0)$, and consider the corresponding switched system (1).

According to the proof of Theorem 3.2, there exists $\Delta_2 > 0$, and for given positive numbers δ_1 and δ_2, $0 < \delta_1 < \delta_2$, one can find $\eta > 0$ and $T > 0$, such that if the initial values of a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) satisfy the conditions $k_0 \geq 0$, $\delta_1 \leq x_i^{(0)} \leq \delta_2$, $i = 1, \ldots, n$, then $0 < x_i(k, \mathbf{x}^{(0)}, k_0) \leq \eta$, $i = 1, \ldots, n$, for $k \geq k_0$, and $0 < x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \ldots, n$, for $k \geq k_0 + T$.

The fulfilment of the Assumption 4.1 implies the existence of positive numbers δ and β, such that $c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \geq \beta$ for $0 < z_i \leq \delta$, $i = 1, \ldots, n$; $s = 1, \ldots, N$. Hence, if $0 < x_i(k, \mathbf{x}^{(0)}, k_0) < \delta$ for some $i \in \{1, \ldots, n\}$, then $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq x_i(k, \mathbf{x}^{(0)}, k_0) \exp(h\beta)$.

Let

$$\omega = \min_{s=1,\ldots,N} \; \min_{i=1,\ldots,n} \; \min_{0 \leq z_i \leq \eta} \left(c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \right),$$

$$\tilde{\omega} = \min_{s=1,\ldots,N} \; \min_{i=1,\ldots,n} \; \min_{0 \leq z_i \leq \Delta_2} \left(c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \right).$$

We obtain that $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq \delta \exp(h\omega)$ for $k \geq k_0$, $x_i(k, \mathbf{x}^{(0)}, k_0) \geq \delta$, $i = 1, \ldots, n$, and $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq \delta \exp(h\tilde{\omega})$ for $k \geq k_0 + T$, $x_i(k, \mathbf{x}^{(0)}, k_0) \geq \delta$, $i = 1, \ldots, n$.

Therefore, there exists $\widetilde{T} \geq T$, such that $\Delta_1 \leq x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \ldots, n$, for $k \geq k_0 + \widetilde{T}$, where $\Delta_1 = \delta \min\{1; \exp(h\tilde{\omega})\}$. This completes the proof.

Consider one more approach for a Lyapunov function constructing which permits to use for the verification of the permanence property system (4) instead of system (3).

Theorem 4.2 *Let Assumptions 3.2 and 4.1 be fulfilled. If system* (4) *admits a positive solution, then there exists $h_0 > 0$ such that system* (1) *is uniformly permanent for any $h \in (0, h_0)$ and for arbitrary switching law.*

Proof. For a constant h_0, let the condition (7) be valid, and $h \in (0, h_0)$. Consider the corresponding switched system (1).

Choose a positive vector $\mathbf{b} = (b_1, \ldots, b_n)^T$ satisfying the inequalities (4). There exists a number $\gamma > 0$, such that $\sum_{i=1}^n \bar{p}_{ij}^{(s)} b_i \leq -\gamma$, $j = 1, \ldots, n$, $s = 1, \ldots, N$.

Construct a CLF for (2) in the form

$$V_3(\mathbf{z}) = \sum_{i=1}^{n} b_i \log z_i. \tag{9}$$

Function $V_3(\mathbf{z})$ is defined and continuous for $\mathbf{z} \in \text{int } R_+^n$.

For some s in $\{1, \ldots, N\}$, consider the difference of the function (9) with respect to the sth subsystem from (2). Let $\hat{\mathbf{x}} \in \text{int } R_+^n$, and $\mathbf{x}(k) = (x_1(k), \ldots, x_n(k))^T$ be the solution of the sth subsystem starting from $\hat{\mathbf{x}}$ at $k = 0$. We obtain

$$\Delta V_3\big|_{(s)} = V_3(\mathbf{x}(k+1)) - V_3(\mathbf{x}(k)) = \sum_{i=1}^{n} b_i \left(\log x_i(k+1) - \log x_i(k)\right)$$

$$= h \sum_{i=1}^{n} b_i \left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(x_j(k))\right) = h \sum_{i=1}^{n} b_i c_i^{(s)} + h \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_i p_{ij}^{(s)}\right) f_j(x_j(k))$$

$$\leq h \sum_{i=1}^{n} b_i c_i^{(s)} - h\gamma \sum_{j=1}^{n} f_j(x_j(k)).$$

Hence, there exists a positive number H, such that $\Delta V_3\big|_{(s)} < 0$ for $\|\mathbf{x}(k)\| > H$ and for all $s = 1, \ldots, N$.

Let

$$A = H \max_{s=1,\ldots,N} \max_{i=1,\ldots,n} \max_{\|\mathbf{z}\| \leq H} \exp\left(h\left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(z_j)\right)\right),$$

$$M_1 = \max_{0 < z_i \leq A, \ i=1,\ldots,n} V_3(\mathbf{z}) = \log A \sum_{i=1}^{n} b_i.$$

In a similar way as in the proof of Theorem 3.2, it can be shown that for any $Q > 0$ there exists $T = T(Q) \geq 0$, such that if the initial values of a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) satisfy the conditions $k_0 \geq 0$, $\mathbf{x}^{(0)} \in B_Q$, then $V_3(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$ for $k \geq k_0 + T$.

The fulfilment of the Assumption 4.1 implies the existence of positive numbers δ and β, such that $c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \geq \beta$ for $0 < z_i \leq \delta$, $i = 1, \ldots, n$; $s = 1, \ldots, N$. Hence, if $0 < x_i(k) < \delta$ for some $i \in \{1, \ldots, n\}$, then $x_i(k+1) \geq x_i(k) \exp(h\beta)$. Without loss of generality, we assume that $\delta < 1$.

In the case when $x_i(k) \geq \delta$, the following estimates hold

$$x_i(k+1) \geq x_i(k) \exp\left(h\left(p_{ii}^{(s)} f_i(x_i(k))\right)\right) \geq x_i(k) \exp\left(-hD\tilde{f}_i(y_i(k))\right)$$

$$\geq x_i(k) \exp\left(-hD(L|y_i(k)| + \tilde{f}_i(0))\right) \geq \lambda \delta^{1+hLD}.$$

Here $y_i(k) = \log x_i(k)$, $\lambda = \exp(-hD \max_{i=1,\ldots,n} f_i(1))$.

Let positive numbers δ_1 and δ_2, $\delta_1 < \delta_2$, be given. Choose the numbers $T_1 = T_1(\delta_1) > 0$ and $T_2 = T_2(\delta_2) > 0$, such that if $k_0 \geq 0$, $\delta_1 \leq x_i^{(0)} \leq \delta_2$, $i = 1, \ldots, n$, then $x_i(k, \mathbf{x}^{(0)}, k_0) \geq \lambda \delta^{1+hLD}$, $i = 1, \ldots, n$, for $k \geq k_0 + T_1$, and $V_3(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$

for $k \geq k_0 + T_2$. By taking $\hat{T} = \max\{T_1; T_2\}$, we obtain $\Delta_1 \leq x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \ldots, n$, for $k \geq k_0 + \hat{T}$. Here $\Delta_1 = \lambda \delta^{1+hLD}$, and

$$\Delta_2 = \max_{i=1,\ldots,n} \left(\exp(M_1)/\Delta_1^{\sum_{j \neq i} b_j} \right)^{1/b_i}.$$

This completes the proof.

Remark 4.1 The fulfilment of Assumption 3.2 (Assumption 3.3) with a single constant L for all $z_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$, is quite severe constraint on the admissible functions $f_1(z_1), \ldots, f_n(z_n)$. It is worth mentioning that in a similar way the conditions of permanence can be obtained in the case when, for every $r > 0$, functions $\tilde{f}_i(z_i)$ satisfy Assumption 3.2 (Assumption 3.3) for $z_i \in (-\infty, r)$, $i = 1, \ldots, n$, with the constant $L(r)$, and $L(r) \to +\infty$ as $r \to +\infty$. However, in this case, we can not guarantee the permanence property for all solutions of (1). For any $Q > 0$, there exists a number $h_0 > 0$, such that for any $h \in (0, h_0)$ the conditions of Definition 2.4 are fulfilled only for $\delta_2 < Q$.

5 Example

In (1) let $n = 3$, and the family (2) consists of two subsystems with the matrices

$$\mathbf{P_1} = \begin{pmatrix} -1 & a & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix}, \qquad \mathbf{P_2} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -1 & 1 \\ d & 0 & -4 \end{pmatrix}.$$

Here a and d are positive parameters. In this case, $\overline{\mathbf{P}}_1 = \mathbf{P_1}$, $\overline{\mathbf{P}}_2 = \mathbf{P_2}$.

On the one hand, it is easy to verify that the system $\overline{\mathbf{P}}_1 \theta < \mathbf{0}$, $\overline{\mathbf{P}}_2 \theta < \mathbf{0}$ admits a positive solution if and only if

$$a < 3, \quad d < 12, \quad ad < 4. \tag{10}$$

On the other hand, for the existence of a positive solution for the system $\overline{\mathbf{P}}_1^T \mathbf{b} < \mathbf{0}$, $\overline{\mathbf{P}}_2^T \mathbf{b} < \mathbf{0}$ it is necessary and sufficient the fulfilment of the inequalities

$$a < 6, \quad d < 9, \quad ad < 18. \tag{11}$$

The regions (10) and (11) in the parameter space are nonoverlapping. Thus, this example shows that Theorems 4.1 and 4.2 complement each other.

6 Conclusion

In this paper, a discrete-time Lotka–Volterra type system with switching of parameter values is studied. The conditions are determined under which the system is ultimately bounded or permanent for any admissible switching law. Two new approaches for Lyapunov functions constructing are proposed. By the usage of these approaches, the theorems on the ultimate boundedness and permanence conditions are proved. These theorems complement each other and relax the known ultimate boundedness conditions found in [3]. The interesting direction for further research is the extension of the obtained results to switched biological models of more general form.

Acknowledgment

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Approximations of Solutions for a Sobolev Type Fractional Order Differential Equation

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Abstract: In this paper, using semigroup theory and Banach fixed point theorem, we establish the existence and uniqueness of approximate solutions of nonlinear Sobolev type fractional order evolution equation in a separable Hilbert space. Also, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results.

Keywords: *analytic semigroup; approximate solution; fractional differential equation; Faedo-Galerkin approximation; Sobolev type evolution equation.*

Mathematics Subject Classification (2010): 34G20, 35R10, 47D06, 47N20.

1 Introduction

In recent few decades, researchers have developed great interest in fractional calculus due to its wide applicability in science and engineering. Tools of fractional calculus have been available and applicable to deal with many physical and real world problems such as anomalous diffusion process, traffic flow, nonlinear oscillation of earthquake, real system characterized by power laws, critical phenomena, scale free process, description of viscoelastic materials and many others. For more details about fractional calculus we refer to [3–5, 7, 10, 12, 13, 16, 18].

In the present paper, we study the convergence of the Faedo-Galerkin approximations of solutions to the nonlinear fractional order Sobolev type evolution equation

$$
\begin{aligned}
\frac{d^q}{dt^q}[u(t) + g(t, u(t))] + Au(t) &= f(t, u(t)),\ 0 < t \le T \le \infty,\ 0 < q \le 1, \\
u(0) &= \phi,
\end{aligned}
\tag{1}
$$

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in a separable Hilbert space $(H, \| \cdot \|, (\cdot, \cdot))$, where A is a closed linear operator defined on $D(A)$ which is dense in H. We assume that linear operator $-A$ is the infinitesimal generator of analytic semigroup $\{S(t);\ t \geq 0\}$ in H. The functions f and g are continuous functions and satisfy certain assumptions stated later in Section 2.

The Feado-Galerkin approximations of solutions of the particular case of (1) in which $g = 0$, have been established by Muslim [9]. Author in [9] has discussed the convergence of Feado-Galerkin approximation of the solution to the equation

$$\frac{d^\beta}{dt^\beta} u(t) + Au(t) = f(t, u(t)),\ t \in [0, T],\ \beta \in (0, 1), \tag{2}$$

$$u(0) = \phi. \tag{3}$$

under the assumption that $-A$ generates an analytic semigroup of bounded linear operators defined on a Banach space H and f satisfies certain conditions.

The existence and uniqueness of solution and approximation of solution of functional differential equation

$$\frac{d}{dt}[u(t) + g(t, u(t))] = -Au(t) + f(t, u(t)),\ \ t > 0,$$

$$u(0) = \phi, \tag{4}$$

have been discussed by D. Bahuguna and Reeta in [2] with the assumption that $-A$ generates an analytic semigroup and f and g satisfy the conditions such that f and $A^\beta g$ satisfy the Lipschitz condition on $C([0, T] \times D(A^\alpha); H)$.

This paper is organized as follows: we present some basic definitions, lemmas, theorems and assumptions required to establish the convergence result as preliminaries in Section 2. The existence and uniqueness of the approximate solutions are proved using semigroup theory and fixed point theorem in Section 3. In Section 4, we prove the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and the convergence of the approximate Feado-Galerkin solutions will be shown in Section 5. In the last section we consider an example as an application.

2   Preliminaries and Assumptions

In this section we provide some basic definitions, results and assumptions on f and g which will be used in the later sections.

Definition 2.1 The fractional derivative of $f : [0, \infty) \to \mathbb{R}$ in the Caputo sense of order q is defined as

$$^cD_t^q f(t) = \frac{1}{\Gamma(m - q)} \int_0^t (t - s)^{m - q - 1} f^m(s) ds, \tag{5}$$

for $m - 1 \leq q < m,\ m \in N,\ t > 0$, with the following property:

$$^cD_t^q f(t) = D_t^q [f(t) - \sum_{k=0}^{m-1} f^k(0) g_{k+1}(t)], \tag{6}$$

where D_t^q denotes the Riemann-Liouville fractional derivative of order q defined as

$$D_t^q f(t) = \frac{d^m}{dt^m} \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f(s) ds, \quad t > 0, \quad m - 1 < q < m \qquad (7)$$

Definition 2.2 [14]. A function $u \in C([0,T]; H)$ is said to be a mild solution of equation (1) if it satisfies

$$
\begin{aligned}
u(t) &= S_q(t)(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g(s,u(s)) ds \\
&\quad + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,u(s)) ds, \ t \in [0,T], \\
u(0) &= \phi,
\end{aligned}
\qquad (8)
$$

where

$$S_q(t) = \int_0^\infty \zeta_q(\theta) S(t^q \theta) d\theta, \quad T_q(t) = q \int_0^\infty \theta \zeta_q(\theta) S(t^q \theta) d\theta.$$

Here $\zeta_q(\theta)$ is a probability density function defined on the interval $(0,\infty)$, satisfying the following properties

- $\zeta_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \zeta_q(\theta) d\theta = 1$;

- $\zeta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \times \psi_q(\theta^{-1/q}) \geq 0$, where

$$\psi_q(\theta) = \frac{1}{\pi} \Sigma_{n=1}^\infty (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \ \theta \in (0,\infty).$$

Now, we consider some assumptions on A, f and g.

Assumptions on A: We assume that linear operator A satisfies the following conditions.

(A1) A is a closed, positive, self-adjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H. We assume that A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e. $Au_i = \lambda_i u_i$ and $< u_i, u_j >= \delta_{ij}$, where δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

These assumptions on A imply that $-A$ generates an analytic semigroup, therefore there exist constants $M \geq 1$ and $\delta \geq 0$ such that

$$\|S(t)\| \leq M e^{-\delta t}, \quad t \geq 0.$$

So $-A$ is an infinitesimal generator of analytic semigroup. We assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e. $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ denotes the resolvent set of $-A$. If required, for $c > 0$

large enough, we may add cI to A, then $-(A + cI)$ is invertible and generates a bounded analytic semigroup. Also for $t > 0$, we have

$$
\begin{aligned}
\|AS(t)\| &\leq M t^{-1}, &(9) \\
\|A^\alpha S(t)\| &\leq M_\alpha t^{-\alpha}. &(10)
\end{aligned}
$$

The set of all continuous functions from $[0, T]$ into X, denoted by $C_T = C([0, T]; X)$ is a Banach space under the supremum norm given by

$$
\|\psi\|_T = \sup_{0 \leq t \leq T} \|\psi(t)\|, \ \psi \in C_T.
$$

Also, it can be shown easily that $C_T^\alpha = X^\alpha(T) = C([0, T]; D(A^\alpha))$ is a Banach space endowed with the supremum norm

$$
\|\psi\|_{T,\alpha} = \sup_{0 \leq t \leq T} \|\psi(t)\|_\alpha, \ \psi \in C_T^\alpha.
$$

It follows that A^α, $0 \leq \alpha \leq 1$, can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ which is dense in H. $D(A^\omega) \hookrightarrow D(A^\alpha)$, for $0 < \alpha < \omega$ such that embedding is continuous. Also, it can be easily shown that $D(A^\alpha)$ is a Banach space with norm $\|x\| = \|A^\alpha x\|$ and this norm is equivalent to the graph norm of A^α. For more details on the fractional powers of closed linear operator, we refer to Pazy [10].

Assumptions on f and g: We list the following assumptions on f and g:

(A2) The nonlinear map $f : [0, T] \times D(A^\alpha) \to H$ satisfies a local Lipschitz-like condition

$$
\|f(t, x) - f(t, y)\| \leq F_R(t) \|x - y\|_{T,\,\alpha} \tag{11}
$$

and

$$
\|f(t, x)\| \leq F_R(t), \tag{12}
$$

for all $t \in [0, T]$, $x, \ y \in B_R(X^\alpha(T), \phi)$, where $B_R(X^\alpha(T), \phi) := \{u \in X^\alpha(T) : \|u - \phi\|_{T,\,\alpha} \leq R\}$, and $F_R(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function depending on R.

(A3) For $(t, x) \in [0, T] \times D(A^\alpha)$, there exist positive constants L and β, $0 < \alpha < \beta < 1$ such that the function $A^\beta g$ is a continuous function satisfying the following conditions

$$
\|A^\beta g(t, x) - A^\beta g(s, y)\| \leq L\{\|t - s\|^\gamma + \|x - y\|_{T,\,\alpha}\} \tag{13}
$$

and

$$
L\|A^{\alpha - \beta}\| \leq 1, \tag{14}
$$

for all $t \in [0, T]$, $\gamma \in (0, 1]$ and $x, y \in B_R(X^\alpha(T), \phi)$, L is a constant.

Lemma 2.1 *[Zhou and Jiao [14]] For any fixed $t \geq 0$, $S_q(t)$ and $T_q(t)$ are bounded linear operators such that $\|S_q(t)x\| \leq M\|x\|$, $\|T_q(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|$ and $\|A^\alpha T_q(t)x\| \leq \frac{qM_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} t^{-q\alpha}$ for all $x \in D(A^\alpha)$, where M is a constant such that $\|S(t)\| \leq M$, for all $t \in [0, T]$.*

3 Existence and Uniqueness

In this section, we establish the existence and uniqueness of the solution to every approximate integral equations of (1) by using Banach fixed point theorem.

Let H_n denote the finite dimensional subspace of the Hilbert space H which is spanned by $\{u_0, u_1, \cdots, u_n\}$ and let $P^n : H \to H_n$ for $n = 1, 2, \cdots$, be the corresponding projection operators. Let $0 < T_0 \leq T < \infty$ be arbitrary but fixed constant chosen is such a way that

$$B = \max_{\{0 \leq t \leq T_0\}} \|A^\beta g(t, \phi)\|, \tag{15}$$

$$\|(S_q(t) - I)A^\alpha(\phi + g_n(0, \phi))\| \leq \frac{(1 - \varsigma)R}{3}, \tag{16}$$

$$\|A^{\alpha-\beta}\|LT_0^\gamma + M_{1+\alpha-\beta}C_1(L\widetilde{R} + B)\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} + M_\alpha F_R(T)C_2\frac{T_0^{q(1-\alpha)}}{(1-\alpha)}$$
$$< (1-\varsigma)\frac{R}{6}, \tag{17}$$

$$M_{1+\alpha-\beta}LC_1\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} + M_\alpha F_R(T)C_2\frac{T_0^{q(1-\alpha)}}{(1-\alpha)} < 1 - \varsigma, \tag{18}$$

where $L\|A^{\alpha-\beta}\| = \varsigma < 1$, $\widetilde{R} = \sqrt{R^2 + \|\phi_\alpha^2\|}$, $C_1 = \frac{\Gamma\{1-(\alpha-\beta)\}}{\Gamma\{1+q(\beta-\alpha)\}}$, $C_2 = \frac{\Gamma(2-\alpha)}{\Gamma\{1+q(1-\alpha)\}}$.

We define

$$g_n : [0, T] \times D(A^\alpha) \to H, \text{ such that } g_n(t, u(t)) = g(t, P^n u(t)) \tag{19}$$

and

$$f_n : [0, T] \times D(A^\alpha) \to H, \text{ such that } f_n(t, u(t)) = f(t, P^n u(t)), \tag{20}$$

for each n.

Now, we consider a map Q_n on $B_R(X^\alpha(T_0), \phi)$ defined by

$$Q_n(u)(t) = S_q(t)(\phi + g_n(0, \phi)) - g_n(t, (t)) + \int_0^t (t-s)^{q-1}AT_q(t-s)g_n(s, u(s))ds$$
$$+ \int_0^t (t-s)^{q-1}T_q(t-s)f_n(s, u(s))ds, \quad t \in [0, T_0], \tag{21}$$

for each $n = 0, 1, 2, \cdots$.

Theorem 3.1 *Let the assumptions* $(A1)$-$(A3)$ *hold. Then there exists a constant* T_0, $0 < T_0 < T$ *and a unique fixed point* $u_n \in B_R(X^\alpha(T_0), \phi)$ *of the operator* Q_n *for all* n *i.e.* u_n *satisfies the approximate integral equations*

$$u_n(t) = S_q(t)(\phi + g_n(0, \phi)) - g_n(t, u_n(t)) + \int_0^t \frac{AT_q(t-s)g_n(s, u_n(s))}{(t-s)^{1-q}}ds$$
$$+ \int_0^t \frac{T_q(t-s)f_n(s, u_n(s))}{(t-s)^{1-q}}ds, \quad t \in [0, T_0], \tag{22}$$

for each $n = 0, 1, 2, \cdots$.

Proof. First we prove the continuity of the map $t \to Q_n u(t)$ from $[0, T_0]$ into $D(A^\alpha)$ with respect to norm $\| \cdot \|_\alpha$. For any $u \in B_R(X^\alpha(T_0), \phi)$ and $t_1, t_2 \in [0, T_0]$ with $t_1 < t_2$, we have

$$
\begin{aligned}
A^\alpha[(Q_n u) t_2 - (Q_n u) t_1] \\
&= A^\alpha[(S_q(t_2) - S_q(t_1))(\phi + g(0, \phi))] \\
&\quad - A^{\alpha-\beta}[A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)] \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} T_q(t_2 - s) A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))] ds \\
&\quad + \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] T_q(t_2 - s) A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))] ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{q-1} [T_q(t_2 - s) - T_q(t_1 - s)] A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))] ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} A^\alpha T_q(t_2 - s) f_n(s, u(s)) ds \\
&\quad + \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] A^\alpha T_q(t_2 - s) f_n(s, u(s)) ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{q-1} A^\alpha [T_q(t_2 - s) - T_q(t_1 - s)] f_n(s, u(s)) ds, \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8.
\end{aligned}
$$

Hence, we have

$$
\|(Q_n u) t_2 - (Q_n u) t_1\| \le \sum_{i=1}^8 \|K_i\|. \tag{23}
$$

We have

$$
\begin{aligned}
K_1 &= A^\alpha[(S_q(t_2) - S_q(t_1))(\phi + g(0, \phi))], \\
&= \int_0^\infty \zeta_q(\theta) \left[\int_{t_1}^{t_2} q \theta t^{q-1} A^\alpha S(t^\beta \theta) A(\phi + g(0, \phi)) dt \right] d\theta,
\end{aligned}
$$

taking norm on both the sides, we get (see [7, p. 101] and [8, p. 437])

$$
\begin{aligned}
\|K_1\| &\le \int_0^\infty \zeta_q(\theta) \left[\int_{t_1}^{t_2} q \theta t^{q-1} \|A^\alpha S(t^\beta \theta)\| \|A(\phi + g(0, \phi))\| dt \right] d\theta, \\
&\le M_\alpha \int_0^\infty \theta^{1-\alpha} \zeta_q(\theta) \int_{t_1}^{t_2} t^{q(1-\alpha)-1} \|A(\phi + g(0, \phi))\| dt d\theta, \\
&\le \frac{C_2 M_\alpha}{(1-\alpha)} \|A(\phi + g(0, \phi))\| (t_2^{q(1-\alpha)} - t_1^{q(1-\alpha)}), \\
&\le C_2 M_\alpha q \|A(\phi + g(0, \phi))\| (t_1 + \kappa(t_2 - t_1))^{q(1-\alpha)-1} (t_2 - t_1), \\
&\le C_2 M_\alpha q \|A(\phi + g(0, \phi))\| \kappa^{q(1-\alpha)-1} (t_2 - t_1)^{q(1-\alpha)}, \tag{24}
\end{aligned}
$$

and

$$
\|K_2\| \le \|A^{\alpha-\beta}\| \|A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)\| \le L \|A^{\alpha-\beta}\| (t_1 - t_2)^\gamma. \tag{25}
$$

Further, we have

$$
\begin{aligned}
\|K_3\| &\leq C_1 q M_{1+\alpha-\beta} \int_{t_1}^{t_2} (t_2 - s)^{\alpha(\beta-\alpha)-1} \|A^\beta g_n(s, u(s))\| ds, \\
&\leq C_1 M_{1+\alpha-\beta}[(L\widetilde{R} + B)] \frac{(t_2 - t_1)^{q(\beta-\alpha)}}{(\beta - \alpha)},
\end{aligned}
\tag{26}
$$

and

$$
K_4 = \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] A^{1+\alpha-\beta} T_q(t_2 - s) A^\beta g_n(s, u(s)) ds.
$$

Taking norm on both the sides, we get

$$
\begin{aligned}
\|K_4\| &\leq C_1 q M_{1+\alpha-\beta} \int_0^{t_1} \{(t_1 - s)^{-q(1+\alpha-\beta)} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] \\
&\quad \times \|A^\beta g_n(s, u(s))\|\} ds, \\
&\leq C_1 q M_{1+\alpha-\beta}[(L\widetilde{R} + B)] \\
&\quad \times \int_0^{t_1} (t_1 - s)^{-q(1+\alpha-\beta)} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds, \\
&\leq C_1 q M_{1+\alpha-\beta}[(L\widetilde{R} + B)] \int_0^{t_1} (t_1 - s)^{\lambda-1} [(t_2 - s)^{-\lambda\mu} - (t_1 - s)^{-\lambda\mu}] ds,
\end{aligned}
$$

where $\lambda = 1 - q(1 + \alpha - \beta)$, $\mu = \frac{q-1}{1-q(1+\alpha-\beta)}$ (see Muslim, [8] and El-Borai [9]).
Hence, after some calculations we get

$$
\|K_4\| \leq C_1 q M_{1+\alpha-\beta}[(L\widetilde{R} + B)] \mu \delta^{\mu-1} (1 - b)^{-\lambda(1-\mu)-1} (t_2 - t_1)^{\lambda(1-\mu)},
\tag{27}
$$

where $b = (1 - (\frac{\mu}{\lambda})^{\frac{1}{\lambda\mu}})$ and $0 \leq \delta \leq 1$.
Similarly,

$$
\|K_5\| \leq C_1 q M_{1+\alpha-\beta}[(L\widetilde{R} + B)] \mu_1 \delta_1^{\mu_1-1} (1 - b_1)^{-q(1-\mu_1)-1} (t_2 - t_1)^{q(1-\mu_1)},
\tag{28}
$$

where $\mu_1 = 1 + \alpha - \beta$, $b_1 = (1 - (\frac{\mu_1}{q})^{\frac{1}{q\mu_1}})$ and $0 \leq \delta_1 \leq 1$ (see [8,9]).

$$
\begin{aligned}
\|K_6\| &\leq \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|A^\alpha T_q(t_2 - s)\| \|f_n(s, u(s))\| ds, \\
&\leq F_R(T) M_\alpha C_2 \frac{(t_2 - t_1)^{q(1-\alpha)}}{(1 - \alpha)}.
\end{aligned}
\tag{29}
$$

Also, we notice that

$$
\begin{aligned}
\|K_7\| &\leq \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1}) \|A^\alpha T_q(t_2 - s)\| \|f_n(s, u(s))\| ds, \\
&\leq M_\alpha C_2 q \int_0^{t_1} ((t_2 - s)^{q-1} - (t_1 - s)^{q-1})(t_1 - s)^{-q\alpha} \|f_n(s, u(s))\| ds, \\
&\leq M_\alpha C_2 F_R(T) q \int_0^{t_1} ((t_2 - s)^{-\mu_2 \lambda_1'} - (t_1 - s)^{-\mu_2 \lambda_1'})(t_1 - s)^{\lambda_1'-1} ds, \\
&\leq M_\alpha C_2 q F_R(T) \mu_2 \delta_2^{\mu_2-1} (1 - b_2)^{-\lambda_1'(1-\mu_2)-1} (t_2 - t_1)^{\lambda_1'(1-\mu_2)},
\end{aligned}
\tag{30}
$$

where $\lambda_1' = 1 - q\alpha$, $\mu_2 = \frac{1-q}{1-q\alpha}$, $b_2 = (1 - (\frac{\mu_2}{\lambda_1'})^{\frac{1}{\mu_2 \lambda_1'}})$, $0 \leq \delta_2 \leq 1$, and

$$
\begin{aligned}
\|K_8\| &\leq \int_0^{t_1} (t_1 - s)^{q-1} \|A^\alpha[T_q(t_2 - s) - T_q(t_1 - s)]\|\|f_n(s, u(s))\| ds, \\
&\leq C_2 q M_\alpha F_R(T) \int_0^{t_1} (t_1 - s)^{q-1} [(t_2 - s)^{-q\alpha} - (t_1 - s)^{-q\alpha}] ds, \\
&\leq C_2 q M_\alpha F_R(T) \alpha \delta_3^{\alpha-1} (1 - b_3)^{-q(1-\alpha)-1} (t_2 - t_1)^{q(1-\alpha)}, \quad (31)
\end{aligned}
$$

where $b_3 = (1 - (\frac{\alpha}{q})^{\frac{1}{q\alpha}})$ and $0 \leq \theta_3 \leq 1$. Using (24)-(31) in (23), we get that $(Q_n u)$ is Hölder continuous on $[0, T_0]$. Hence the continuity of the map $t \to (Q_n u)(t)$ is proved. Next we show that $Q_n(B_R(X^\alpha(T_0), \phi)) \subseteq B_R(X^\alpha(T_0), \phi)$. For any element $u \in B_R(X^\alpha(T_0), \phi)$, we have

$$
\begin{aligned}
\|(Q_n u)(t) - \phi\|_\alpha &\leq \|(S_q(t) - I)A^\alpha(\phi + g_n(0, \phi))\| \\
&\quad + \|A^{\alpha-\beta}\|\|A^\beta g_n(0, \phi) - A^\beta g_n(t, u(t))\| \\
&\quad + \int_0^t (t - s)^{q-1} \|A^{1+\alpha-\beta} T_q(t - s)\|\|A^\beta g_n(s, u(s))\| ds \\
&\quad + \int_0^t (t - s)^{q-1} \|T_q(t - s)\|_\alpha \|f_n(s, u(s))\| ds, \\
&\leq \|(S_q(t) - I)A^\alpha(\phi + g_n(0, \phi))\| \\
&\quad + \|A^{\alpha-\beta}\|\|A^\beta g_n(0, \phi) - A^\beta g_n(t, u(t))\| \\
&\quad + M_{1+\alpha-\beta} C_1 q \int_0^t (t - s)^{q(\beta-\alpha)-1} \|A^\beta g_n(s, u(s))\| ds \\
&\quad + M_\alpha C_2 q \int_0^t (t - s)^{q(1-\alpha)-1} \|f_n(s, u(s))\| ds, \\
&\leq \|(S_q(t) - I)(\phi + g_n(0, \phi))\| + \|A^{\alpha-\beta}\|L\{T_0^\gamma + \|u(t) - \phi\|\} \\
&\quad + M_{1+\alpha-\beta} C_1 \{(L\tilde{R} + B)\} \frac{T_0^{q(\beta-\alpha)}}{(\beta - \alpha)} \\
&\quad + M_\alpha C_2 F_R(T) \frac{T_0^{q(1-\alpha)}}{(1 - \alpha)}. \\
&\leq R.
\end{aligned}
$$

Taking supremum over $[0, T_0]$, we get

$$
\|(Q_n u) - \phi\|_{T_0, \alpha} \leq R. \quad (32)
$$

This implies that $Q_n(B_R(X^\alpha(T_0), \phi)) \subseteq B_R(X^\alpha(T_0), \phi)$.

In the next step, our aim is to show that Q_n is a strict contraction mapping on $B_R(X^\alpha(T_0), \phi)$. Let for all $t \in [0, T_0]$ and $u_1, u_2 \in B_R(X^\alpha(T_0), \phi)$, we have

$$
\begin{aligned}
\|(Q_n u_1)(t) - (Q_n u_2)(t)\|_\alpha &\leq \|A^{\alpha-\beta}\|\|A^\beta g_n(t, u_1) - A^\beta g_n(t, u_2)\| \\
&+ \int_0^t (t - s)^{q-1} \|A^{1+\alpha-\beta} T_q(t - s)\|[\|A^\beta g_n(s, u_1(s)) - A^\beta g_n(s, u_2(s))\|] ds \\
&+ \int_0^t (t - s)^{q-1} \|A^\alpha T_q(t - s)\|\|f_n(s, u_1(s)) - f_n(s, u_2(s))\| ds, \quad (33)
\end{aligned}
$$

From the assumptions $(A2)$ - $(A3)$, we have

$$\|A^\beta g_n(t, u_1) - A^\beta g_n(t, u_2)\| \le L\|u_1(t) - u_2(t)\|_\alpha \le L\|u_1 - u_2\|_{T_0,\,\alpha}, \qquad (34)$$

$$\|f_n(s, u_1) - f_n(s, u_2)\| \le F_R(T)\|u_1(s) - u_2(s)\|_\alpha \le F_R(T)\|u_1 - u_2\|_{T_0,\,\alpha}. \qquad (35)$$

Using inequalities (34) and (35) in (33), we get

$$
\begin{aligned}
\|Q_n u_1(t) - Q_n u_2(t)\|_\alpha \;\le\; & [\|A^{\alpha-\beta}\|\, L + M_{1+\alpha-\beta} L C_1 \frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} \\
& + M_\alpha F_R(T) C_2 \frac{T_0^{q(1-\alpha)}}{(1-\alpha)}]\|u_1(t) - u_2(t)\|_\alpha. \qquad (36)
\end{aligned}
$$

Taking supremum over $[0, T_0]$, we get

$$
\begin{aligned}
\|Q_n u_1 - Q_n u_2\|_{T_0,\,\alpha} \;\le\; & [\|A^{\alpha-\beta}\|\, L + M_{1+\alpha-\beta} L C_1 \frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} \\
& + M_\alpha F_R(T) C_2 \frac{T_0^{q(1-\alpha)}}{(1-\alpha)}]\|u_1 - u_2\|_{T_0,\,\alpha}. \qquad (37)
\end{aligned}
$$

We use (15)–(18) in the inequality (37) and get that Q_n is a strict contraction on $B_R(X^\alpha(T_0), \phi)$. Hence, by the fixed point theorem, there exists a unique $u_n \in B_R(X^\alpha(T_0), \phi)$ such that $Q_n u_n = u_n$. which implies that u_n satisfies the integral equation (22) for each $n = 1, 2, \cdots$. This completes the proof of the theorem.

Lemma 3.1 *Suppose that assumptions $(A1) - -A(3)$ are satisfied. If $\phi \in D(A^\alpha)$, where $0 < \alpha < 1$, then $u_n(t) \in D(A^v)$ for all $t \in (0, T_0]$ with $0 \le v < 1$. Furthermore, if $\phi \in D(A)$ then $u_n(t) \in D(A^v)$ for all $t \in [0, T_0]$ with $0 \le v < 1$.*

From Theorem 3.1, we have that there exists a unique $u_n \in B_R(X^\alpha(T_0), \phi)$ such that u_n satisfies the equation (22). Theorem 2.6.13 in Pazy [10] implies that $T(t) : H \to D(A^v)$ for $t > 0$ and $0 \le v < 1$ and for $0 \le v \le \eta < 1$, $D(A^\eta) \subseteq D(A^v)$. From the assumption $(A3)$ we have that the map $t \mapsto A^\beta g(t, u_n(t))$ is Hölder continuous on $[0, T_0]$ with the exponent $\rho = min\{\gamma, v\}$. It is easy to see that Hölder continuity of u_n can be established using the similar arguments from equation (23), (30)-(31). From Theorem 4.3.2 in Pazy [10], we have

$$\int_0^t (t-s)^{q-1} T_q(t-s) A^\beta g_n(s, u_n) ds \in D(A).$$

Also from Theorem 1.2.4 in Pazy [10], we have that $T(t)x \in D(A)$ if $x \in D(A)$. The result follows from these facts and the fact that $D(A) \subseteq D(A^v)$ for $0 \le v \le 1$.

Corollary 3.1 *Suppose that $(A1)$, $(A2)$ and $(A3)$ are satisfied. If $\phi \in D(A^\alpha)$ with $0 < \alpha < 1$, then for any $t_0 \in (0, T_0]$ there exists a constant U_{t_0} such that*

$$\|A^v u_n(t)\| \le U_{t_0}$$

for all $t_0 \le t \le T_0$ independent of n, where $0 < \alpha < v < \beta$.

Proof. Let us assume that $\phi \in D(A^\alpha)$. Applying A^υ on both the sides of (22) and using (9)–(10) for $t \in [t_0, T_0]$ and $\alpha < \upsilon < \beta$, we obtain

$$
\begin{aligned}
\|u_n(t)\|_\upsilon &\leq \|A^\upsilon S_q(t)(\phi + g_n(0, \phi))\| + \|A^{\upsilon-\beta}\|\|A^\beta g_n(t, u_n)\| \\
&\quad + \int_0^t (t-s)^{q-1} \|A^{1+\upsilon-\beta} T_q(t-s)\|\|A^\beta g_n(s, u_n)\| ds \\
&\quad + \int_0^t (t-s)^{q-1} \|A^\upsilon T_q(t-s)\|\|f_n(s, u_n)\| ds, \\
&\leq M_\upsilon t_0^{-q\upsilon}(\|\phi\| + \|g_n(0, \phi)\|) + \|A^{\upsilon-\beta}\|[(L\widetilde{R} + B)] \\
&\quad + M_{1+\upsilon-\beta}(L\widetilde{R} + B)C_3 \frac{T_0^{q(\beta-\upsilon)}}{(\beta-\upsilon)} + M_\upsilon C_4 F_R(T) \frac{T_0^{q(1-\upsilon)}}{(1-\upsilon)} \\
&\leq U_{t_0},
\end{aligned}
$$

where $C_3 = \frac{\Gamma(1-\upsilon+\beta)}{\Gamma 1 + q(-\upsilon+\beta)}$, $C_4 = \frac{\Gamma(2-\upsilon)}{\Gamma 1 + q(1-\upsilon)}$.

Again, for $t \in [0, T_0]$ and $0 < \upsilon \leq \alpha$, $\phi \in D(A^\upsilon)$ and

$$
\begin{aligned}
\|u_n(t)\|_\upsilon &\leq M(\|A^\upsilon \phi\| + \|g_n(0, \phi)\|_\upsilon) + \|A^{\upsilon-\beta}\|[LR + B] \\
&\quad + M_{1+\upsilon-\beta}(L\widetilde{R} + B)C_3 \frac{T_0^{q(\beta-\upsilon)}}{(\beta-\upsilon)} + M_\upsilon C_4 F_R(T) \frac{T_0^{q(1-\upsilon)}}{(1-\upsilon)} \\
&\leq U_{t_0}.
\end{aligned}
$$

Furthermore, we have if $\phi \in D(A^\beta)$ then $\phi \in D(A^\upsilon)$ for $0 < \upsilon \leq \beta$ and required result can be proved easily.

4   Convergence of Solution

In this section we will show the convergence of the solution $u_n \in X^\alpha(T_0)$ of the approximate integral equations (22) to a unique solution $u(\cdot) \in X^\alpha(T_0)$ of the equation (8).

Theorem 4.1 *Let the assumptions* (A1)–(A3) *hold. If* $\phi \in D(A^\alpha)$, *then for any* $t_0 \in (0, T_0]$,

$$
\lim_{n \to \infty} \sup_{\{n \geq m, \, t_0 \leq t \leq T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.
$$

Proof. For $n \geq m$, we have

$$
\begin{aligned}
A^\alpha[u_n(t) - u_m(t)] &= S_q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi)) \\
&\quad + \int_0^t (t-s)^{q-1} \times [A^{\alpha+1} T_q(t-s)\{g_n(s, u_n) - g_m(s, u_m)\}] ds \\
&\quad + \int_0^t (t-s)^{q-1} A^\alpha T_q(t-s)[f_n(s, u_n) - f_m(s, u_m)] ds. \quad (38)
\end{aligned}
$$

Now, let $0 < \alpha < \nu < \beta$, then we have

$$
\begin{aligned}
\|f_n(t, u_n) - f_m(t, u_m)\| &\leq \|f_n(t, u_n) - f_n(t, u_m)\| + \|f_n(t, u_m) - f_m(t, u_m)\|, \\
&\leq F_R(T)\|u_n(t) - u_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_\alpha.
\end{aligned}
$$

Also,

$$\|(P^n - P^m)u_m(t)\|_\alpha \quad \leq \quad \|A^{\alpha-\beta}(P^n - P^m)A^\nu u_m(t)\| \leq \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|.$$

Thus, we have

$$\|f_n(t, u_n) - f_m(t, u_m)\| \leq F_R(T)[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|]. \qquad (39)$$

Similarly,

$$\|A^\beta g_n(t, u_n) - A^\beta g_m(t, u_m)\| \quad \leq \quad L[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|]. \quad (40)$$

From (38), (39) and (40) and for $0 < t_0' < t_0$, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|S_q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\|$$

$$\leq \quad \|S_q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\| + \|A^{\alpha-\beta}\|\|A^\beta g_n(t, u_n) - A^\beta g_m(t, u_m)\|$$

$$+(\int_0^{t_0'} + \int_{t_0'}^t)(t-s)^{q-1}\|A^{1+\alpha-\beta}T_q(t-s)\| \times [\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|]ds$$

$$+(\int_0^{t_0'} + \int_{t_0'}^t)(t-s)^{q-1}\|A^\alpha T_q(t-s)\|\|f_n(s, u_n) - f_m(s, u_m)\|ds. \qquad (41)$$

The first term of (41) is estimated as

$$\|S^q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\| \quad \leq \quad M\|A^{\alpha-\beta}\|\|A^\beta g(0, P^n\phi) - A^\beta g(0, P^m\phi)\|$$
$$\leq \quad M\|A^{\alpha-\beta}\|L\|(P^n - P^m)A^\alpha\phi\|. \qquad (42)$$

We estimate the first and third integrals as

$$\int_0^{t_0'}(t-s)^{q-1}\|A^{1+\alpha-\beta}T_q(t-s)\|\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|ds$$

$$\leq 2M_{1+\alpha-\beta}C_1 q(LR_1 + B_1) \times (t_0 - t_0')^{q(\beta-\alpha)-1}t_0', \qquad (43)$$

$$\int_0^{t_0'}(t-s)^{q-1}\|A^\alpha T_q(t-s)\| \quad \times \quad \|f_n(s, u_n) - f_m(s, u_m)\|ds$$

$$\leq 2M_\alpha C_2 F_R(T)q(t_0 - t_0')^{q(1-\alpha)-1}t_0'. \qquad (44)$$

From the second and fourth integrals, we have

$$\int_{t_0'}^t(t-s)^{q-1}\|A^{1+\alpha-\beta}T_q(t-s)\|\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|ds$$

$$\leq M_{1+\alpha-\beta}LC_1 q \int_{t_0'}^t(t-s)^{q(\beta-\alpha)-1}[\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(s)\|]ds,$$

$$\leq M_{1+\alpha-\beta}LC_1 q(\frac{U_{t_0'}T_0^{q(\beta-\alpha)}}{\lambda_m^{\nu-\alpha}q(\beta-\alpha)} + \int_{t_0'}^t(t-s)^{q(\beta-\alpha)-1}\|u_n(s) - u_m(s)\|_\alpha ds), \quad (45)$$

and

$$\int_{t_0'}^{t} (t-s)^{q-1} \|A^\alpha T_q(t-s)\| \|f_n(s,u_n) - f_m(s,u_m)\| ds$$

$$\leq M_\alpha F_R(T) C_2 q \int_{t_0'}^{t} (t-s)^{q(1-\alpha)-1} [\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}} \|A^\nu u_m(s)\|] ds,$$

$$\leq M_\alpha F_R(T) C_2 q \left(\frac{U_{t_0'} T_0^{q(1-\alpha)}}{\lambda_m^{\nu-\alpha} q(1-\alpha)} + \int_{t_0'}^{t} (t-s)^{q(1-\alpha)-1} \|u_n(s) - u_m(s)\|_\alpha ds \right). \quad (46)$$

Using (42)–(46) in (41), we obtain

$$
\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha \leq \ & M\|A^{\alpha-\beta}\| \|(P^n - P^m)A^\alpha \phi\| \\
& + \|A^{\alpha-\beta}\| L[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}} \|A^\nu u_m(t)\|] \\
& + 2\left(\frac{M_{1+\alpha-\beta} C_1 q(LR_1 + B_1)}{(t_0 - t_0')^{q(\alpha-\beta)-1}} + \frac{M_\alpha C_2 F_R(T) q}{(t_0 - t_0')^{q(\alpha-1)-1}} \right) t_0' + M_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}} \\
& + \int_{t_0'}^{t} \left(\frac{M_\alpha q C_2 F_R(T)}{(t-s)^{q(\alpha-1)+1}} + \frac{M_{1+\alpha-\beta} q L C_1}{(t-s)^{q(\alpha-\beta)+1}} \right) \\
& \quad \times [\|u_n(s) - u_m(s)\|_\alpha] ds, \quad (47)
\end{aligned}
$$

where

$$M_{\alpha,\beta} = M_\alpha F_R(T) C_2 \frac{T_0^{q(1-\alpha)}}{(1-\alpha)} + M_{1+\alpha-\beta} L C_1 \frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)}. \quad (48)$$

Also, we have $\|A^{\alpha-\beta}\| L < 1$. Therefore inequality (47) becomes

$$
\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha \leq \ & \frac{1}{(1 - \|A^{\alpha-\beta}\| L)} \{ M\|(P^n - P^m)A^\alpha \phi\| + \|A^{\alpha-\beta}\| L \frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}} \\
& + 2\left(\frac{M_{1+\alpha-\beta} C_1 q(LR_1 + B_1)}{(t_0 - t_0')^{q(\alpha-\beta)-1}} + \frac{M_\alpha C_2 q F_R(T)}{(t_0 - t_0')^{q(\alpha-1)-1}} \right) t_0' + M_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}} \\
& + \int_{t_0'}^{t} \left(\frac{M_\alpha q C_2 F_R(T)}{(t-s)^{q(\alpha-1)+1}} + \frac{M_{1+\alpha-\beta} L C_1 q}{(t-s)^{q(\alpha-\beta)+1}} \right) \\
& \quad \times [\|u_n(s) - u_m(s)\|_\alpha] ds \}. \quad (49)
\end{aligned}
$$

Taking supremum over $[t_0, T_0]$, we get

$$
\begin{aligned}
\sup_{t \in [t_0, T_0]} \|u_n(t) - u_m(t)\|_\alpha & \\
\leq \ & \frac{1}{(1 - \|A^{\alpha-\beta}\| L)} \{ M\|(P^n - P^m)A^\alpha \phi\| + \|A^{\alpha-\beta}\| L \frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}} \\
& + 2\left(\frac{M_{1+\alpha-\beta} C_1 q(LR_1 + B_1)}{(t_0 - t_0')^{q(\alpha-\beta)-1}} + \frac{M_\alpha C_2 q F_R(T)}{(t_0 - t_0')^{q(\alpha-1)-1}} \right) t_0' + M_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}} \\
& + \int_{t_0'}^{t} \left(\frac{M_\alpha q C_2 F_R(T)}{(t-s)^{q(\alpha-1)+1}} + \frac{M_{1+\alpha-\beta} L q C_1}{(t-s)^{q(\alpha-\beta)+1}} \right) \|u_n - u_m\|_{T_0, \alpha} ds \}. \quad (50)
\end{aligned}
$$

Applying Gronwall's inequality to the above inequality, taking $m \to \infty$, we obtain

$$\lim_{m \to \infty} \sup_{\{n \geq m, \, t_0 \leq \, t \, \leq T_0\}} \|u_n(t) - u_m(t)\|_\alpha$$

$$\leq \frac{2}{(1 - \|A^{\alpha - \beta}\|L)}\Big[\frac{M_{1 + \alpha - \beta}C_1(LR_1 + B_1)}{(t_0 - t_0')^{q(\alpha - \beta) - 1}} + \frac{M_\alpha C_2 F_R(T)}{(t_0 - t_0')^{q(\alpha - 1) - 1}}\Big]t_0' \times C, \quad (51)$$

where C is arbitrary constant. The right hand side of inequality (51) may be made as small as possible by taking t_0' (as t_0' is arbitrary) sufficiently small. This completes the proof of the theorem.

Corollary 4.1 *Let assumptions $(A1) - (A3)$ hold. If $\phi \in D(A)$, then*

$$\sup_{\{n \geq m, \, 0 \leq t \leq T_0\}} \|A^\alpha[u_n(t) - u_m(t)]\| \to 0,$$

as $m \to \infty$.

Proof. In this case, we have

$$\|S_q(t)\phi\|_\alpha \leq M\|\phi\|_\alpha. \quad (52)$$

Then from the inequality (52), Lemma (3.1) and Corollary (3.1) we get that in the proof of Theorem (4.1), we can take $t_0 = 0$ to get the required result.

Theorem 4.2 *Suppose that $(A1) - (A3)$ are satisfied and $\phi \in D(A^\alpha)$. Then, there exist T_0, $0 < T_0 \leq T$ and a unique function $u \in X^\alpha(T_0)$ such that $u_n \to u$ as $n \to \infty$ in $X^\alpha(T_0)$ and $u \in X^\alpha(T_0)$ satisfies the equation (8) on $[0, T_0]$.*

Proof. Let $\phi \in D(A^\alpha)$. Since $A^\alpha u_n(t) \to A^\alpha u(t)$ as $n \to \infty$, for $0 < t \leq T_0$ and $u_n(0) = u(0) = \phi$ for all n. Since $u_n \in B_R(X^\alpha(T_0), \phi)$, it follows that $u \in B_R(X^\alpha(T_0), \phi)$. Further, for any $0 < t_0 \leq T_0$, we have

$$\sup_{\{t_0 \leq t \leq T_0\}} \|u_n(t) - u(t)\|_\alpha = 0.$$

Also,

$$\begin{aligned}\|f_n(t, u_n) - f(t, u)\| &= \|f(t, P^n u_n) - f(t, u)\|, \\ &\leq F_R(T)[\|u_n - u\|_\alpha + \|(P^n - I)u\|_\alpha], \quad (53)\end{aligned}$$

and

$$\begin{aligned}\|A^\beta g_n(t, u_n) - A^\beta g(t, u)\| &= \|A^\beta g(t, P^n u_n) - A^\beta g(t, u)\|, \\ &\leq L[\|u_n - u\|_\alpha + \|(P^n - I)u\|]. \quad (54)\end{aligned}$$

Taking supremum on $[t_0, T_0]$, we get

$$\begin{aligned}\sup_{\{t_0 \leq t \leq T_0\}} \|f_n(t, u_n) - f(t, u)\| &\leq F_R(T)[\|u_n - u\|_{T_0, \, \alpha} + \|(P^n - I)u\|_{T_0, \, \alpha}], \\ &\to 0,\end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned}\sup_{\{t_0 \leq t \leq T_0\}} \|A^\beta g_n(t, u_n) - A^\beta g(t, u)\| &\leq L[\|u_n - u\|_{T_0, \, \alpha} + \|(P^n - I)u\|_{T_0, \alpha}], \\ &\to 0,\end{aligned}$$

as $n \to \infty$. Now, for $0 < t_0 < t$, we may rewrite (22) as

$$u_n(t) = S_q(\phi + g_n(0, \phi)) - g_n(t, u_n) + (\int_0^{t_0} + \int_{t_0}^t)(t-s)^{q-1}AT_q(t-s)g_n(s, u_n)ds$$

$$+ (\int_0^{t_0} + \int_{t_0}^t)(t-s)^{q-1}T_q(t-s)f_n(s, u_n)ds. \tag{55}$$

We have

$$\| \int_0^{t_0} (t-s)^{q-1}AT_q(t-s)g_n(s, u_n)ds \| \leq \int_0^{t_0} (t-s)^{q-1}\|A^{1-\beta}T_q(t-s)\|$$

$$\times [\|A^\beta g_n(s, u_n)\|]ds,$$

$$\leq M_{1-\beta}C_1'\{(L\widetilde{R} + B)\}T_0^{q\beta-1}t_0, \tag{56}$$

and

$$\| \int_0^{t_0} (t-s)^{q-1}T_q(t-s)f_n(s, u_n)ds \| \leq \int_0^{t_0} (t-s)^{q-1}\|T_q(t-s)\|\|f_n(s, u_n)\|ds,$$

$$\leq MC_1'\{(L\widetilde{R} + B)\}T_0^{q\beta-1}t_0, \tag{57}$$

where $C_1' = \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}$ and $C_2' = \frac{q}{\Gamma(1+q)}$. Thus, we have

$$\|u_n(t) - S_q(t)(\phi + g_n(0, \phi)) + g_n(t, u_n) - \int_{t_0}^t (t-s)^{q-1}AT_q(t-s)g_n(s, u_n)ds$$

$$- \int_{t_0}^t (t-s)^{q-1}T_q(t-s)f_n(s, u_n)ds\|$$

$$\leq M_{1-\beta}C_1'\{(L\widetilde{R} + B)\}T_0^{q\beta-1}t_0 + MC_2'F_R(T)T_0^{q-1}t_0.$$

Let $n \to \infty$, in the above inequality, we get

$$\|u(t) - S_q(t)(\phi + g(0, \phi)) + g(t, u(t)) - \int_{t_0}^t (t-s)^{q-1}AT_q(t-s)g(s, u(s))ds$$

$$- \int_{t_0}^t (t-s)^{q-1}T_q(t-s)f(s, u(s))ds\|$$

$$\leq M_{1-\beta}C_1'\{(L\widetilde{R} + B)\}T_0^{q\beta-1}t_0 + MC_2'F_R(T)T_0^{q-1}t_0. \tag{58}$$

Since $0 < t_0 \leq T_0$ is arbitrary, we get that u satisfies the integral equation (8).

Now, let $\phi \in D(A)$. Corollary 4.1 implies that there exists $u \in X^\alpha(T_0)$ such that $u_n \to u$ in $X^\alpha(T_0)$. Since $u_n \in B_R(X^\alpha(T_0), \phi)$ for each n, u is also in $B_R(X^\alpha(T_0), \phi)$. Further, we have

$$\sup_{\{0 \leq t \leq T_0\}} \|f_n(t, u_n) - f(t, u)\| \leq F_R(T)[\|u_n - u\|_{T_0, \alpha} + \|(P^n - I)u\|_{T_0, \alpha}],$$

$$\to 0, \text{ as } n \to \infty, \tag{59}$$

and

$$\sup_{\{0 \leq t \leq T_0\}} \|A^\beta g_n(t, u_n) - A^\beta g(t, u)\| \leq L[\|u_n - u\|_{T_0, \alpha} + \|(P^n - I)u\|_{T_0, \alpha}],$$

$$\to 0, \text{ as } n \to \infty. \tag{60}$$

Using (59), (60) and (22), we obtain

$$u(t) = S_q(t)(\phi + g(0, \phi)) - g(t, u(t)) + \int_0^t (t - s)^{q-1} A T_q(t - s) g(s, u(s)) ds$$
$$+ \int_0^t (t - s)^{q-1} T_q(t - s) f(s, u(s)) ds. \qquad (61)$$

Hence, this completes the proof of the theorem.

Now, we shall show the uniqueness of the solution to equation (61). Let u_1 and u_2 be the two solutions of (61). We have

$$\begin{aligned}
u_1(t) - u_2(t) &= -\{g(t, u_1(t)) - g(t, u_2(t))\} \\
&\quad + \int_0^t (t - s)^{q-1} A T_q(t - s)[g(s, u_1) - g(s, u_2)] ds \\
&\quad + \int_0^t (t - s)^{q-1} T_q(t - s)[f(s, u_1) - f(s, u_2)] ds,
\end{aligned}$$

and thus

$$\begin{aligned}
\|A^\alpha[u_1(t) - u_2(t)]\| &\leq \|A^{\alpha-\beta}\| \|A^\beta g(t, u_1(t)) - A^\beta g(t, u_2(t))\| \\
&\quad + \int_0^t (t - s)^{q-1} \|A^{1+\alpha-\beta} T_q(t - s)\| \|A^\beta g(s, u_1) - A^\beta g(s, u_2)\| ds \\
&\quad + \int_0^t (t - s)^{q-1} \|A^\alpha T_q(t - s)\| \|f(s, u_1) - f(s, u_2)\| ds, \\
&\leq \|A^{\alpha-\beta}\| L \|u_1(t) - u_2(t)\|_\alpha \\
&\quad + M_{1+\alpha-\beta} C_1 L q \int_0^t (t - s)^{q(\beta-\alpha)-1} \|u_1(t) - u_2(t)\|_\alpha ds \\
&\quad + M_\alpha F_R(T) C_2 q \int_0^t (t - s)^{q(1-\alpha)-1} \|u_1(t) - u_2(t)\|_\alpha ds.
\end{aligned}$$

Since $\|A^{\alpha-\beta}\| L < 1$, therefore we obtain
$\|u_1(t) - u_2(t)\|_\alpha$

$$\leq \frac{1}{(1 - L\|A^{\alpha-\beta}\|)} \left[\int_0^t \left\{ \frac{M_{1+\alpha-\beta} C_1 q L}{(t - s)^{1-q(\beta-\alpha)}} + \frac{M_\alpha F_R(T) C_2 q}{(t - s)^{1-q(1-\alpha)}} \right\} \|u_1(t) - u_2(t)\|_\alpha ds \right].$$

Applying Gronwall's inequality, we obtain

$$\|u_1(t) - u_2(t)\|_\alpha = 0$$

for all $0 \leq t < T_0$. From the fact

$$\|u_1(t) - u_2(t)\| \leq \frac{1}{\lambda_0^\alpha} \|u_1(t) - u_2(t)\|_\alpha,$$

therefore, $u_1 = u_2$ on $[0, T_0]$. The proof of the theorem is complete.

5 Faedo-Galerkin Approximation

In this section, we will discuss the Faedo-Galerkin approximations of solutions and prove some convergence result for such approximations.

We know that for any $0 < T_0 < T$, we have a unique $u \in X^\alpha(T_0)$ satisfying the integral equation

$$
\begin{aligned}
u(t) \; = \; & S_q(t)(\phi + g(0, \phi)) - g(t, u(t)) + \int_0^t (t - s)^{q-1} A T_q(t - s) g(s, u(s)) ds \\
& + \int_0^t (t - s)^{q-1} T_q(t - s) f(s, u(s)) ds,
\end{aligned}
\tag{62}
$$

Also, there is a unique solutions $u_n \in X^\alpha(T_0)$ of the approximate integral equations

$$
\begin{aligned}
u_n(t) \; = \; & S_q(t)(\phi + g_n(0, \phi)) - g_n(t, u_n(t)) + \int_0^t (t - s)^{q-1} A T_q(t - s) g_n(s, u_n(s)) ds \\
& + \int_0^t (t - s)^{q-1} T_q(t - s) f_n(s, u_n(s)) ds.
\end{aligned}
\tag{63}
$$

We apply the projection on the above equation, then Faedo-Galerkin approximation is given by $v_n(t) = P^n u_n(t)$ satisfying

$$
\begin{aligned}
P^n u_n(t) \; = \; v_n(t) = \; & S_q(t)(P^n \phi + P^n g(0, P^n \phi)) - P^n g(t, v_n(t)) \\
& + \int_0^t (t - s)^{q-1} A T_q(t - s) P^n g(s, v_n(s)) ds \\
& + \int_0^t (t - s)^{q-1} T_q(t - s) P^n f(s, v_n(s)) ds.
\end{aligned}
\tag{64}
$$

Let the solution u of (62) and v_n of (64) have the representation

$$
u(t) \; = \; \sum_{i=0}^{\infty} \alpha_i(t) u_i, \quad \alpha_i(t) = (u(t), u_i) \;\; i = 0, 1, 2, \cdots,
\tag{65}
$$

$$
v_n(t) \; = \; \sum_{i=0}^{n} \alpha_i^n(t) u_i, \quad \alpha_i^n(t) = (v_n(t), u_i) \;\; i = 0, 1, 2, \cdots,
\tag{66}
$$

Using (66) in (64), we obtain a system of fractional order integro-differential equation of the form

$$
\frac{d^q}{dt^q}(\alpha_i^n(t) + g_i^n(t, \alpha_0^n(t), \alpha_1^n(t)..., \alpha_n^n)) + \lambda_i \alpha_i^n(t) = f_i^n(\alpha_0^n(t), \alpha_1^n(t)..., \alpha_n^n),
\tag{67}
$$

$$
\alpha_i^n(0) = \phi_i,
\tag{68}
$$

where

$$
g_i^n(t, \alpha_0^n(t), \alpha_1^n(t)..., \alpha_n^n)) = (g(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i),
$$

$$
f_i^n(\alpha_0^n(t), \alpha_1^n(t)..., \alpha_n^n) = (f(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i),
$$

and $\phi_i = (\phi, u_i)$, for $i = 1, 2, \cdots, n$. The system (67)–(68) determines the $\alpha_i^n(t)$'s.

As a consequence of Theorems 3.1 and 4.1, we have the following convergence result.

Theorem 5.1 *Let* $(A1) - (A3)$ *hold and* $\phi \in D(A^\alpha)$. *Then there exist functions* $v_n \in C([0, T_0], D(A^\alpha))$,

$$v_n(t) = S_q(t)(P^n\phi + P^n g(0, P^n\phi)) - P^n g(t, v_n(t))$$

$$+ \int_0^t (t - s)^{q-1} A T_q(t - s) P^n g(s, v_n(s)) ds$$

$$+ \int_0^t (t - s)^{q-1} T_q(t - s) P^n f(s, v_n(s)) ds, \;\; t \in [0, T_0]$$

and $u \in C([0, T_0], D(A^\alpha))$,

$$u(t) = S_q(t)(\phi + g(0, \phi)) - g(t, u(t)) + \int_0^t (t - s)^{q-1} A T_q(t - s) g(s, u(s)) ds$$

$$+ \int_0^t (t - s)^{q-1} T_q(t - s) f(s, u(s)) ds, \;\; t \in [0, T_0]$$

such that $v_n \to u$ *in* $C([0, T_0], D(A^\alpha))$ *as* $n \to \infty$.

Now, we show the convergence of $\alpha_i^n(t) \to \alpha_i(t)$. Consider the following

$$A^\alpha[u(t) - v_n(t)] = A^\alpha \Big[\sum_{i=0}^\infty (\alpha_i(t) - \alpha_i^n(t)) u_i\Big] = \sum_{i=0}^\infty \lambda_i^\alpha (\alpha_i(t) - \alpha_i^n(t)) u_i.$$

Therefore, we have

$$\|A^\alpha[u(t) - v_n(t)]\|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2.$$

We have the following convergence theorem.

Theorem 5.2 *We have the following result:*
(a) If $\phi \in D(A^\alpha)$ *for all* $t_0 \in (0, T_0]$, *then*

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T_0} \Big[\sum_{i=0}^n \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2\Big] = 0.$$

(b) If $\phi \in D(A)$ *for all* $t \in [0, T_0]$, *then*

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T_0} \Big[\sum_{i=0}^n \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2\Big] = 0.$$

The assertion of this theorem follows from the facts mentioned above and the following result.

Proposition 5.1 *Let* $(H1) - (H3)$ *hold and let* T_0 *be any number such that* $0 < T_0 < T$, *then we have the following.*

(a) If $\phi \in D(A^\alpha)$ *for all* $t_0 \in (0, T_0]$ *then*

$$\lim_{n\to\infty} \sup_{n \geq m,\, t_0 \leq t \leq T} \| A^\alpha[v_n(t) - v_m(t)]\| = 0. \tag{69}$$

(b) If $\phi \in D(A)$ for all $t_0 \in [0, T_0]$ then

$$\lim_{n \to \infty} \sup_{n \geq m,\, 0 \leq t \leq T} \| A^\alpha [v_n(t) - v_m(t)] \| = 0. \tag{70}$$

Proof. For $n \geq m$, we have

$$
\begin{aligned}
\| A^\alpha [v_n(t) - v_m(t)] \| &= \| A^\alpha [P^n u_n(t) - P^m u_m(t)] \|, \\
&\leq \| P^n [u_n(t) - u_m(t)] \|_\alpha + \| (P^n - P^m) u_m(t) \|_\alpha, \\
&\leq \| [u_n(t) - u_m(t)] \|_\alpha + \frac{1}{\lambda_m^{\vartheta - \alpha}} \| A^\vartheta u_m \|.
\end{aligned}
\tag{71}
$$

If $\phi \in D(A^\alpha)$, then the result in (a) follows from Theorem 4.1. If $\phi \in D(A)$, (b) follows from Corollary 4.1.

6   Application

Consider the following partial differential equation of fractional order of the form

$$\frac{d^q}{dt^q}[u(t,x) - \Delta u(x,t)] + \Delta^2 u(x,t) = F(x,t,u(t,x)), 0 < q \leq 1, \tag{72}$$

$$u(x,0) = u_0, \quad x \in \Omega, \tag{73}$$

with the homogenous boundary conditions. Were Ω is a bounded domain in the $\mathbb{R}^N$ with sufficiently smooth boundary $\partial \Omega$ and Δ is N-dimensional Laplacian and function h is sufficiently smooth in all arguments. We take $X = L^2(\Omega)$ and let A be the operator defined as $-Au = \Delta u$ with the domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{74}$$

Then equation (72) can be written as

$$\frac{d^q}{dt^q}[v(t) + Av(t)] + A^2 v(t) = F(t, v(t)), \tag{75}$$

$$v(0) = u_0. \tag{76}$$

It is well known that A is not invertible but $(A + cI)$ is invertible and $\|(A + cI)^{-1}\| \leq C$ for large enough $c > 0$. Therefore equation (75) can be written of the form (1) with $g(t, v) = (1 - c)(A + cI)^{-1}v$ and $f(t, v) = cA(A + cI)^{-1}v + F(t, (A + cI)^{-1}v)$. It is easy to see that operator A satisfies $(A1)$ and f and g satisfy $(A2)$ and $(A3)$ respectively. By applying the results of the earlier sections, we have the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (72)-(73).

Acknowledgment

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Stability of Dynamic Graph on Time Scales

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Abstract: This paper is aimed at establishing stability conditions for a dynamic graph on a time scale in terms of the matrix Lyapunov function and the principle of comparison. Dynamic graphs on time scales are defined in a linear spaces as a one-parameter mapping of the space of graphs with N nodes into itself. In the analysis of the dynamic graph this mapping is referred to as a motion of the corresponding dynamic graph. A notion of motion stability of a dynamic graph is introduced together with a notion of stability of an equilibrium adjacent matrix of dynamic graph. The dynamic graph on a time scale is considered for the first time and the necessity of introducing these objects is caused by the presence of a series of unsolved problems on stability of complex systems, whose subsystem interconnections are changing in time continuous-discrete mode. A method of matrix-valued function is proposed to solve the motion stability problem for the dynamic graph on a time scale. The essence of this method is that the problem on stability of an equilibrium graph of the given dynamic graph is replaced by a simpler problem on stability of the equilibrium state of a matrix equation. The application of the theory of dynamic graphs to the modeling of time-varying interconnections between subsystems of complex system of Lotka-Volterra type is proposed for the first time. A mathematical model is constructed in the form of a dynamic graph for the equilibrium adjacency matrix of which the existence conditions are established as well as the sufficient stability conditions.

Keywords: *dynamic graphs on time scales; matrix Lyapunov functions; comparison principle; Lotka-Volterra systems; stability.*

Mathematics Subject Classification (2010): 34A34, 34A40, 34D20, 39A13, 39A11.

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1 Introduction

The notion of a dynamic graph (not on a time scale) was introduced by D.D.Siljak (see [1] and bibliography therein). This notion was justified by the fact that it makes possible to present the effect of interconnections between subsystems of a complex system on its whole dynamics in a more precise way (see [3]). In a series of works that followed paper [1] (see [2] and bibliography therein) the idea of a dynamic graph for continuous complex system was extended for controlled and other systems.

This paper is aimed at establishing stability conditions for a dynamic graph on a time scale (see Bohner and Peterson [4] and bibliography therein) in terms of the matrix Lyapunov function and the principle of comparison (see [5] and bibliography therein). The paper is arranged as follows.

Section 1 presents a notion of dynamic graph as a one-parameter mapping of the space of graphs with N nodes into itself. In the analysis of the dynamic graph this mapping is referred to as a motion of the corresponding dynamic graph.

In Section 2 a notion of motion stability of a dynamic graph is introduced together with a notion of stability of an equilibrium adjacent matrix of dynamic graph. The latter is considered in the case when the properties of the dynamic graph are studied in terms of the adjacent matrix.

Section 3 deals with a partial case of the dynamic graph, i.e. the dynamic graph on a time scale. This type of dynamic graphs is considered for the first time and the necessity of introducing these objects is caused by the presence of a series of unsolved problems on stability of complex systems, whose subsystem interconnections are changing in time continuous-discrete mode.

In Section 4 a method of matrix-valued function is proposed to solve the motion stability problem for the dynamic graph on a time scale. The essence of this method is that the problem on stability of an equilibrium graph of the given dynamic graph is replaced by a simpler problem on stability of the equilibrium state of a matrix equation. The answer to the question when the solution of the second problem guarantees the solution of the first one is given in Section 5. Also, in this section the procedure of constructing an auxiliary equation is specified.

In Section 6 the application of the theory of dynamic graphs to the modeling of time-varying interconnections between subsystems of complex system of Lotka-Volterra type is proposed for the first time. A mathematical model is constructed in the form of a dynamic graph for the equilibrium adjacency matrix of which the existence conditions are established as well as the sufficient stability conditions.

2 The Description of a Dynamic Graph

Consider a weighted directed graph (later referred to as a graph) $D = (V, E)$ which is an ordered pair where V is a nonempty finite set of N nodes and E is a set of the ribs of the graph. The nodes $(v_1, v_2, \ldots, v_N)$ tie the ribs of the graph (v_j, v_i) so that each rib is oriented from v_j to v_i at all $(i, j) \in \mathcal{N} = \{1, 2, \ldots, N\}$. Each rib (v_j, v_i) is put in correspondence with the weight e_{ij}, if the rib $(v_j, v_i) \in D$ while $e_{ij} = 0$ if $(v_j, v_i) \neq D$. Put the concept of isomorphism $N \times N$ of the matrix $E = (e_{ij})$ in correspondence with the digraph D. Later we will use this concept of isomorphism and the permutation of the symbols D and E as applied to the concerned situation.

Now define the space of graphs D with the fixed number N of nodes, as a linear space

above the field $\mathcal{F}$ of real numbers. For any $D_1, D_2 \in \mathcal{D}$ there exists a single graph

$$D_1 + D_2 \in \mathcal{D}, \tag{1}$$

which is called a sum of graphs D_1 and D_2, and for any $D \in \mathcal{D}$ and an arbitrary number $\alpha \in \mathcal{F}$ there exists a single graph

$$\alpha D \in \mathcal{D}. \tag{2}$$

If in the formula (2) we assume $\alpha = 0$, then $\alpha D = 0$, which corresponds to the zero graph $D = 0 \in \mathcal{D}$. This graph consists of N disconnected nodes, and therefore the matrix E is empty.

The above operations defining $\mathcal{D}$ as a linear space can be interpreted in the context of a linear space $\mathcal{C}$ of adjacent matrices. For the two $N \times N$ matrices $E_1 = (e_{ij}^1)$ and $E_2 = (e_{ij}^2)$ the sum is

$$(e_{ij}^1) + (e_{ij}^2) = (e_{ij}^1 + e_{ij}^2) \in \mathcal{C} \tag{3}$$

and for any $N \times N$ matrix $E = (e_{ij}) \in \mathcal{C}$ and a scalar quantity $\alpha \in \mathcal{F}$ obtain

$$\alpha e_{ij} = (\alpha e_{ij}) \in \mathcal{C}. \tag{4}$$

Note that the zero element of the space $\mathcal{C}$ is an $N \times N$ matrix $E = 0 \in \mathcal{C}$.

Now, in order to introduce the notion of the motion of the graph and its stability in the space $\mathcal{D}$, introduce the norm of the graph $\nu(D)$ with the following properties:

(a) $\nu(D) > 0$ at all $D \in \mathcal{D} \, (D \neq 0)$;
(b) $\nu(\alpha D) = |\alpha| \nu(D)$ at all $D \in \mathcal{D}$ and $\alpha \in \mathcal{F}$; $\qquad$ (5)
(c) $\nu(D_1 + D_2) \leq \nu(D_1) + \nu(D_2)$ at all $(D_1, D_2) \in \mathcal{D}$.

For the space of adjacent matrices $\mathcal{C}$ isomorphic to the space $\mathcal{D}$, consider the matrix norm $\nu \colon \mathbb{R}^{N \times N} \to \mathbb{R}_+$ in the space $\mathbb{R}^{N \times N}$ with the properties:

(a) $\nu(E) > 0$ at all $E \in \mathbb{R}^{N \times N} \, (E \neq 0)$;
(b) $\nu(\alpha E) = |\alpha| \nu(E)$ at all $E \in \mathbb{R}^{N \times N}$ and at all $\alpha \in \mathcal{F}$; $\qquad$ (6)
(c) $\nu(E_1 + E_2) \leq \nu(E_1) + \nu(E_2)$ at all $(E_1, E_2) \in \mathbb{R}^{N \times N}$.

Using these norms, introduce the metric in the space $\mathcal{D}$ by the formula

$$\rho(D_1, D_2) = \nu(D_1 - D_2) \ \text{ at all } \ (D_1, D_2) \in \mathcal{D}. \tag{7}$$

and in the matrix space $\mathcal{D}$ by the formula

$$\rho(E_1, E_2) = \nu(E_1 - E_2) \ \text{ at all } \ (E_1, E_2) \in \mathcal{C}. \tag{8}$$

Taking into account some of the results of the monograph [6], consider the axiomatic specification of a dynamic graph as a mapping of the abstract space $\mathcal{D}$ into itself.

Let the family of mappings $\Phi(t, \mathcal{D})$ in the space $\mathcal{D}$ for any $D \in \mathcal{D}$ and an arbitrary $t \in \mathbb{R}$ be put into correspondence with some graph $\Phi \in \mathcal{D}$.

Definition 2.1 A dynamic graph D is a one-parameter mapping $\Phi \colon \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ of the space $\mathcal{D}$ into itself, which satisfies the following axioms:

(a) $\Phi(t_0, D_0) = D_0$ at all $t_0 \in \mathbb{R}$ and at all $D_0 \in \mathcal{D}$;

(b) $\Phi(t, D)$ is continuous at all $t \in \mathbb{R}$ and at all $D \in \mathcal{D}$;

(c) $\Phi(t_2, \Phi(t_1, D)) = \Phi(t_1 + t_2, D)$ at all $(t_1, t_2) \in \mathbb{R}$ and at all $D \in \mathcal{D}$.

The axiom (a) establishes the fact of the existence of an initial graph $D(t_0) = D_0$. The axiom (b) specifies the continuity of the mapping $\Phi(t, D)$ with respect to all t and all D, including t_0 and D_0. The axiom (b) determines that the dynamic graph is a one-parameter group of transformations of the space $\mathcal{D}$ into itself.

In applications of the theory of dynamic graphs the notion of an adjacent matrix plays a key role, therefore the introduction of such a notion is justified.

Definition 2.2 A dynamic adjacent matrix E is a one-parameter mapping $\Psi \colon \mathbb{R} \times \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$ of the space $\mathbb{R}^{N \times N}$ into itself, satisfying the following axioms:

(a) $\Psi(t_0, E_0) = E_0$ at all $t_0 \in \mathbb{R}$ and at all $E_0 \in \mathbb{R}^{N \times N}$;

(b) the mapping $\Psi(t, E)$ is continuous at all $t \in \mathbb{R}$ and at all $E \in \mathbb{R}^{N \times N}$;

(c) $\Psi(t_2, \Psi(t_1, E)) = \Psi(t_1 + t_2, E)$ at all $(t_1, t_2) \in \mathbb{R}$ and at all $E \in \mathbb{R}^{N \times N}$.

In the process of the analysis of the dynamic graph $\Phi(t, D)$ the mapping is called the motion of the dynamic graph D, while $\Psi(t, E)$ is called the motion of the adjacent matrix E. The graph of stationary motion determined by the formula

$$\Phi(t, D^e) = D^e \quad \text{at all} \quad t \in \mathbb{R}. \tag{9}$$

is of interest. The graph D^e will also be called the equilibrium graph.

Analogously, the adjusent equilibrium matrix is determined by the formula

$$\Psi(t, E^e) = E^e \quad \text{at all} \quad t \in \mathbb{R}. \tag{10}$$

Now consider the notion of stability (instability) of a dynamic graph, if a graph of stationary motion (equilibrium) is specified.

3   Setting of a Problem of Stability of a Dynamic Graph

The analysis of the form and the character of motions of a graph in the neighbourhood of an equilibrium graph or an equilibrium adjacent matrix is of interest, since this analysis allows to identify the conditions for the conservation in time of a certain structure of a complex system described by the specified graph. Introduce some definitions, taking into account the notion of stability in the Lyapunov sense and the two metrics: $\rho_0(\cdot, D^e)$ and $\rho(\cdot, D^e)$ for the characteristic of the initial and the current state of the dynamic graph.

Definition 3.1 The equilibrium graph D^e is called

(a) (ρ_0, ρ)-stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\Delta = \Delta(t_0, \epsilon) > 0$ such that the inequality

$$\rho_0(D_0, D^e) < \Delta \tag{11}$$

implies the estimate

$$\rho(D(t, D_0), D^e) < \epsilon \tag{12}$$

at all $t \geq t_0$;

(b) uniformly (ρ_0, ρ)-stable, if in the conditions of Definition 3.1 (a) the quantity Δ does not depend on $t_0 \in \mathbb{R}$;

(c) asymptotically (ρ_0, ρ)-stable, if it is (ρ_0, ρ)-stable and for any $t_0 \in \mathbb{R}$ there exists $\eta > 0$ such that at

$$\rho_0(D_0, D^e) < \eta \tag{13}$$

the following relation holds:

$$\lim_{t \to \infty} \rho(D(t, D_0), D^e) = 0; \tag{14}$$

(d) globally asymptotically (ρ_0, ρ)-stable if the conditions of Definition 3.1 (c) are satisfied at an arbitrary large η and at all $D \in \mathcal{D}$;

(e) (ρ_0, ρ)-unstable if the conditions of Definition 3.1 (a) are not satisfied.

In the case when the properties of a dynamic graph are studied on the basis of an adjacent equilibrium matrix it makes sense to consider the following definition.

Definition 3.2 An equilibrium adjacent matrix $E^e \in \ \in \mathbb{R}^{N \times N}$ is said to be:

(a) (ρ_0, ρ)-stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\Delta = \Delta(t_0, \epsilon) > 0$ such that the inequality

$$\rho_0(E_0, E^e) < \Delta \tag{15}$$

implies the estimate

$$\rho(E(t, E_0), E^e) < \epsilon \tag{16}$$

at all $t \geq t_0$;

(b) uniformly (ρ_0, ρ)-stable if all the conditions of Definition 3.2 (a) are satisfied with Δ not depending on $t_0 \in \mathbb{R}$;

(c) asymptotically (ρ_0, ρ)-stable if it is (ρ_0, ρ)-stable and for any $t_0 \in \mathbb{R}$ there exists $\zeta > 0$ such that at

$$\rho_0(E_0, E^e) < \zeta \tag{17}$$

the following relation holds:

$$\lim_{t \to \infty} \rho(E(t, E_0), E^e) = 0;$$

(d) globally asymptotically (ρ_0, ρ)-stable if the conditions of Definition 3.2 (c) are satisfied at an arbitrary fixed ζ and at any matrix $E_0 \in \mathbb{R}^{N \times N}$.

Remark 3.1 Since for the selection of two measures some variants are admissible, Definitions 3.1 and 3.2 can have different interpretations. Let us dwell on some of them:

(1) let $D^e = 0$ and $\rho_0(t, \cdot) = \rho(t, \cdot) = \|D\|$, where $\| \cdot \|$ is an Euclidean norm. Then Definition 3.1 characterises the stability of a dynamic graph with respect to the zero graph;

(2) let $E^e = 0$ and $\rho_0(t, \cdot) = \rho(t, \cdot) = \|E\|$. Then Definition 3.2 characterises the stability of the dynamic adjacent matrix with respect to the zero adjacent matrix $E = 0 \in \mathcal{C}$.

4 The Evolution of a Dynamic Graph on a Time Scale

Let a time scale $\mathbb{T}$ with a graininess function $\mu(t) = \sigma(t) - t$, where $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ be specified. The function $\sigma(t)$ determines the operator of a jump forward $\sigma \colon \mathbb{T} \to \mathbb{T}$. Determine $\mathbb{T}^k$ by the formula $\mathbb{T}/\{M\}$, if $\mathbb{T}$ has a right scattered maximum M, and in the rest cases $\mathbb{T}^k = \mathbb{T}$ (see [5] and the bibliography therein).

Definition 4.1 Fix $t \in \mathbb{T}^k$ and let $D \colon \mathbb{T} \to \mathcal{D}$. Determine some matrix $D^\Delta(t)$ (provided that it exists) with the following properties: for any $\epsilon > 0$ there exists a neighbourhood W of a point t for which

$$\|[D(\sigma(t)) - D(s)] - D^\Delta(t)[\sigma(t) - s]\| \le |\sigma(t) - s|$$

at all $s \in W$.

In this case we will say that $D^\Delta(t)$ is a delta derivative of the graph $D(t)$ in a point t.

The evolution of the dynamic graph on a time scale $\mathbb{T}$ will be described by the matrix equation

$$D^\Delta(t) = G(t, D), \quad D(t_0) = D_0 \in \mathcal{D}, \tag{18}$$

where $G \colon \mathbb{T} \times \mathcal{D} \to \mathcal{D}$. In terms of the dynamic matrix of adjacency $E(t)$ the equation (18) takes the form

$$E^\Delta(t) = F(t, E), \quad E(t_0) = E_0 \in \mathbb{R}^{N \times N}, \tag{19}$$

where $F \colon \mathbb{T} \times \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and $E^\Delta = \frac{dE}{dt}$ and the initial problem (19) becomes the initial problem for the matrix ordinary differential equation

$$\frac{dE}{dt} = F(t, E), \quad E(t_0) = E_0 \in \mathbb{R}^{N \times N}. \tag{20}$$

If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ and $E^\Delta = \Delta E(t) = E(t + 1) - E(t)$ and the initial problem (19) becomes the initial problem for the matrix difference equation

$$E(t + 1) - E(t) = F(t, E(t)), \quad E(t_0) = E_0 \in \mathbb{R}^{N \times N}. \tag{21}$$

The objective of the qualitative analysis of a dynamic graph is the study of the solutions of the matrix system of dynamic equations (19).

5 The Application of Matrix-Valued Functions Method in the Study of Stability

Now, connect with the system (19) the matrix-valued function $V(t, E) \colon \mathbb{T} \times \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$ and its full dynamic derivative along the solutions of the system (19)

$$
\begin{aligned}
V^\Delta(t, E) = V_t^\Delta(t, E(\sigma(t))) + \\
+ \int_0^1 \dot{V}_E\big(t, E(t) + H\mu(t)E^\Delta(t)\big)dH\, E^\Delta(t) = \\
= V_t^\Delta(t, E(\sigma(t))) + \\
+ \int_0^1 \dot{V}_E\big(t, E(t) + H\mu(t)F(t, E(t))\big)dH\, F(t, E(t)),
\end{aligned}
\tag{22}
$$

where V_t^Δ is calculated as a Δ-derivative of the matrix- valued function $V(t, E)$ with respect to t in compliance with Definition 5.4.5, and $\dot{V}_E$ is a partial derivative of the matrix- valued function $V(t, E)$ with respect to the matrix argument $E \in \mathbb{R}^{N \times N}$.

Assume that for the expression (22) there exists a matrix- valued function $G(t, V(t, E))$ such that

$$V^\Delta(t, E)|_{(19)} \leq G(t, V(t, E)). \tag{23}$$

Along with the matrix inequality (23) consider the matrix equation

$$M^\Delta(t) = G(t, M(t)), \quad M(t_0) = M_0 \in \mathbb{R}^{N \times N}, \tag{24}$$

where $M(t) = U(t, E(t))$, $E(t) = E(t; t_0, E_0)$ at all $t \in \mathbb{T}$.

Now introduce some notions and definitions for the dynamic equations (19) and (24).

Assume that for the system (19) a time scale $\mathbb{T}$ with the graininess function $\mu(t)$ is chosen. Let $X_1 = \mathbb{R}^{N \times N}$ and $A_1 \subset X_1$ be the space of initial data E_0, such that $E(t_0; t_0, E_0) = E_0 \in A_1$. Denote S_E which is a family of motions of the dynamic graph on the time scale $\mathbb{T}$.

Then the sequence of sets and spaces $\{\mathbb{T}, X_1, A_1, I, S_E\}$ determines the evolution of the dynamic graph on a time scale.

Analogously, for the system (24) keep the time scale $\mathbb{T}$ with the same graininess function $\mu(t)$ and denote $X_2 = \mathbb{R}^{N \times N}$, $A_2 \subset X_2$ is a space of initial values M_0 such that $M(t_0; t_0, M_0) = M_0 \in A_2$. Let S_M be a family of motions of the matrix system (24).

Then the sequence $\{\mathbb{T}, X_2, A_2, I, S_M\}$ determines the evolution of the matrix dynamic equation (24) on a time scale.

Let the sets $N_1 \subset X_1$ and $N_2 \subset X_2$ be invariant with respect to the families of motions S_E and S_M respectively.

By the matrix mapping $U \colon \mathbb{T} \times X_1 \to X_2$ connect the sets N_2 and N_1 by the relation

$$N_2 = U(\mathbb{T} \times N_1) = \{M \in X_2 \colon M = U(t_*, E_1) \\ \text{for some } E_1 \subset N_1 \text{ and } t^* \in \mathbb{T}\}. \tag{25}$$

The family of motions S_M of the system (24) and the family of motions S_E of the dynamic graph (19) will be connected by the relation

$$S_m = \mathcal{M}(S_E), \tag{26}$$

where $\mathcal{M}(S_E) = \{M(\cdot; t_0, B) \colon M(t; t_0, B) = U(t, E(t; t_0, A))$ for any $E(t; t_0, A) \in S_E$, $B = U(t_0, A)$, $A \in A_1$ and $t_0 \in \mathbb{T}\}$.

It seems interesting to obtain conditions under which the dynamic properties of the pairs (S_M, N_2) and (S_E, N_1) would be equivalent.

Note that the systems (19) and (24) are determined in the same space of variables $\mathbb{R}^{N \times N}$, but the system (24), in view of its construction according to the inequality (23), can prove to be more traceable compared with the initial system (19).

6   The Comparison Principle

Before we start obtaining the conditions for the stability of the system of dynamic equations (24), formulate a lemma determining the connection between the dynamic properties of the pairs (S_M, N_2) and (S_E, N_1). Let $\nu_1(E, N_1)$ be a metric in a space X_1 and $\nu_2(U(t, E), N_2)$ be a metric in a space X_2.

The function $\psi\colon [0, r_1] \to \mathbb{R}_+$ (respectively $\psi\colon [0, \infty] \to \mathbb{R}_+$) belongs to the Hahn class if $\psi(0) = 0$ and $\psi(r)$ is strictly increasing over $[0, r_1]$ (on $\mathbb{R}_+$). Functions of this class play the part of comparison functions in the theory of stability of motion.

Lemma 6.1 *Assume that evolutions of the systems* (19) *and* (24) *are determined and there exists a matrix-valued function* $U\colon \mathbb{T} \times X_1 \to X_2$, *such that:*

(a) *the sets of motions* S_M *and* S_E *are connected by the relation* (26);

(b) *the sets* V_1 *and* N_2 *are closed and connected by the relation* (25);

(c) *there exist comparison functions* $\psi_1, \psi_2 \in K$-*class, such that*

$$\psi_1(\nu_1(E, N_1)) \leq \nu_2(U(t, E), N_2) \leq \psi_2(\nu_1(E, N_1))$$

at all $t \in \mathbb{T}$ *and* $E \in \mathbb{R}^{N \times N}$.

Then the following statements hold:

(a) *the invariance of the pair* (S_E, N_1) *implies the invariance of the pair* (S_M, N_2);

(b) *the stability of a certain type of the pair* (S_M, N_2) *implies the stability of the same type of the pair* (S_E, N_1);

(c) *the exponential stability of the pair* (S_M, N_2) *implies the exponential stability of the pair* (S_E, N_1) *if the comparison functions have the form* $\psi_i(r) = a_i r^{b_0}$, $a_i > 0$, $b_0 > 0$, $i = 1, 2$.

Proof. Consider the statement (b) and assume that the pair (S_M, N_2) is stable. Here for any $\epsilon_2 > 0$ and any $t_0 \in \mathbb{T}$ one can find $\Delta_2 = \Delta_2(\epsilon_2, t_0) > 0$ such that $\nu_2(M(t; t_0, B), N_2) < \epsilon_2$ at all $M(\cdot; t_0, B) \in S_M$ and at all $t \in T(B, t_0) \subset \mathbb{T}$ as soon as $\nu_2(B, N_2) < \Delta_2$.

To prove the stability of the pair (S_E, N_1) for an arbitrary $\epsilon > 0$ and $t_0 \in \mathbb{T}$ choose $\epsilon_2 = \psi_1(\epsilon)$ and $\Delta = \psi_2^{-1}(\Delta_2)$. If $\nu_1(A, N_1) < \Delta$, then, according to the condition (c) of Lemma 6.1 obtain $\nu_2(B, N_2) \leq \psi_2(\nu_1(A, N_1)) < \psi_2(\Delta) = \Delta_2$. It means that for any solution $M(t, t_0, B) \in S_M$ the estimate $\nu_2(M(t; t_0, B)) < \epsilon_2$ is true at all $t \in T(B, t_0)$. From the conditions (a), (b) of Lemma 6.1 obtain that $E(\cdot; t_0, A) \in N_1$ at all $t \in T(A, t_0) = T(B, t_0)$, where $B = U(t_0, A)$. From the condition (c) of Lemma 6.1 it follows that

$$\nu_1(E(t; t_0, A), N_1) \leq \psi^{-1}(U(t, E(t; t_0, A)), N_2) =$$
$$= \psi^{-1}(\nu_2(M(t; t_0, B), N_2) \leq \psi^{-1}(\epsilon_2) = \epsilon,$$

at all $t \in T(A, t_0) = T(B, t_0)$ as soon as $\nu_1(A, N_1) < \Delta$. The statement (b) is proved.

The proof of the other statements of the comparison principle is performed in a similar way. $\square$

To obtain the sufficient conditions for the stability of a dynamic graph on the basis of the analysis of the system (24) define concretely the choice of the matrix-valued function $U(t, E)$ and the matrix of the function $G(t, U)$ in the inequality (23).

Let

$$U(t, E) = EE^T \text{ and } G(t, U) = AU, \tag{27}$$

where A is an $N \times N$-constant matrix, and $E \in \mathbb{R}^{N \times N}$.

Taking into account the relation

$$E(\sigma(t)) = E(t) + \mu(t)E^{\Delta}(t)$$

on a time scale $\mathbb{T}$ with the graininess $\mu(t)$, obtain

$$U^{\Delta}(E(t)) = EF^T(t, E) + F(t, E)E^T + \mu(t)F(t, E)F^T(t, E). \qquad (28)$$

Taking into account (28), the inequality (23) takes the form

$$U^{\Delta}(E(t))|_{(19)} \leq AU(E(t)) \qquad (29)$$

at all $t \in \mathbb{T}$, and the matrix comparison equation (24)

$$M^{\Delta}(t) = AM(t), \quad M(t_0) = M_0 \in \mathbb{R}^{N \times N} \qquad (30)$$

is linear.

7   Applications

From the analysis of the literature on complex systems [1,3], mathematical biology [7] etc., it becomes clear, that complex systems with the time-varying interaction between subsystems have not been researched. Indeed, in the literature complex systems are described by the system of differential equations:

$$\frac{dx_i}{dt} = g_i(t, x_i) + h_i(t, e_{i1}x_1, e_{i2}x_2, ..., e_{iN}x_N), \quad i = 1, 2, ..., N. \qquad (31)$$

where equations

$$\frac{dx_i}{dt} = g_i(t, x_i), \quad i = 1, 2, ..., N,$$

describe motion of the disconnected subsystems. Functions h_i describe action of all subsystems of the complex system on the i-th subsystem. Parameter e_{ik} replies for the action of the k-th subsystem on the i-th one and e_{ik} is *constant*. So, *the actual problem is to construct the mathematical model and research the complex systems with time-varying interconnection between their subsystems.*

Since interconnection matrix $E = [e_{ij}]_{i,j=1}^N$ in the complex system (31) may be considered as an adjacent matrix of some graph $G = (V, \mathcal{E})$, where $V = \{V_1, V_2, ..., V_N\}$ is a nonempty finite set of N nodes and $\mathcal{E} = \{(V_i, V_j)|V_i, V_j \in V, i, j = \overline{1, N}\}$ is a set of ribs, then the earlier mentioned problem is to construct the example of complex systems, in which interconnections between subsystems would assign some *time-varying* or, perhaps, *dynamic graph* [6].

Following the setting problem, consider the generalization of the well-known in mathematical biology and ecology Volterra model of the community of n species. The generalized system is described by the system of dynamic equations on some time scale $\mathbb{T}$:

$$N_i^{\Delta}(t) = N_i\Big(\varepsilon_i - \sum_{j=1}^n \gamma_{ij}N_j\Big), \quad i = 1, 2, ..., n, \qquad (32)$$

where $N_i(t)$ is a number of individuals of the i-th species at the moment $t \in \mathbb{T}$, $N_i^{\Delta}(t)$ is a delta derivative of the function $N_i(t)$ in a point $t \in \mathbb{T}$. In the case when $\mathbb{T} = \mathbb{R}$

(when the number of the species changes quickly enough, such scales are considered; communities of bacteria are an example), $N_i^\Delta(t) = \frac{dN_i}{dt}$. Such a case is considered in [7]. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$ (when the number of the species changes over long periods of time such scales are considered; communities of higher animals are an example), then $N_i^\Delta(t) = \Delta N_i(t) = N_i(t + h) - N_i(t)$. When the number of species changes with the different intensity on the different time intervals, the scale with inconstant graininess function $\mu(t)$ ($\mu(t) \equiv 0$, when $\mathbb{T} = \mathbb{R}$, and $\mu(t) \equiv h$, when $\mathbb{T} = h\mathbb{Z}$) can be applied to such species dynamics modelling. The intensity can be affected, for example, by habitat conditions (climate, geography, forage base, etc.)

In addition, in (32) ε_i denotes a rate of natural growth or mortality of the i-th species in the absence of other species. The sign and the absolute value of γ_{ij} $(i \neq j)$ represent the nature and intensity of influence of the j-th species to i-th; γ_{ii} is an indicator of infraspecific competition.

We assume now, that n species whose dynamics are described by the system (32), are the preys and identify interconnections in a community of m species, where the individuals are predators, feeding on individuals of preys.

Denote by S_k $(k = 1, 2, ..., m)$ the set of those n species of the preys community, which form the forage base of the k-th species of the predator community. Also define $N(S_k)$ by the formula:

$$N(S_k) = \sum_{i \in S_k} N_i,$$

that is, $N(S_k)$ is equal to the volume of the k-th predator's forage base. Predator's community dynamics can be described by the system (32):

$$M_i^\Delta(t) = M_i\left(\alpha_i - \sum_{j=1}^{m} \beta_{ij} M_j\right), \quad i = 1, 2, ..., m, \tag{33}$$

where $M_i(t)$ is a number of individuals of the i-th species at the moment $t \in \mathbb{T}$, $M_i^\Delta(t)$ is a delta derivative of the function $M_i(t)$. Also in (33) α_i denotes a rate of natural growth or mortality of the i-th species in the absence of other species, and β_{ij} represent the nature and intensity of influence of the j-th species to the i-th. In this case, it seem natural to assume that the effect of the j-th to the i-th is dependent on percentage of the species, forming the mutual forage base, in the j-th species forage base. That is:

$$\beta_{ij} = \beta_{ij}\left(\frac{N(S_i \cap S_j)}{N(S_j)}\right).$$

The more large the ratio $\frac{N(S_i \cap S_j)}{N(S_j)}$ is (the interval $[0, 1]$ is the range of the ratio), the larger the j-th species makes bids for the mutual with the i-th species forage base, thereby affecting on the i-th species of community of the predators.

So, we have constructed an example of the complex system, that is described by the system of equations

$$M_i^\Delta(t) = M_i\left(\alpha_i - \sum_{j=1}^{m} \beta_{ij}\left(\frac{N(S_i \cap S_j)}{N(S_j)}\right) M_j\right), \quad i = 1, 2, ..., m, \tag{34}$$

and the interconnections between the subsystems are described by the system of equations (32).

So, adjacent matrix $E(t) = [e_{ij}]_{i,j=1}^m$ of some dynamic graph $\mathfrak{G}$ is constructed. The matrix satisfies the following system of equations:

$$E(t) = B\left(\frac{N(S_i \cap S_j)}{N(S_j)}\right),$$

$$N_i^\Delta(t) = N_i\left(\varepsilon_i - \sum_{j=1}^n \gamma_{ij} N_j\right), \quad i = 1, 2, ..., n, \tag{35}$$

Let us consider the particular case when the functions β_{ij} are linear:

$$\beta_{ij}\left(\frac{N(S_i \cap S_j)}{N(S_j)}\right) = Q_{ij}\frac{N(S_i \cap S_j)}{N(S_j)}.$$

Let the community of the preys consist of the 3 species z_1, z_2, z_3, and the community of the predators consist of 2 species. Suppose that the forage base S_1 of the first species of the predators is $\{z_1, z_2\}$, and the forage base S_2 of the second species of the predators is $\{z_2, z_2\}$. Then the interconnections parameters β_{ij} satisfy the following relations:

$$\beta_{11} = Q_{11}, \quad \beta_{12} = Q_{12}\frac{N_2}{N_2 + N_3},$$

$$\beta_{21} = Q_{21}\frac{N_2}{N_1 + N_2}, \quad \beta_{22} = Q_{22}, \tag{36}$$

$$N_i^\Delta(t) = N_i\left(\varepsilon_i - \sum_{j=1}^3 \gamma_{ij} N_j\right), \quad i = 1, 2, 3.$$

The equations (36) describe the evolution of a dynamic graph, consisting of two preys. The value $\beta_{ij}(t)$, as it was mentioned, denotes the weight of the edge (V_i, V_j).

For the dynamic graph $\mathfrak{G}$, which is represented by equations (36), consider the problem of existence of the adjacent equilibrium matrix and of its stability in terms of Definition 3.2.

As we see from the formula (36), the value of the adjacent equilibrium matrix E^e is assigned by the equilibrium state of the system of dynamic equations on the time scale (32). That is, adjacent equilibrium matrix E^e equals

$$E^e = \begin{pmatrix} Q_{11} & \beta_{12}^e \\ \beta_{21}^e & Q_{22} \end{pmatrix}$$

if and only if components N_i^e $(i = 1, 2, 3)$ of the equilibrium vector of the system (32) satisfies the system of equations:

$$\begin{cases} N_i\left(\varepsilon_i - \sum_{j=1}^3 \gamma_{ij} N_j\right) = 0, \quad i = 1, 2, 3, \\ \frac{Q_{12}N_2}{N_2 + N_3} = \beta_{12}^e, \\ \frac{Q_{21}N_2}{N_1 + N_2} = \beta_{21}^e. \end{cases} \tag{37}$$

Suppose now, that the adjacent matrix equals to $E^e = E^*$ and let $N^* = (N_1^*, N_2^*, N_3^*)^T$ be a corresponding state vector of the system (32) (that is, the solution of the system (37)). Establish the stability conditions of the state N^*. It is easy to see, that stability conditions of the state N^* of the system (32) are also stability conditions of the

equilibrium matrix $E^e = E^*$. In the system (32) replace the value N_i to x_i by the formula:

$$x_i = N_i - N_i^*, \quad i = 1, 2, 3, \tag{38}$$

to obtain stability conditions. We obtain the system of dynamic equations

$$x_i^\Delta = N_i^\Delta = (x_i + N_i^*)\Big(\varepsilon_i - \sum_{j=1}^{3} \gamma_{ij}(x_j + N_j^*)\Big) =$$

$$= \Big(x_i\Big(\varepsilon_i - \sum_{j=1}^{3} \gamma_{ij} N_j^*\Big) - \sum_{j=1}^{3} N_i^* \gamma_{ij} x_j\Big) - \sum_{j=1}^{3} \gamma_{ij} x_i x_j, \quad i = 1, 2, 3, \tag{39}$$

and

$$\begin{cases} x_1^\Delta = \Big(\varepsilon_1 - \sum_{j=1}^{3} \gamma_{1j} N_j^* - N_1^* \gamma_{11}\Big)x_1 - N_1^* \gamma_{12} x_2 - N_1^* \gamma_{13} x_3 - \sum_{j=1}^{3} \gamma_{1j} x_1 x_j, \\ x_2^\Delta = -N_2^* \gamma_{21} x_1 + \Big(\varepsilon_2 - \sum_{j=1}^{3} \gamma_{2j} N_j^* - N_2^* \gamma_{22}\Big)x_2 - N_2^* \gamma_{23} x_3 - \sum_{j=1}^{3} \gamma_{2j} x_2 x_j, \\ x_3^\Delta = -N_3^* \gamma_{31} x_1 - N_3^* \gamma_{32} x_2 + \Big(\varepsilon_3 - \sum_{j=1}^{3} \gamma_{3j} N_j^* - N_3^* \gamma_{33}\Big)x_3 - \sum_{j=1}^{3} \gamma_{3j} x_3 x_j. \end{cases} \tag{40}$$

Denoting

$$x = (x_1, x_2, x_3)^T,$$

$$A = \begin{pmatrix} \varepsilon_1 - \sum_{j=1}^{3} \gamma_{1j} N_j^* - N_1^* \gamma_{11} & -N_1^* \gamma_{12} & -N_1^* \gamma_{13} \\ -N_2^* \gamma_{21} & \varepsilon_2 - \sum_{j=1}^{3} \gamma_{2j} N_j^* - N_2^* \gamma_{22} & -N_2^* \gamma_{23} \\ -N_3^* \gamma_{31} & -N_3^* \gamma_{32} & \varepsilon_3 - \sum_{j=1}^{3} \gamma_{3j} N_j^* - N_3^* \gamma_{33} \end{pmatrix},$$

$$F(x) = (F_1(x), F_2(x), F_3(x))^T, \quad F_i(x) = -\sum_{j=1}^{3} \gamma_{ij} x_i x_j,$$

we obtain the vector form of the system (40):

$$x^\Delta = Ax + F(x), \tag{41}$$

with the conditions

$$\lim_{\|x\| \to 0} \|F(x)\| = 0. \tag{42}$$

Now the stability conditions of the equilibrium state N^* of the system (32) are the stability conditions of the trivial equilibrium of the system (41), which can be obtained by the generalized Lyapunov's direct method [5]. According to the method, consider the positive definite function:

$$v(x) = x^T x = x_1^2 + x_2^2 + x_3^2,$$

and compute the total Δ-derivative of $v(x)$ with respect to the solutions of the system (41). Using the product rule (see [5]), we find:

$$
\begin{aligned}
v^{\Delta}\Big|_{(41)} &= \left(x^{\Delta}\right)^{T} x^{\sigma} + x^{T} x^{\Delta}\Big|_{(41)} = \left(x^{\Delta}\right)^{T}\left(x + \mu(t)x^{\Delta}\right) + x^{T} x^{\Delta}\Big|_{(41)} = \\
&= (Ax + F(x))^{T}(x + \mu(t)(Ax + F(x))) + x^{T}(Ax + F(x))\Big|_{(41)} = \\
&= x^{T}(A^{T} + A + \mu(t)A^{T}A)x + \Psi(\mu(t), x) = x^{T}(A^{T} \oplus A)x + \Psi(\mu(t), x),
\end{aligned}
\tag{43}
$$

where

$$
\Psi(\mu(t), x) = F^{T}(x)x + x^{T}F(x) + \mu(t)(x^{T}A^{T}F(x) + F^{T}(x)Ax + F^{T}(x)F(x)).
$$

Here we have used a symbol of regressive sum: $A^{T} \oplus A = A^{T} + A + \mu(t)A^{T}A$.

Now if there exists the negative definite matrix $B \in \mathbb{R}^{3 \times 3}$ such that inequality:

$$
x^{T}(A^{T} \oplus A)x \leq x^{T}Bx, \quad \forall t \in \mathbb{T}, \quad \forall x \in D \subseteq \mathbb{R}^{3},
\tag{44}
$$

holds, then the equilibrium state $x = 0$ is stable by Theorem 3.3.2 from [5]. Indeed, conditions (1), (2) and (2b) for the function $v(x)$ hold. From (43) and (44) we obtain:

$$
v^{\Delta}\Big|_{(41)} \leq x^{T}Bx + \Psi(\mu(t), x),
$$

where the function $\Psi(\mu(t), x)$ satisfies the inequality:

$$
\|\Psi(\mu(t), x)\| \leq 2\|F(x)\|\|x\|(1 + \mu(t)\|A\|).
$$

Using the equality (42), we compute

$$
\lim_{\|x\| \to 0} \frac{\|\Psi(\mu(t), x)\|}{\|x\|} \leq \lim_{\|x\| \to 0} 2\|F(x)\|(1 + \mu(t)\|A\|) = 0.
$$

That is, conditions (2b) and (2c) of Theorem 3.3.2 hold, therefore by Theorem 3.3.2 the equilibrium state $x = 0$ of the system (41) is asymptotically stable which implies the asymptotical stability of the state $N = N^{*}$ of the system (32).

So, in the case when the system (37) can be solved with respect to N_1, N_2 and N_3, there exists the equilibrium matrix

$$
E^{e} = \begin{pmatrix} Q_{11} & \beta_{12}^{e} \\ \beta_{21}^{e} & Q_{22} \end{pmatrix},
$$

which is asymptotically stable, when (44) holds.

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Adaptive Hybrid Function Projective Synchronization of Chaotic Space-Tether System

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Abstract: In this paper, we have achieved adaptive hybrid function projective synchronization between two identical chaotic space-tether systems with uncertain time-varying parameters and with each system evolving from different initial conditions by applying adaptive control technique. Based on Lyapunov stability theory, adaptive control laws and parameter update laws for estimating the uncertain, time-varying parameters are derived to make the states of the two identical chaotic systems asymptotically synchronized. Complete synchronization, antisynchronization, hybrid projective synchronization are obtained as special cases from the above synchronization method. The control techniques and the proposed update laws are verified by numerical simulation results.

Keywords: *adaptive control; parameter estimation; hybrid function projective synchronization; Lyapunov stability theory; space-tether system, celestial mechanics.*

Mathematics Subject Classification (2010): 93C40, 70F15, 37N05, 93D20.

1 Introduction

Two identical chaotic systems with different initial conditions were first made to synchronize in 1990 by Pecora and Carroll [25]. Since then, chaos synchronization has attracted a great deal of attention from various scientific fields. The idea of synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically. Many methods and techniques for handling chaos control and synchronization of various chaotic systems have been developed such as PC method [25], OGY method [19], time-delay feedback approach [24], feedback approach [9, 14], backstepping design technique [29],

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adaptive method [5, 7, 15, 21, 27, 28], linear control method [16, 22], nonlinear control scheme [21, 23].

Till now, different types of synchronization phenomenon have been presented such as complete synchronization (CS) [11], generalized synchronization (GS) [8], lag synchronization [26], anticipated synchronization [18], phase synchronization [2], hybrid synchronization (HS) [6] and antiphase synchronization [13], etc. Among all kinds of chaos synchronization schemes, projective synchronization characterized by a scaling factor that two systems synchronize proportionally has been of recent interest as it can be used to obtain faster communication with its proportional feature. Recently, a new kind of synchronization, Function Projective Synchronization (FPS) was introduced [4]. FPS is a more general definition of Projective Synchronization where the drive system and the response system can be synchronized upto a scaling function which is not a constant. Another synchronization phenomenon called a Hybrid Projective Synchronization (HPS) has also been investigated where the different state variables of the two systems synchronize up to different state factors [10]. Combining these two, we have a new kind of synchronization phenomenon called a Hybrid Function Projective Synchronization (HFPS) which is of latest interest [12, 20, 30]. Here, the different state vectors of the drive and response system synchronize up to different scaling functions which are not scalars. Thus, it is the most modified and generalised form of Projective Synchronization.

Motivated by the aforementioned research, we have formulated Hybrid Function Projective Synchronization (HFPS) of two identical chaotic systems with different initial conditions using adaptive control scheme where the response system has uncertain time-varying parameters. Based on Lyapunov stability theory, adaptive control law and the parameter update law are derived using which HFPS between the two systems is achieved.

Application of chaos synchronization is varied. We consider its application in the field of celestial mechanics. In the recent decades, this field has slowly gained interest and some work has followed [1, 3, 17, 31]. The model we choose in this manuscript as identical chaotic systems is that of a space-tether system. The dynamics of space-tether system has recently been of great interest due to its vast applicability in the field of celestial mechanics. A tether is a long cable used to couple spacecrafts to each other or to other masses such as rocket, space station etc., so that their dynamics can be connected. So, a space-craft together with a tether forms a space-tether system and depending upon the objective and mission, there always arise problems of synchronizing its motion with other spacecrafts using a tether itself or with another space-tether system altogether. Here, in this manuscript, we consider the problem where there is a need to synchronize two identical space-tether systems. A space-tether system can have numerous applications like creation of artificial gravitation on board of the spacecraft, maintainance of spacecraft with electric power, study of upper atmosphere, in research of distant space and many more. Thus, the study of dynamics of a space-tether system is an important topic in celestial mechanics.

Consequently, the paper is organized as follows. In Section 2, model of the space-tether system is explained, in Section 3, adaptive HFPS (AHFPS) between the aforementioned two systems is studied in details. In Section 4, numerical simulations are presented following which observations are made. Finally, in Section 5, conclusion is drawn.

2   Model Explaination

The dynamics of a space-tether system can be developed using different kinds of mathematical models which describe its motion. In this paper, we have chosen the model where tether is considered as massless rod. It is given by equation (1).
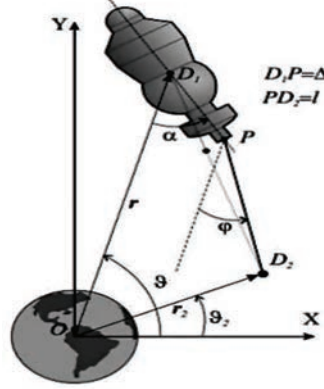
Figure 1: The space-tether problem where tether is considered a massless rod.

$$
\begin{aligned}
\frac{d^2\alpha}{dt^2} &= \frac{3\omega^2}{2}\frac{A-B}{C}\sin 2\alpha - \frac{\Delta c}{C}(l-l_o)\sin(\alpha-\varphi), \\
\frac{d^2l}{dt^2} &= -\frac{c(l-l_o)}{2C}[\Delta^2 + \frac{2C}{m} - \Delta^2\cos(2\alpha-2\varphi)] \\
&\quad +3\omega^2\cos\varphi(l\cos\varphi + \Delta\cos\alpha) + (\frac{d\varphi}{dt})^2 l + 2l\omega\frac{d\varphi}{dt} \\
&\quad +\frac{d\alpha}{dt}\Delta(\frac{d\alpha}{dt} + 2\omega)\cos(\alpha-\varphi) + \\
&\quad \frac{3\Delta\omega^2\sin 2\alpha\sin(\alpha-\varphi)}{2C}, \\
\frac{d^2\varphi}{dt^2} &= \frac{\Delta^2 c}{2lC}(l-l_o)\sin(2\alpha-2\varphi) + \\
&\quad \frac{\frac{d\alpha}{dt}\Delta(\frac{d\alpha}{dt}+2\omega)\sin(\alpha-\varphi)}{l} - \\
&\quad \frac{3\omega^2\sin\varphi(l\cos\varphi + \Delta\cos\alpha)}{l} - \\
&\quad \frac{2\frac{dl}{dt}(\omega + \frac{d\varphi}{dt})}{l} - \frac{3\Delta\omega^2}{2l}\frac{A-B}{C}\sin 2\alpha,
\end{aligned}
\tag{1}
$$

where the parameters are defined as follows:
A, B, C = principal of moments of inertia of the spacecraft;
l_o = length of unstrained tether;
α= angle which the line joining the centres of mass of earth and spacecraft makes with a fixed axis through the center of mass of earth;
l= variable length of the strained tether;

$\varphi =$ inclination of the oscillating plane of the orbit of the center of mass of the system with the plane of ecliptic;

$\alpha =$ angle which the line joining centers of mass of earth and spacecraft makes with the tether;

$\Delta =$ distance between the center of mass of the spacecraft and the position on the spacecraft to which the tether is attached;

$m =$ mass of the spacecraft;

$\omega =$ angular velocity of the carrying spacecraft in circular orbit.

3 Adaptive Control Scheme for AHFPS

For the applicability of the adaptive control scheme, the system is identified in the form of first order differential equations. For this, we make the following substitution:

$$\alpha(t) = x_1(t), \frac{d\alpha}{dt} = x_2(t), l(t) = x_3(t), \frac{dl}{dt} = x_4(t), \varphi(t) = x_5(t), \frac{d\varphi}{dt} = x_6(t).$$

Also, we rename the parameters in the following manner:

$$\frac{3\omega^2}{2}\frac{A - B}{C} = a, \frac{\Delta c}{C} = b, \frac{\Delta c l_o}{C} = d, \frac{c}{2C}[\Delta^2 + \frac{2C}{m}] = e,$$

$$\frac{\Delta^2 c}{2C} = f, \frac{\Delta^2 c l_o}{2C} = g, 3\omega^2 = h, 3\omega^2\Delta = j, 2\omega\Delta = k,$$

$$\frac{3\Delta\omega^2}{2C} = n, \frac{3\Delta\omega^2}{2}\frac{A - B}{C} = p, \frac{c l_o}{2C}[\Delta^2 + \frac{2C}{m}] = q.$$

Based on these substitutions, the system of equations is given as:

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= a\sin 2x_1 - bx_3\sin(x_1 - x_5) + d\sin(x_1 - x_5), \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= -ex_3 + fx_3\cos(2x_1 - 2x_5) - g\cos(2x_1 - 2x_5) + hx_3\cos^2 x_5 + \\
&\quad j\cos x_1 \cos x_5 + x_3 x_6^2 + 2\omega x_3 x_6 + \Delta x_2^2 \cos(x_1 - x_5) + \\
&\quad kx_2\cos(x_1 - x_5) + n\sin 2x_1 \sin(x_1 - x_5) + q, \\
\frac{dx_5}{dt} &= x_6, \\
\frac{dx_6}{dt} &= f\sin(2x_1 - 2x_5) - g\frac{\sin(2x_1 - 2x_5)}{x_3} + \frac{\Delta x_2^2 \sin(x_1 - x_5)}{x_3} + \\
&\quad k\frac{x_2\sin(x_1 - x_5)}{x_3} - h\sin x_5 \cos x_5 - j\frac{\cos x_1 \sin x_5}{x_3} - \\
&\quad \frac{2\omega x_4}{x_3} - \frac{2x_4 x_6}{x_3} - \frac{p\sin 2x_1}{x_3}.
\end{aligned}
\tag{2}
$$

The system of equations (2) is considered as our master system. Then the identical slave system is given by:

$$\frac{dy_1}{dt} = y_2 + u_1,$$

$$\frac{dy_2}{dt} = a_1 \sin 2y_1 - b_1 x_3 \sin(y_1 - y_5) + d_1 \sin(y_1 - y_5) + u_2,$$

$$\frac{dy_3}{dt} = y_4 + u_3,$$

$$\frac{dy_4}{dt} = -e_0 y_3 + f_1 x_3 \cos(2y_1 - 2y_5) - g_1 \cos(2y_1 - 2y_5) +$$
$$h_1 y_3 \cos^2 y_5 + j_1 \cos y_1 \cos y_5 + y_3 y_6^2 + 2\omega_1 y_3 y_6 +$$
$$\Delta_1 y_2^2 \cos(y_1 - y_5) + k_1 y_2 \cos(y_1 - y_5) +$$
$$n_1 \sin 2y_1 \sin(y_1 - y_5) + u_4 + q_1,$$

$$\frac{dy_5}{dt} = y_6 + u_5,$$

$$\frac{dy_6}{dt} = f_1 \sin(2y_1 - 2y_5) - g_1 \frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1 y_2^2 \sin(y_1 - y_5)}{y_3} +$$
$$k_1 \frac{y_2 \sin(y_1 - y_5)}{y_3} - h_1 \sin y_5 \cos y_5 - j_1 \frac{\cos y_1 \sin y_5}{y_3} -$$
$$\frac{2\omega_1 y_4}{y_3} - \frac{2y_4 y_6}{y_3} - \frac{p_1 \sin 2y_1}{y_3} + u_6, \tag{3}$$

where x_i, y_i stand for the state variables of the master system and slave system respectively, $a_1, b_1, d_1, e_0, f_1, g_1, h_1, j_1, k_1, n_1, p_1, q_1, \Delta_1, \omega_1$ are the uncertain time-varying parameters of the slave system which are to be estimated and $u_1, u_2, u_3, u_4, u_5, u_6$ are the time-dependent non-linear controls which are also to be determined.

Let us now suppose that that the time-varying scaling function matrix be given by $A(t) = \text{diag}\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t))$ where $\alpha_i(t) \neq 0; i = \overline{1,6}$. The synchronization errors are defined by

$$e_r(t) = x_r(t) - \alpha_r(t) y_r(t), \quad r = \overline{1,6}. \tag{4}$$

AHFPS between the two systems (2) and (3) will be achieved up to the desired scaling function matrix $A(t)$ if $\lim_{t\to\infty} ||e_r(t)|| = 0, r = \overline{1,6}$. Following these, the error dynamics is given by:

$$\frac{de_1}{dt} = x_2 - \alpha_1 y_2 - \alpha_1 u_1 - \frac{d\alpha_1}{dt} y_1,$$

$$\frac{de_2}{dt} = a \sin 2x_1 - b x_3 \sin(x_1 - x_5) + d \sin(x_1 - x_5) -$$
$$\alpha_2 [a_1 \sin 2y_1 - b_1 x_3 \sin(y_1 - y_5) + d_1 \sin(y_1 - y_5)] -$$
$$\alpha_2 u_2 - \frac{d\alpha_2}{dt} y_2,$$

$$\frac{de_3}{dt} = x_4 - \alpha_3 y_4 - \alpha_3 u_3 - \frac{d\alpha_3}{dt} y_3,$$

$$
\begin{aligned}
\frac{de_4}{dt} &= -ex_3 + fx_3\cos(2x_1 - 2x_5) - g\cos(2x_1 - 2x_5) + q + \\
&\quad hx_3\cos^2 x_5 + j\cos x_1\cos x_5 + x_3 x_6^2 + 2\omega x_3 x_6 + \\
&\quad \Delta x_2^2\cos(x_1 - x_5) + kx_2\cos(x_1 - x_5) + \\
&\quad n\sin 2x_1\sin(x_1 - x_5) - \alpha_4[-e_0 y_3 + f_1 x_3\cos(2y_1 - 2y_5) - \\
&\quad g_1\cos(2y_1 - 2y_5) + q_1 + h_1 y_3\cos^2 y_5 + j_1\cos y_1\cos y_5 + \\
&\quad y_3 y_6^2 + 2\omega_1 y_3 y_6 + \Delta_1 y_2^2\cos(y_1 - y_5) + k_1 y_2\cos(y_1 - y_5) + \\
&\quad n_1\sin 2y_1\sin(y_1 - y_5)] - \alpha_4 u_4 - \frac{d\alpha_4}{dt} y_4, \\
\frac{de_5}{dt} &= x_6 - \alpha_5 y_6 - \alpha_5 u_5 - \frac{d\alpha_5}{dt} y_5, \\
\frac{de_6}{dt} &= f\sin(2x_1 - 2x_5) - g\frac{\sin(2x_1 - 2x_5)}{x_3} + \frac{\Delta x_2^2\sin(x_1 - x_5)}{x_3} + \\
&\quad k\frac{x_2\sin(x_1 - x_5)}{x_3} - h\sin x_5\cos x_5 - j\frac{\cos x_1\sin x_5}{x_3} - \\
&\quad \frac{2\omega x_4}{x_3} - \frac{2x_4 x_6}{x_3} - \frac{p\sin 2x_1}{x_3} - \alpha_6[f_1\sin(2y_1 - 2y_5) - \\
&\quad g_1\frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1 y_2^2\sin(y_1 - y_5)}{y_3} + k_1\frac{y_2\sin(y_1 - y_5)}{y_3} - \\
&\quad h_1\sin y_5\cos y_5 - j_1\frac{\cos y_1\sin y_5}{y_3} - \frac{2\omega_1 y_4}{y_3} - \frac{2y_4 y_6}{y_3} - \\
&\quad \frac{p_1\sin 2y_1}{y_3}] - \alpha_6 u_6 - \frac{d\alpha_6}{dt} y_6.
\end{aligned}
\tag{5}
$$

When we have two identical chaotic systems without controls (i.e. $u_i = 0$), if they evolve from different initial conditions, the trajectories of the two systems eventually separate from each other and become unindentifiable and irrelevant. But when we have two controlled chaotic systems, the two systems will approach synchronization for any initial condition by appropriate control gain and update laws for uncertain time-varying parameters. So, taking $[k_i; i = \overline{1,20}]$ as control gains which are positive constants and letting $e_a = a_1 - a, e_b = b_1 - b, e_d = d_1 - d, e_e = e_0 - e, e_f = f_1 - f, e_g = g_1 - g, e_h = h_1 - h, e_j = j_1 - j, e_k = k_1 - k, e_n = n_1 - n, e_p = p_1 - p, e_q = q_1 - q, e_\Delta = \Delta_1 - \Delta, e_\omega = \omega_1 - \omega$, the following adaptive control laws and parameter update laws are proposed:

Adaptive control laws:

$$
\begin{aligned}
-\alpha_1 u_1 &= -x_2 + \alpha_1 y_2 + \frac{d\alpha_1}{dt} y_1 - k_1 e_1, \\
-\alpha_2 u_2 &= -[a_1\sin 2x_1 - b_1 x_3\sin(x_1 - x_5) + d_1\sin(x_1 - x_5)] + \\
&\quad \alpha_2[a_1\sin 2y_1 - b_1 x_3\sin(y_1 - y_5) + d_1\sin(y_1 - y_5)] + \\
&\quad \frac{d\alpha_2}{dt} y_2 - k_2 e_2, \\
-\alpha_3 u_3 &= -x_4 + \alpha_3 y_4 + \frac{d\alpha_3}{dt} y_3 - k_3 e_3,
\end{aligned}
$$

$$
\begin{aligned}
-\alpha_4 u_4 \;=\; & -[-e_0 x_3 + f_1 x_3 \cos(2x_1 - 2x_5) - g_1 \cos(2x_1 - 2x_5) + q_1 + \\
& h_1 x_3 \cos^2 x_5 + j_1 \cos x_1 \cos x_5 + x_3 x_6^2 + 2\omega_1 x_3 x_6 + \\
& \Delta_1 x_2^2 \cos(x_1 - x_5) + k_1 x_2 \cos(x_1 - x_5) + \\
& n_1 \sin 2x_1 \sin(x_1 - x_5)] + \alpha_4[-e_0 y_3 + f_1 x_3 \cos(2y_1 - 2y_5) - \\
& g_1 \cos(2y_1 - 2y_5) + q_1 + h_1 y_3 \cos^2 y_5 + j_1 \cos y_1 \cos y_5 + \\
& y_3 y_6^2 + 2\omega_1 y_3 y_6 + \Delta_1 y_2^2 \cos(y_1 - y_5) + \\
& k_1 y_2 \cos(y_1 - y_5) + n_1 \sin 2y_1 \sin(y_1 - y_5)] + \\
& \frac{d\alpha_4}{dt} y_4 - k_4 e_4,
\end{aligned}
$$

$$
-\alpha_5 u_5 \;=\; -x_6 + \alpha_5 y_6 + \frac{d\alpha_5}{dt} y_5 - k_5 e_5,
$$

$$
\begin{aligned}
-\alpha_6 u_6 \;=\; & -\Big[-f_1 \sin(2x_1 - 2x_5) - g_1 \frac{\sin(2x_1 - 2x_5)}{x_3} + \\
& \frac{\Delta_1 x_2^2 \sin(x_1 - x_5)}{x_3} + k \frac{x_2 \sin(x_1 - x_5)}{x_3} - h \sin x_5 \cos x_5 - \\
& j \frac{\cos x_1 \sin x_5}{x_3} - \frac{2\omega x_4}{x_3} - \frac{2x_4 x_6}{x_3} - \frac{p_1 \sin 2x_1}{x_3}\Big] + \\
& \alpha_6\Big[f_1 \sin(2y_1 - 2y_5) - g_1 \frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1 y_2^2 \sin(y_1 - y_5)}{y_3} + \\
& k_1 \frac{y_2 \sin(y_1 - y_5)}{y_3} - h_1 \sin y_5 \cos y_5 - j_1 \frac{\cos y_1 \sin y_5}{y_3} - \frac{2\omega_1 y_4}{y_3} - \\
& \frac{2y_4 y_6}{y_3} - \frac{p_1 \sin 2y_1}{y_3}\Big] + \frac{d\alpha_6}{dt} y_6 - k_6 e_6. \qquad\qquad (6)
\end{aligned}
$$

While, parameter update laws are:

$$
\begin{aligned}
\frac{da_1}{dt} \;=\;& \sin 2x_1 e_2 - k_7 e_a, \\
\frac{db_1}{dt} \;=\;& -x_3 \sin(x_1 - x_5) e_2 - k_8 e_b, \\
\frac{dd_1}{dt} \;=\;& \sin(x_1 - x_5) e_2 - k_9 e_d, \\
\frac{de_0}{dt} \;=\;& -x_3 e_4 - k_{10} e_e, \\
\frac{df_1}{dt} \;=\;& x_3 \cos(2x_1 - 2x_5) e_4 + \sin(2x_1 - 2x_5) e_6 - k_{11} e_f, \\
\frac{dg_1}{dt} \;=\;& -\cos(2x_1 - 2x_5) e_4 - \frac{\sin(2x_1 - 2x_5)}{x_3} e_6 - k_{12} e_g, \\
\frac{dh_1}{dt} \;=\;& x_3 \cos^2 x_5 e_4 - \sin x_5 \cos x_5 e_6 - k_{13} e_h, \\
\frac{dj_1}{dt} \;=\;& \cos x_1 \cos x_5 e_4 - \frac{\cos x_1 \sin x_5}{x_3} e_6 - k_{14} e_j, \\
\frac{dk_1}{dt} \;=\;& x_2 \cos(x_1 - x_5) e_4 + \frac{x_2 \sin(x_1 - x_5)}{x_3} e_6 - k_{15} e_k,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dn_1}{dt} &= \sin 2x_1 \sin(x_1 - x_5)e_4 - k_{16}e_f, \\[2mm]
\frac{dp_1}{dt} &= -\frac{\sin 2x_1}{x_3}e_6 - k_{17}e_p, \\[2mm]
\frac{dq_1}{dt} &= e_4 - k_{18}e_q, \\[2mm]
\frac{d\Delta_1}{dt} &= x_2^2 \cos(x_1 - x_5)e_4 + \frac{x_2^2 \sin(x_1 - x_5)}{x_3}e_6 - k_{19}e_\Delta, \\[2mm]
\frac{d\omega_1}{dt} &= 2x_3 x_6 e_4 - \frac{2x_4}{x_3} - k_{20}e_\omega.
\end{aligned}
\tag{7}
$$

Now we have the following theorem which shows the stability and control performance of the adaptive control scheme:

Theorem 3.1 *For a given scaling function matrix*

$$
A(t) = diag\left(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)\right),
$$

where $\alpha_i(t) \neq 0$, $i = \overline{1,6}$, and any initial conditions $x_i(0), y_i(0)$, $i = \overline{1,6}$, the adaptive control law (6) and parameter update law (7) warrant that the error functions $e_i(t)$ are asymptotically convergent to zero, i.e. $\lim_{t\to\infty} ||e_i(t)|| = 0$, $i = \overline{1,6}$.

Proof. We choose a Lyapunov function as follows:

$$
\begin{aligned}
V &= \frac{1}{2}[e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_a^2 + e_b^2 + e_d^2 + e_e^2 + e_f^2 + \\
&\quad e_g^2 + e_h^2 + e_j^2 + e_k^2 + e_n^2 + e_p^2 + e_q^2 + e_\Delta^2 + e_\omega^2].
\end{aligned}
$$

We substitute the values of the controls u_i using adaptive control laws (6) into error dynamical system (5) and also note that for each uncertain parameter say, $a_1, \dot{e}_a = \dot{a}_1$ (where $(\cdot)$ represents differentiation with respect to t) and its value is given by the first equation of parameter update laws (7). Similarly, it follows for the other parameters. Using all these values, it can be shown that the time derivative of the Lyapunov function along the trajectory of the error system (5) is given by:

$$
\frac{dV}{dt} = e^T \frac{de}{dt} = -e^T Q e.
\tag{8}
$$

where $e = (e_1, e_2, e_3, e_4, e_5, e_6, e_a, e_b, e_d, e_e, e_f, e_g, e_h, e_j, e_k, e_n, e_p, e_q, e_\Delta, e_\omega)^T$ and $Q = \text{diag}\,(k_i; i = \overline{1, 20})$.

Clearly, Q is a positive definite matrix and hence, $V(t)$ is negative definite. Based on the Lyapunov stability theory, the error dynamical system (5) is globally and asymptotically stable at the origin and we have $\lim_{t\to\infty} ||e_r(t)|| = 0; r = \overline{1,6}$. Thus, AHFPS between the master system (2) and slave system (3) is achieved. This proves the theorem. □

4 Numerical Simulation Results and Discussions

In this section, we verify and demonstrate the effectiveness of the proposed method by displaying and discussing the simulation results. We find by simulating that the system given by (2) shows chaotic behavior for the following sets of values : $a = 0, b =$

$10^{-10}, d = 10^{-11}, e = 10^{-6}, f = 5 \times 10^{-20}, g = 5 \times 10^{-21}, h = 0.03, j = 3. \times 10^{-9}, k = 2. \times 10^{-8}, n = 1.5 \times 10^{-10}, p = 0, q = 10^{-7}, \Delta = 0.0000001, \Omega = 0.1$ with initial conditions chosen as $x_1(0) = 0.8, x_2(0) = 1.09, x_3(0) = 0.8, x_4(0) = 1.9, x_5(0) = 0.8, x_6(0) = 1.9$. With these values, we take the resulting system as the master system (2) (see Figure 2(a)). Now, we take the initial values of the unknown estimated parameters as
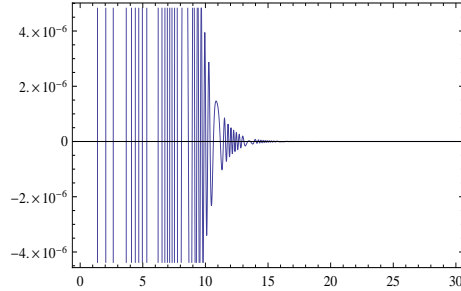

(a) Master system.   (b) Slave system (without controls).
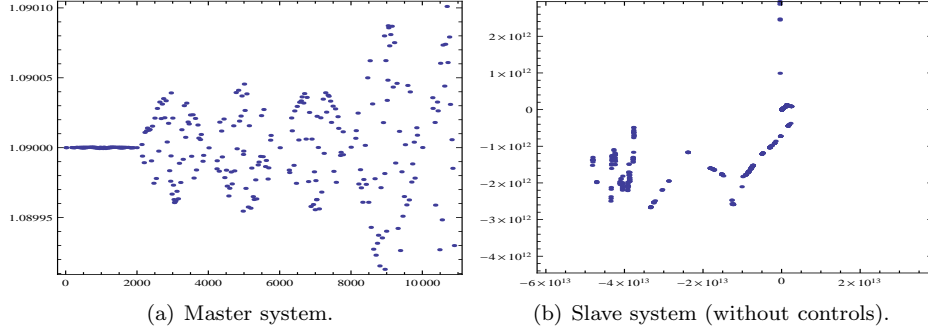
Figure 2: Poincare map showing chaotic master and slave systems.

$a_1(0) = -0.00136336, b_1(0) = 9. \times 10^{-7}, d_1(0) = 4.5 \times 10^{-6}, e_1(0) = 0.00130435, f_1(0) = 4.5 \times 10^{-11}, g_1(0) = 2.25 \times 10^{-10}, h_1(0) = 0.030603, j_1(0) = 3.0603 \times 10^{-6}, k_1(0) = 0.0000202, n_1(0) = 1.53015 \times 10^{-6}, p_1(0) = -1.36336 \times 10^{-7}, q_1(0) = 0.00652174, \Delta_1(0) = 0.0001, \Omega_1(0) = 0.101$ with initial conditions chosen as $y_1(0) = 1.3, y_2(0) = 0.5, y_3(0) = 0.8, y_4(0) = 3.01, y_5(0) = -0.8, y_6(0) = 1.1$. We find that when the system is considered with these values, without the controls, then the system again is chaotic. Thus, this is chosen as our slave system (3) which is to be controlled using the adaptive controllers $u_i(t); i = \overline{1,6}$ (see Figure 2(b)). Also, we choose the control gains as $k_i = 1; i = \overline{1, 20}$. With these values, we now test AHFPS between systems (2) and (3). We can have numerous cases of AHFPS, to test, let us as an example, choose the scaling function matrix as $A(t) = diag \ (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (5 \sin t - 6, 2, 5, 0.9e^{-t}, 1, 10)$. Clearly, $\alpha_i(t) \neq 0; i = \overline{1,6}$; for all t. Accordingly, the initial values of the error variables are: $e_1(0) = 8.6, e_2(0) = 0.09, e_3(0) = -3.2, e_4(0) = -0.809, e_5(0) = 1.6, e_6(0) = -9.1$.
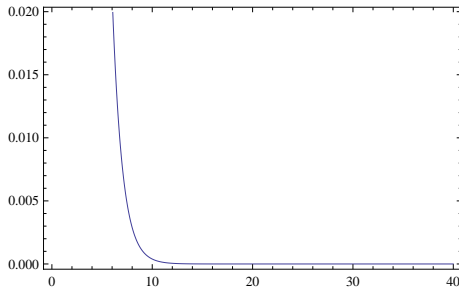
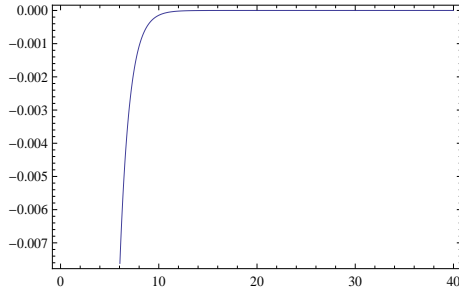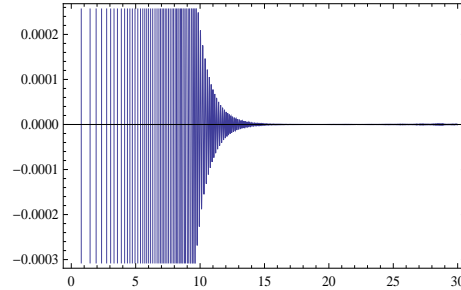(a) **Figure 3**: Time Series Analysis of $e_1(t)$.   (b) **Figure 4**: Time Series Analysis of $e_2(t)$.

The time-evolution graphs of the error variables $e_i(t), i = \overline{1,6}$, are plotted in Figures 3 to 8 while time-evolution graphs of the estimated parameters $a_1, b_1, d_1, e_0, f_1, g_1, h_1, j_1, k_1, n_1, p_1, q_1, \Delta_1, \omega_1$ are presented in Figures 9 to 22. It is clear
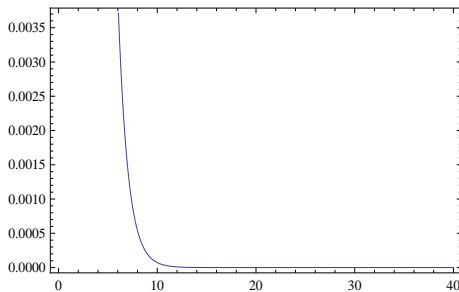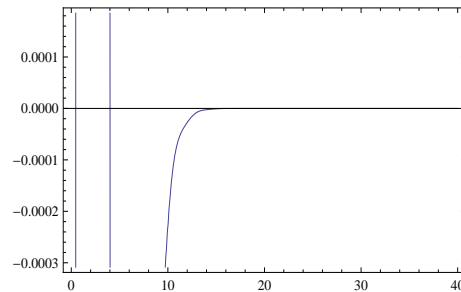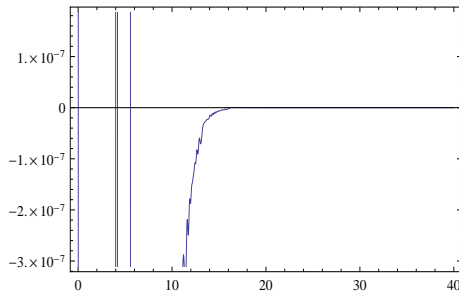
(c) **Figure 5**: Time Series Analysis of $e_3(t)$.

(d) **Figure 6**: Time Series Analysis of $e_4(t)$.

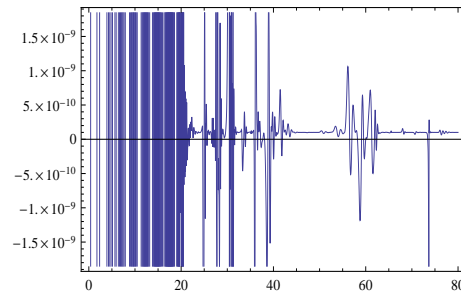(e) **Figure 7**: Time Series Analysis of $e_5(t)$.
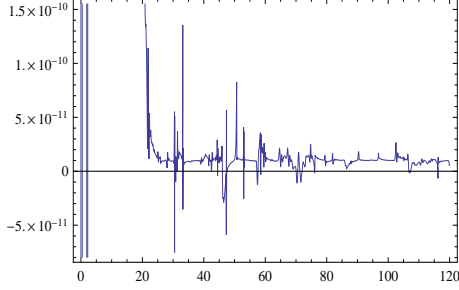

(f) **Figure 8**: Time Series Analysis of $e_6(t)$.

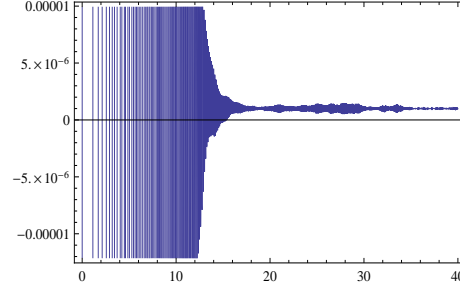(g) **Figure 9**: Time Series Analysis of $a_1(t)(a_1 \rightarrow a = 0)$.

(h) **Figure 10**: Time Series Analysis of $b_1(t)(b_1 \rightarrow b = 10^{-10})$.

from time-evolution graphs of all error variables in Figures 3 to 8 that they converge to zero asymptotically while Figures 9 to 22 show that $a_1 \rightarrow a, b_1 \rightarrow b, d_1 \rightarrow d, e_0 \rightarrow e, f_1 \rightarrow f, g_1 \rightarrow g, h_1 \rightarrow h, j_1 \rightarrow j, k_1 \rightarrow k, n_1 \rightarrow n, p_1 \rightarrow p, q_1 \rightarrow q, \Delta_1 \rightarrow \Delta, \omega_1 \rightarrow \omega$, respectively. Hence parameter update law is verified. All these graphs together indicate the achievement of AHFPS between systems (2) and (3).
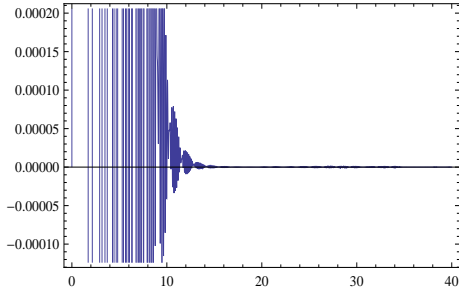
By choosing different scaling function matrices $A(t)$, we can obtain different synchronization phenomenon between the systems (2) and (3) as special cases:

(i) **Figure 11**: Time Series Analysis of $d_1(t)(d_1 \to d = 10^{-11})$.

(j) **Figure 12**: Time Series Analysis of $e_0(t)(e_0 \to e = 10^{-6})$.

(k) **Figure 13**: Time Series Analysis of $f_1(t)(f_1 \to f = 5 \times 10^{-20})$.
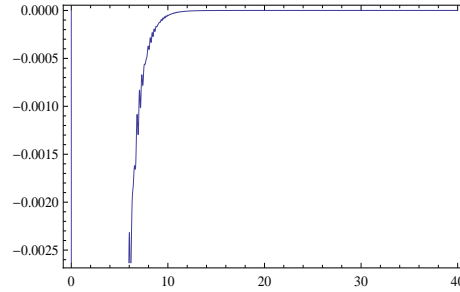
(l) **Figure 14**: Time Series Analysis of $g_1(t)(g_1 \to g = 5 \times 10^{-21})$.
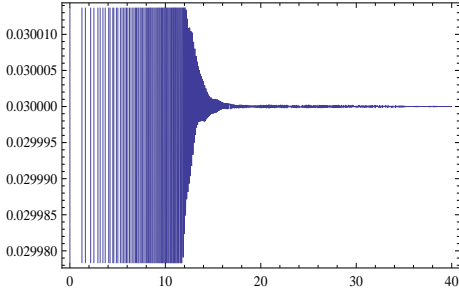
(m) **Figure 15**: Time Series Analysis of $h_1(t)(h_1 \to h = 0.03)$.

(n) **Figure 16**: Time Series Analysis of $j_1(t)(j_1 \to j = 3. \times 10^{-9})$.

4.1   Complete Synchronization

We choose $A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (1, 1, 1, 1, 1, 1)$.
Accordingly, the initial values of the error variables are: $e_1(0) = -0.5, e_2(0) = 0.59, e_3(0) = 0, e_4(0) = -1.11, e_5(0) = 1.6, e_6(0) = 0.8$.

4.2   Antisynchronization

We choose

$$A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (-1, -1, -1, -1, -1, -1).$$
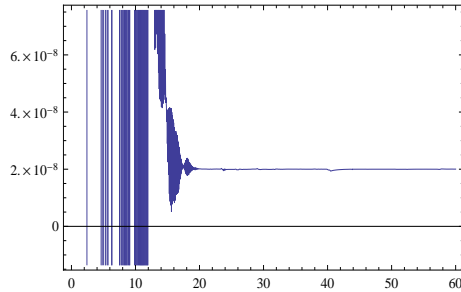
(o) **Figure 17**: Time Series Analysis of $k_1(t)(k_1 \to k = 2. \times 10^{-8})$.
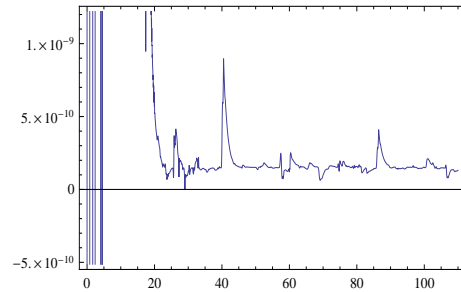
(p) **Figure 18**: Time Series Analysis of $n_1(t)(n_1 \to n = 1.5 \times 10^{-10})$.
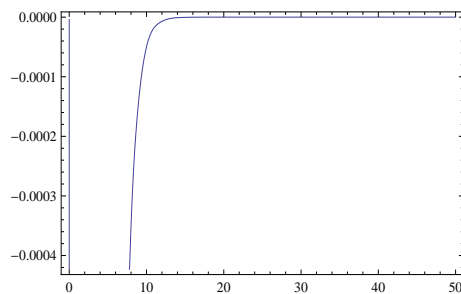
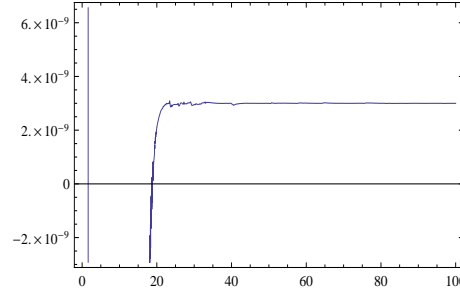
(q) **Figure 19**: Time Series Analysis of $p_1(t)(p_1 \to p = 0)$.
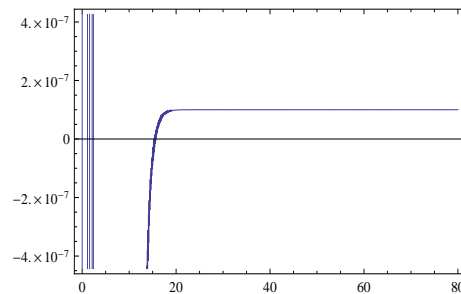
(r) **Figure 20**: Time Series Analysis of $q_1(t)(q_1 \to q = 10^{-7})$.
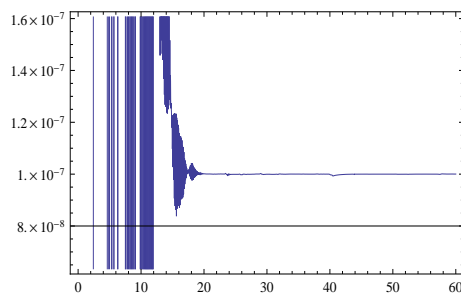
(s) **Figure 21**: Time Series Analysis of $\Delta_1(t)(\Delta_1 \to \Delta = 0.0000001)$.
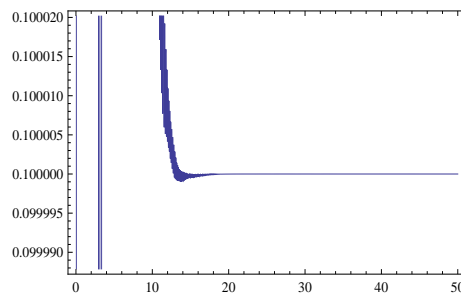
(t) **Figure 22**: Time Series Analysis of $\omega_1(t)(\omega_1 \to \omega = 0.1)$.

Accordingly, the initial values of the error variables are: $e_1(0) = 2.1, e_2(0) = 1.59, e_3(0) = 1.6, e_4(0) = 4.91, e_5(0) = 0, e_6(0) = 3.0$.

4.3 Hybrid Projective Synchronization (HPS)

We can have numerous cases of HPS, as an example let us choose

$$A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (1, 2, 5, 90, 10, 0.1).$$

Accordingly, the initial values of the error variables are: $e_1(0) = 0.7935, e_2(0) = 1.0875, e_3(0) = 0.796, e_4(0) = 1.88495, e_5(0) = 0.804, e_6(0) = 1.8945$.   When

the time-evolution graphs of $e_i(t); i = \overline{1,6}$ and the uncertain parameters $a_1, b_1, d_1,$ $e_0, f_1, g_1, h_1, j_1, k_1, n_1, p_1, q_1, \Delta_1, \omega_1$ are plotted in each of the above cases, we find they are similar to those plotted in Figures 3 to 22. Clearly, then, complete synchronization, antisynchronization, hybrid projective synchronization, all can be achieved as special cases of AHFPS.

5 Conclusion

In this paper, we have presented an application of adaptive control technique in the field of celestial mechanics. The control method has been applied to two identical chaotic space-tether systems, where each system starts from different initial conditions and the response system contains uncertain parameters so that AHFPS is achieved between them. Based on Lyapunov stability theory, adaptive control laws and parameter update laws are designed to make the states between the drive and response systems synchronized asymptotically and they have also been used to estimate the uncertain time-varying parameters. Both theoretical analysis and numerical simulation confirm the effectiveness of our proposed method.

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On the New Concepts of Solutions and Existence Results for Impulsive Integro-Differential Equations with a Deviating Argument

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Abstract: In this paper, we prove the existence of $\mathcal{PC}$-mild solutions for impulsive integro-differential equations with a deviating argument in a Banach space H. The results are obtained by using the analytic semigroup theory and the fixed point methods.

Keywords: *impulsive integro-differential equation; deviating argument; analytic semigroup; fixed point theorems.*

Mathematics Subject Classification (2010): 34K45, 34A60, 35R12, 45J05.

1 Introduction

In the theory of differential equations with deviating arguments, we study the differential equations involving variables (arguments) as well as unknown functions and its derivative, generally speaking, under different values of the variables (arguments). It is a very important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences, and many other areas of science and technology. The book [3] by El'sgol'ts and Norkin provides a comprehensive study of differential equations with deviated arguments. The existence, uniqueness, almost automorphic solutions and asymptotic behaviors of differential equations with deviating arguments have been studied by many authors like Driver [4], Obreg [5], Grimm [6], Gal [7], Haloi [8, 10, 11] (see [12–16] and references cited therein).

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Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process, such phenomena may also be called impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations since such equations are mathematical approaches for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, economics and so on. Chang et al. [27] have studied the existence of $\mathcal{PC}$-mild solutions for first order impulsive neutral integro-differential inclusions with nonlocal initial conditions. Ding et al. [17] discussed a class of second-order impulsive differential equations with integral boundary values. By using Krasnoselskii's fixed point theorem, the existence of solutions for the system is obtained. For more details, one can see ([18, 20, 21, 24–26, 28]) and references cited therein.

On the other hand, due to theoretical and practical difficulties, the study of impulsive differential equations with deviating arguments has been developed rather slowly. Recently, the study of impulsive differential equations with deviating arguments has been found in some papers. For example, in [32], Jankowski discussed the existence of solutions for second order impulsive differential equations with deviating arguments. Guobing et al. [29] established the existence solution of periodic boundary value problems for a class of impulsive neutral differential equations with multi-deviation arguments (see also [30–35] and the references therein).

The existence and uniqueness of abstract integro-differential equations have been discussed by many authors (see [9, 10, 19, 22, 23] and references cited therein). Bahuguna [2] proved the existence, uniqueness, regularity and continuation of solutions to the following integro-differential equations in an arbitrary Banach space H:

$$\left. \begin{aligned} \frac{du(t)}{dt} + Au(t) &= f(t, u(t)) + K(u)(t), \quad t > t_0, \\ u(t_0) &= u_0, \end{aligned} \right\} \tag{1}$$

where

$$K(u)(t) = \int_{t_0}^{t} a(t-s)g(s, u(s))ds.$$

Under the assumptions that $-A$ generates an analytic semigroup $S(t)$, $t \geq 0$ on H, the function a is real-valued and locally integrable on $[0, \infty)$, the nonlinear maps f and g are defined on $[0, \infty) \times H$ into H.

Gal [7] proved the global existence and uniqueness to the following differential equation with deviated argument in a Banach space $(X, \|.\|)$:

$$\left. \begin{aligned} \frac{du}{dt} &= Au(t) + f(t, u(t), u([h(u(t), t)])), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \right\} \tag{2}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. He proved the results under the following assumptions on f and h:

1. $f : [0, \infty) \times X_\alpha \times X_{\alpha-1} \to X$ satisfies

$$\|f(t, x, x') - f(s, y, y')\| \leq L_f\{|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x' - y'\|_{\alpha-1}\} \tag{3}$$

for all $x, y \in X_\alpha$, $x', y' \in X_{\alpha-1}$, $s, t \in [0, \infty)$, for some constants $L_f > 0$ and $0 < \theta_1 \leq 1$.

2. $h : X_\alpha \times [0, \infty) \to [0, \infty)$ satisfies

$$|h(x, t) - h(y, s)| \leq L_h\{\|x - y\|_\alpha + |t - s|^{\theta_2}\} \tag{4}$$

for all $x, y \in X_\alpha$, $s, t \in [0, \infty)$, for some constants $L_h > 0$ and $0 < \theta_2 \leq 1$.

Here $\|x\|_\alpha = \|(A)^\alpha x\|$, denotes the norm on X_α, the domain of A^α, for $0 < \alpha \leq 1$.

In this paper, we extend the Cauchy problem (1) for integro-differential equations to the Cauchy problems for the impulsive integro-differential equations with a deviated argument in a Banach space $(H, \|.\|)$:

$$\left.\begin{aligned}
\frac{d}{dt}u(t) + Au(t) &= f(t, u(t), u[w(t, u(t))]) + \int_0^t a(t, \tau)g(\tau, u(\tau))d\tau, \\
& \quad t \in I = [0, T_0], \quad t \neq t_k, \\
u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\
u(0) &= u_0,
\end{aligned}\right\} \tag{5}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $S(t)$, $t \geq 0$ on H. Functions f, a, g and w are suitably defined and satisfying certain conditions to be stated later. $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T_0$, $I_k \in C(H, H)(k = 1, 2, ..., m)$, are bounded functions. $I_k(u(t_k)) = u(t_k^+) - u(t_k^-)$, $u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t = t_k$, respectively.

The paper is organized as follows. In "Preliminaries and Assumptions" we provide some basic definitions, notations, lemmas and proposition which are used throughout the paper. In "Local existence of mild solution" we will prove some existence and uniqueness results concerning the $\mathcal{PC}$-mild solutions. At last (i.e., in "Application"), we give an example to demonstrate the application of the main results.

2   Preliminaries and Assumptions

In this section, we will introduce some basic definitions, notations, lemmas and proposition which are used throughout this paper.

It is assume that $-A$ generates an analytic semigroup of bounded operators, denoted by $\{S(t)\}_{t \geq 0}$. It is known that there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq \tilde{M}e^{\omega t}, \quad t \geq 0.$$

If necessary, we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ for $t \geq 0$, and $0 \in \rho(-A)$, i.e., $-A$ is invertible. In this case, it is possible to define the fractional power A^α for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$. Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|$$

defines a norm on $D(A^\alpha)$. Henceforth, we denote the space $D(A^\alpha)$ by H_α endowed with the norm $\|\cdot\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α with the norm

$$\|x\|_{-\alpha} = \|A^{-\alpha}x\|.$$

Then $H_{-\alpha}$ is a Banach space endowed with this norm. For more details, we refer to the book by Pazy [1].

Lemma 2.1 *[1, pp. 72,74,195-196] Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following:*

(i) *H_α is a Banach space for $0 \leq \alpha \leq 1$;*

(ii) *For any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;*

(iii) *The operator $A^\alpha S(t)$ is bounded for every $t > 0$ and*

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}.$$

We define the following space

$$X = \mathcal{PC}(H_\alpha) = \{u : [0, T_0] \to H_\alpha : u \in C((t_k, t_{k+1}], H_\alpha), k = 0, 1, \cdots, m,$$
$$\text{and there exists } u(t_k^-), u(t_k^+) \text{ and } u(t_k^-) = u(t_k)\}.$$

X is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in I} \|u(t)\|_\alpha.$$

We shall use the following conditions on f and w in its arguments:

(H1) Let $W \subset \text{Dom}(f)$ be an open subset of $\mathbb{R}_+ \times H_\alpha \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u, v) \in W$, there is a neighborhood $V_1 \subset W$ of (t, u, v), such that the nonlinear map satisfies the following condition,

$$\|f(t, u, v) - f(s, u_1, v_1)\| \leq L_f\{|t - s|^{\theta_1} + \|u - u_1\|_\alpha + \|v - v_1\|_{\alpha-1}\},$$

for all (t, u, v), $(s, u_1, v_1) \in V_1$, $L_f = L_f(t, u, v, V_1) > 0$ and $0 < \theta_1 \leq 1$ are constants.

(H2) Let $U \subset \text{Dom}(w)$ be a open subsets of $\mathbb{R}_+ \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u) \in U$, there is a neighborhood $V_2 \subset U$ of (t, u), $w(\cdot, 0) = 0$ such that

$$|w(t, u) - w(s, v)| \leq L_w \{\|u - v\|_{\alpha-1} + |t - s|^{\theta_2}\},$$

for all (t, u), $(s, v) \in V_2$, $L_w = L_w(u, t, U) > 0$ and $0 < \theta_2 \leq 1$ are constants.

(H3) Let W_1 be an open subset of $\mathbb{R}_+ \times H_\alpha$. For each $(t, x) \in W_1$ there exists a neighborhood $V_3 \subset W_1$ of (t, x) and a positive constant $L_g = L_g(t, x, V_3)$ such that

$$\|g(t, x) - g(s, y)\| \leq L_g\|x - y\|_\alpha,$$

for all $(t, x), (s, y) \in V_3$.

(H4) Let $a : [0, T_0] \times [0, T_0] \to [0, T_0]$ be a continuous function that satisfies the Holder condition uniformly in the first variable, i.e., there exist positive constants $L_a > 0$ and $0 < \theta_3 \leq 1$, such that

$$|a(t, s) - a(\tau, s)| \leq L_a|t - \tau|^{\theta_3},$$

for all $t, \tau, s \in [0, T_0]$.

(H5) The functions $I_k : H_\alpha \to H_\alpha$ are continuous and there exists D_k such that $\|I_k(u)\|_\alpha \leq D_k, \ k = 0, 1, \cdots, m$.

(H6) There exists continuous nondecreasing $d_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|I_k(u) - I_k(v)\|_\alpha \leq d_k \|u - v\|_\alpha, k = 1, 2, \cdots, m.$$

New concept of solutions. Here, we prove a new concept of solutions for the following problem (6)

$$\begin{cases} u'(t) + Au(t) & = \ r(t) + \int_0^t a(t, \tau) g(\tau, u(\tau)) d\tau, \ \ t \in [0, T_0], \ t \neq t_k, \\ u(0) & = \ u_0, \\ u(t_k) & = \ I_k(u(t_k^-)), \ \ k = 1, 2 \cdots, m, \end{cases} \tag{6}$$

where $r \in \mathcal{PC}(I, H)$.

Let

$$\begin{cases} v'(t) + Av(t) & = \ r(t) + \int_0^t a(t, \tau) g(\tau, u(\tau)) d\tau, \ \ t \in [0, T_0], \\ v(0) & = \ v_0, \end{cases} \tag{7}$$

and

$$\begin{cases} w'(t) + Aw(t) & = \ 0, \ \ t \in [0, T_0], \ \ t \neq t_k, \\ w(0) & = \ 0, \\ w(t_k) & = \ I_k(u(t_k^-)), \ \ k = 1, 2, \cdots, m, \end{cases} \tag{8}$$

be the decomposition of $u(.) = v(.) + w(.)$, where v is the continuous mild solution of (7) and w is the $\mathcal{PC}$ mild solution of (8).

By a mild solution for (7), we mean a continuous function $v : [0, T_0] \to H$ satisfying the following integral equation (For more details we refer to [2] and [10])

$$v(t) = S(t)v_0 + \int_0^t S(t - s) \Big[r(s) + \Upsilon v(s) \Big] ds, \ \ t \in [0, T_0], \tag{9}$$

where

$$\Upsilon v(t) = \int_0^t a(t, \tau) g(\tau, u(\tau)) d\tau.$$

and by a $\mathcal{PC}$ mild solution for (8), we mean a function $w \in \mathcal{PC}([0, T_0], D(A))$ satisfying the following integral equation (see [20, Lemma 2.3])

$$w(t) = \begin{cases} - \int_0^t Aw(s) ds, & t \in [0, t_1], \\ \\ I_1(u(t_1^-)) - \int_0^t Aw(s) ds, & t \in (t_1, t_2], \\ \quad \vdots \\ \sum_{i=1}^k I_i(u(t_i^-)) - \int_0^t Aw(s) ds, & t \in (t_k, t_{k+1}], \\ \quad\quad\quad k = 1, 2, \cdots, m. \end{cases} \tag{10}$$

The above equation (10) can be expressed as

$$w(t) = \sum_{i=1}^{k} \chi_i(t) I_i(w(t_i^-)) - \int_0^t A w(s) ds, \tag{11}$$

for $t \in [0, T_0]$, where

$$\chi_i(t) = \begin{cases} 0, & \text{for} & t \in [0, t_1], \\ 1, & \text{for} & t \in (t_k, t_{k+1}], \quad k = 1, 2, 3, \cdots, m. \end{cases} \tag{12}$$

Taking Laplace transform of (11), we obtain

$$w(p) = \sum_{i=1}^{k} \frac{e^{-t_i p}}{p} I_i - \frac{A w(p)}{p},$$

this gives

$$w(p) = \sum_{i=1}^{k} e^{-t_i p} (pI + A)^{-1} I_i, \tag{13}$$

Also, we note that $(pI + A)^{-1} = \int_0^\infty e^{-pt} S(t) dt$. Thus we can derive the mild solution for (8)

$$w(t) = \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(w(t_i^-)).$$

Hence, the mild solution for the problem (6) is given by

$$u(t) = S(t) u_0 + \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(u(t_i^-)) + \int_0^t S(t - s) \Big[r(s) + \Upsilon u(s) \Big] ds. \tag{14}$$

We can rewrite (14) as

$$u(t) = \begin{cases} S(t) u_0 + \int_0^t S(t - s) \Big[r(s) + \Upsilon u(s) \Big] ds, & t \in [0, t_1], \\ \\ S(t) u_0 + S(t - t_1) I_1(u(t_1^-)) \\ + \int_0^t S(t - s) \Big[r(s) + \Upsilon u(s) \Big] ds, & t \in (t_1, t_2], \\ \vdots \\ S(t) u_0 + \sum_{i=1}^{k} S(t - t_i) I_i(u(t_i^-)) \\ + \int_0^t S(t - s) \Big[r(s) + \Upsilon u(s) \Big] ds, & t \in (t_k, t_{k+1}], \quad k = 1, 2, \cdots, m. \end{cases} \tag{15}$$

3   Local Existence of Mild Solutions

In this section, we will prove the existence and uniqueness results concerning $\mathcal{PC}$-mild solutions for system (5). For $0 \leq \alpha < 1$, we define

$$X_1 = \{u \in X : \|u(t) - u(s)\|_{\alpha-1} \leq L|t - s|, \forall \ t, s \in (t_k, t_{k+1}], k = 0, 1, \cdots, m\},$$

where L is a suitable positive constant to be specified later.

Definition 3.1 A continuous function $u : [0, T_0] \to H$ solution of problem (5)

$$u(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in [0, t_1], \\ \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ \quad + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1, t_2], \\ \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ \quad + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k, t_{k+1}], \\ \qquad k = 1, 2, \cdots, m. \end{cases} \tag{16}$$

is said to be a mild solution.

For a fixed $R > 0$, we define

$$\mathcal{W} = \{u \in X \cap X_1 : u(0) = u_0, \ \ \|u - u_0\|_{\mathcal{PC}} \leq R\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of X_1 and is a Banach space.
Let

$$N_1 = \sup_{0 \leq t \leq T_0} \|f(0, u_0, u_0)\|, \tag{17}$$

$$N_2 = \sup_{0 \leq t \leq T_0} \|g(0, u_0)\| \tag{18}$$

and

$$a_{T_0} = \int_0^{T_0} |a(s)|ds. \tag{19}$$

Now we define a map $\mathcal{G} : \mathcal{W} \to \mathcal{W}$ by

$$(\mathcal{G}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \ t \in [0, t_1], \\ \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ \quad + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1, t_2], \\ \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ \quad + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k, t_{k+1}], \\ \qquad k = 1, 2, \cdots, m. \end{cases} \tag{20}$$

Theorem 3.1 *Let $u_0 \in H_\alpha$ and the assumptions (H1) – (H4) hold. Then the problem (5) has a mild solution provided that*

$$C_\alpha[(N_f + a_{T_0} N_g)]\frac{T_0^{1-\alpha}}{1-\alpha} + M \sum_{i=1}^{k} D_i \leq \frac{R}{2} \tag{21}$$

and

$$C_\alpha\{L_f(2 + LL_w) + a_{T_0} L_g\}\frac{T_0^{1-\alpha}}{(1-\alpha)} + M \sum_{0}^{m} d_i < 1, \tag{22}$$

Proof. We begin with showing that $\mathcal{G}u \in X_1$ for each $u \in X_1$. Clearly, $\mathcal{G} : X \to X$. Let $u \in X_1$, then for each $\tau_1, \tau_2 \in [0, t_1], \quad \tau_1 < \tau_2$ and $0 \leq \alpha < 1$, we have

$$
\begin{aligned}
\|(\mathcal{G}u)&(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\
&\leq \|[S(\tau_2) - S(\tau_1)]u_0\|_{\alpha-1} \\
&\quad + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\| \ \|f(s, u(s), u(w(s, u(s))))\|ds \\
&\quad + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\|\left\{ \int_0^s |a(s, \tau)| \ \|g(\tau, u(\tau))\|d\tau \right\}ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2 - s)\| \ \|f(s, u(s), u(w(s, u(s))))\|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2 - s)\|\left\{ \int_0^s |a(s, \tau)| \ \|g(\tau, u(\tau))\|d\tau \right\}ds. \tag{23}
\end{aligned}
$$

Since $f(t, u(t), u(w(u(t), t)))$ and $g(t, u(t))$ are continuous, together with the assumptions (H1), (H2) and (H3), there exist constants N_f and N_g, such that

$$\left.\begin{aligned}\|f(t, u(t), u(w(t, u(t))))\| &\leq N_f, \\ \|g(t, u(t))\| &\leq N_g\end{aligned}\right\}, u \in X, \ t \in [0, T_0], \tag{24}$$

where $N_f = L_f\left\{T_0^{\theta_1} + R(1 + LL_w) + LL_w T_0^{\theta_2}\right\} + N_1$ and $N_f = L_g R + N_2$.

For the first term on the right hand side of (23), we have

$$
\begin{aligned}
\|A^{\alpha-1}[S(\tau_2) - S(\tau_1)]u_0\| &\leq \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S'(s)u_0\|ds \\
&= \int_{\tau_1}^{\tau_2} \|A^\alpha S(s)u_0\|ds \\
&= \int_{\tau_1}^{\tau_2} \|S(s)\| \ \|u_0\|_\alpha ds \\
&\leq M\|u_0\|_\alpha(\tau_2 - \tau_1). \tag{25}
\end{aligned}
$$

For the second and third term on the right hand side of (23), we have the following estimate

$$
\begin{aligned}
\|(S(\tau_2 - s) - S(\tau_1 - s))\|_{\alpha-1} &\leq \int_0^{\tau_2 - \tau_1} \|A^{\alpha-1} S'(l) S(\tau_1 - s)\| dl \\
&= \int_0^{\tau_2 - \tau_1} \|S(l) A^\alpha S(\tau_1 - s)\| dl \\
&\leq MC_\alpha (\tau_2 - \tau_1)(\tau_1 - s)^{-\alpha}.
\end{aligned}
\tag{26}
$$

Then using the inequality (26), we get the following bounds for the second and third term on the right hand side of (23) as

$$
\int_0^{\tau_1} \|(S(\tau_2 - s) - S(\tau_1 - s)) A^{\alpha-1}\| \|f(s, u(s), u(w(s, u(s))))\| ds
$$
$$
\leq N_f M C_\alpha \frac{T_0^{1-\alpha}}{1 - \alpha}(\tau_2 - \tau_1).
\tag{27}
$$

$$
\int_0^{\tau_1} \|(S(\tau_2 - s) - S(\tau_1 - s)) A^{\alpha-1}\| \left\{ \int_0^s |a(s, \tau)| \ \|g(\tau, u(\tau))\| d\tau \right\} ds
$$
$$
\leq M N_g C_\alpha a_{T_0} \frac{T_0^{1-\alpha}}{1 - \alpha}(\tau_2 - \tau_1).
\tag{28}
$$

The fourth and fifth term on the right side of (23) are estimated as

$$
\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s) A^{\alpha-1}\| \|f(s, u(s), u(w(s, u(s))))\| ds
$$
$$
\leq \|A^{\alpha-1}\| M N_f(\tau_2 - \tau_1).
\tag{29}
$$

$$
\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s) A^{\alpha-1}\| \left\{ \int_0^s |a(s, \tau)| \ \|g(\tau, u(\tau))\| d\tau \right\} ds
$$
$$
\leq \|A^{\alpha-1}\| a_{T_0} M N_g(\tau_2 - \tau_1).
\tag{30}
$$

Thus from the inequalities (25) and (27)-(30), we see that

$$
\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \leq M \Big\{ \|u_0\|_\alpha + C_\alpha(N_f + a_{T_0} N_g) \frac{T_0^{1-\alpha}}{1 - \alpha}
$$
$$
+ (N_f + a_{T_0} N_g) \|A^{\alpha-1}\| \Big\}(\tau_2 - \tau_1).
\tag{31}
$$

For $\tau_1, \tau_2 \in (t_1, t_2], \ \ \tau_1 < \tau_2$ and $0 \leq \alpha < 1$, we have

$$
\begin{aligned}
&\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\
\leq \ & \|[S(\tau_2) - S(\tau_1)]u_0\|_{\alpha-1} + \|A^{\alpha-1}[S(\tau_2 - t_1) - S(\tau_1 - t_1)]I_1(u(t_1^-))\| \\
& + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\| \Big\{ \|f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\| \Big\} ds \\
& + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1} S(\tau_2 - s)\| \Big\{ \|f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\| \Big\} ds.
\end{aligned}
\tag{32}
$$

The second term on the right side of (32) is estimated as

$$
\begin{aligned}
\|A^{\alpha-1}[S(\tau_2 - t_1) - S(\tau_1 - t_1)]I_1(u(t_1^-))\| &\leq \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S'(t - t_1)\|\|I_1(u(t_1^-))\|ds \\
&= \int_{\tau_1}^{\tau_2} \|A^{\alpha}S(t - t_1)\| \ \|I_1(u(t_1^-))\|ds \\
&\leq M \ \|I_1(u(t_1^-))\|_{\alpha}(\tau_2 - \tau_1).
\end{aligned}
\tag{33}
$$

Thus, from the inequalities (25), (27)-(30) and (33), we see that

$$
\begin{aligned}
&\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\
&\leq M\Big\{\|u_0\|_{\alpha} + \|I_1(u(t_1^-))\|_{\alpha} + C_{\alpha}(N_f + a_{T_0}N_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\
&\qquad\qquad + (N_f + a_{T_0}N_g)\ \|A^{\alpha-1}\|\Big\}(\tau_2 - \tau_1).
\end{aligned}
\tag{34}
$$

Similarly, for $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\ \tau_1 < \tau_2, k = 1, 2, \cdots, m$ and $0 \leq \alpha < 1$, we have

$$
\begin{aligned}
&\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\
&\leq M\Big\{\|u_0\|_{\alpha} + \sum_{i=1}^{k}\|I_i(u(t_i^-))\|_{\alpha} + C_{\alpha}(N_f + a_{T_0}N_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\
&\qquad\qquad + (N_f + a_{T_0}N_g)\ \|A^{\alpha-1}\|\Big\}(\tau_2 - \tau_1).
\end{aligned}
\tag{35}
$$

Thus, for each $\tau_1, \tau_2 \in [0, T_0]$, $\ \tau_1 < \tau_2$ and $0 \leq \alpha < 1$, we have

$$
\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \leq L(\tau_2 - \tau_1),
\tag{36}
$$

where $L = \max\{M\|u_0\|_{\alpha}, M\ \sum_{i=1}^{m}\|I_i(u(t_i^-))\|_{\alpha}, (N_f + a_{T_0}N_g)MC_{\alpha}\frac{T_0^{1-\alpha}}{1-\alpha}, (N_f + a_{T_0}N_g)M\|A^{1-\alpha}\|\}$.

Therefore, $\mathcal{G}$ is piecewise Lipschitz continuous on $[0, T_0]$ and so $\mathcal{G} : X_1 \to X_1$.

Next we will show that $\mathcal{G} : \mathcal{W} \to \mathcal{W}$.

Let $u \in X \cap X_1$ and $t \in [0, t_1]$, we have

$$
\begin{aligned}
\|(\mathcal{G}u)(t) - u_0\|_{\alpha} &\leq \|(S(t) - I)A^{\alpha}u_0\| \\
&\quad + \int_0^t \|S(t - s)A^{\alpha}\|\|f(s, u(s), u(w(s, u(s))))\|ds \\
&\quad + \int_0^t \|S(t - s)A^{\alpha}\|\Big[\int_0^s |a(s, \tau)|\ \|g(s, u(s))\|d\tau\Big]ds \\
&\leq \frac{R}{2} + C_{\alpha}[(N_f + a_{T_0}N_g)]\frac{T_0^{1-\alpha}}{1-\alpha} \\
&\leq R.
\end{aligned}
\tag{37}
$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1 \cdots, m$, we have

$$
\begin{aligned}
\|(\mathcal{G}u)(t) - u_0\|_\alpha \quad &\leq \quad \|(S(t) - I)A^\alpha u_0\| \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \|f(s, u(s), u(w(s, u(s))))\| ds \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \Big[\int_0^s |a(s,\tau)| \|g(s, u(s))\| d\tau \Big] ds \\
&\quad + \sum_{i=1}^k \|A^\alpha S(t-t_i) I_i(u(t_i^-))\| \\
&\leq \quad \frac{R}{2} + C_\alpha[(N_f + a_{T_0} N_g)]\frac{T_0^{1-\alpha}}{1-\alpha} + M \sum_{i=1}^k \|I_i(u(t_i^-))\|_\alpha \\
&\leq \quad R. \hspace{4cm} (38)
\end{aligned}
$$

Thus, from (37), (38) and (21), it is clear that

$$
\|\mathcal{G}u - u_0\|_{\mathcal{PC}} \leq R.
$$

Therefore, $\mathcal{G} : \mathcal{W} \to \mathcal{W}$ is well defined.

Finally, we will claim that $\mathcal{G}$ is a contraction on $\mathcal{W}$. If $[0, t_1], \quad u, v \in \mathcal{W}$, then we have

$$
\begin{aligned}
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha &\leq \int_0^t \|S(t-s)A^\alpha\| \|f(s, u(s), u(w(s, u(s)))) \\
&\quad - f(s, v(s), u(v(s, v(s))))\| ds + \int_0^t \|S(t-s)A^\alpha\| \\
&\quad \Big[\int_0^s |a(s,\tau)| \|g(\tau, u(\tau)) - g(\tau, v(\tau))\| d\tau \Big] ds. \hspace{1cm} (39)
\end{aligned}
$$

We also note that

$$
\begin{aligned}
&\|f(s, u(s), u(w(s, u(s)))) - f(s, v(s), u(v(s, v(s))))\| \\
&\leq L_f \Big\{ \|u(s) - v(s)\|_\alpha + \|u(w(s, u(s))) - u(w(s, v(s)))\|_{\alpha-1} \\
&\quad + \|u(w(s, v(s))) - v(w(s, v(s)))\|_{\alpha-1} \Big\} \\
&\leq L_f(2 + LL_w)\|u - v\|_{\mathcal{PC}}. \hspace{4cm} (40)
\end{aligned}
$$

and

$$
\|g(\tau, u(\tau)) - g(\tau, v(\tau))\|_\alpha \leq L_g \|u - v\|_{\mathcal{PC}}. \hspace{2cm} (41)
$$

We use (40) and (41) into (39), we get

$$
\begin{aligned}
&\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \\
&\leq \frac{C_\alpha}{(1-\alpha)} \Big\{ L_f(2 + LL_w) + a_{T_0} L_g \Big\} T_0^{1-\alpha} \|u - v\|_{\mathcal{PC}}.
\end{aligned}
$$

For $t \in (t_1, t_2]$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha \{ L_f(2 + LL_w) + a_{T_0} L_g \} \frac{T_0^{1-\alpha}}{(1-\alpha)}$$
$$+ M \| I_1(u(t_1^-)) \|_\alpha \Big] \| u - v \|_{\mathcal{PC}}.$$

For $t \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots, m$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha \{ L_f(2 + LL_w) + a_{T_0} L_g \} \frac{T_0^{1-\alpha}}{(1-\alpha)}$$
$$+ M \sum_{i=1}^{k} \| I_i(u(t_i^-)) \|_\alpha \Big] \| u - v \|_{\mathcal{PC}}.$$

Thus, for each $t \in [0, T_0]$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha \{ L_f(2 + LL_w) + a_{T_0} L_g \} \frac{T_0^{1-\alpha}}{(1-\alpha)}$$
$$+ M \sum_{i=1}^{m} d_i \Big] \| u - v \|_{\mathcal{PC}}. \tag{42}$$

Therefore, the map $\mathcal{G}$ is a contraction map, hence G has a unique fixed point $u \in \mathcal{W}$. That is, problem (5) has a unique mild solution.

4   Further Existence Results

Theorem 3.1 can be proved if we drop the hypothesis (H1),(H2) and (H3). In that case the proof is based on the idea of Wang et al. [21].

Theorem 4.1 *Assume the conditions (H4)-(H6) hold. The semigroup $\{S(t)\}_{t \geq 0}$ is compact, $f : I \times H \times H \to H$ and $g : I \times H \to H$ are continuous. Let $u_0 \in H_\alpha$ there exists a constant $r > 0$ such that*

$$M \Big\{ \| u_0 \|_\alpha + \sum_{i=1}^{k} \| I_i(u(t_i^-)) \|_\alpha \Big\} + C_\alpha (M_f + a_{T_0} M_g) \frac{T_0^{1-\alpha}}{1-\alpha} \leq r, \tag{43}$$

where

$$M_f = \sup_{s \in I, u \in \Omega} \| f(s, u(s), u(w(s, u(s)))) \|, \quad M_g = \sup_{s \in I, u \in \Omega} \| g(s, u(s)) \| \tag{44}$$

and

$$\Omega = \{ v \in \mathcal{PC}(H_\alpha) : \| v \|_{\mathcal{PC}} \leq r \}.$$

Then there exists a mild solution $u \in \mathcal{PC}(H_\alpha)$ of the problem (5).

Proof. Let us define a map $\mathcal{F} : \mathcal{PC}(H_\alpha) \to \mathcal{PC}(H_\alpha)$, by

$$(\mathcal{F}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \ t \in [0,t_1], \\ \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1,t_2], \\ \qquad \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k,t_{k+1}], \\ \qquad k = 1,2,\cdots,m. \end{cases}$$

Step 1. First we show that $\mathcal{F}$ is continuous. It follows from the continuity of f and g that

$$\|f(s,u_n(s),u_n(w(s,u_n(s)))) - f(s,u(s),u(w(s,u(s))))\| \leq \epsilon, \text{ as } n \to \infty,$$
$$\|g(s,u_n(s)) - g(s,u(s))\| \leq \epsilon, \text{ as } n \to \infty,$$

for $s \in [0,t], \ t \in [0,T_0]$.

Now, for each $t \in [0,t_1]$, we have

$$\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha \leq C_\alpha(1+a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{ as } n \to \infty. \tag{45}$$

For, $t \in (t_1,t_2]$, we have

$$\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha$$
$$\leq M\|I_1(u_n(t_1^-)) - I_1(u(t_1^-))\|_\alpha + C_\alpha(1+a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{ as } n \to \infty. \tag{46}$$

Similarly, for each $t \in (t_k,t_{k+1}], k = 1,2,\cdots,m$,

$$\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha$$
$$\leq M\sum_{i=1}^k \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|_\alpha + C_\alpha(1+a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{ as } n \to \infty. \tag{47}$$

Thus, from the inequalities (45)-(47), we see that $\mathcal{F}$ is continuous.

Step 2. Next we show that $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{PC}(H_\alpha)$.

Let $u \in \Omega$, then for $t \in [0,t_1]$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \leq M\|u_0\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{48}$$

For each $t \in (t_1,t_2]$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \leq M\Big\{\|u_0\|_\alpha + \|I_1(u(t_1^-))\|_\alpha\Big\} + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{49}$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \leq M\Big\{\|u_0\|_\alpha + \sum_{i=1}^k \|I_i(u(t_i^-))\|_\alpha\Big\} + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{50}$$

Thus, from inequalities (43) and (48)-(50), we see that $\mathcal{F} : \Omega \to \Omega$.

Step 3. In this step, we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets in $\mathcal{PC}(H_\alpha)$. Let $\tau_1, \tau_2 \in [0, t_1], \tau_1 < \tau_2$, we have

$$
\begin{aligned}
&\|(\mathcal{F}u)(\tau_2) - (\mathcal{F}u)(\tau_1)\|_\alpha \\
&\leq M\Big\{\|u_0\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\
&\quad + \|A^{\alpha-1}\|(M_f + a_{T_0}M_g)\Big\}(\tau_2 - \tau_1).
\end{aligned} \tag{51}
$$

Similarly, for each $\tau_1, \tau_2 \in (t_k, t_{k+1}], \tau_1 < \tau_2, k = 1, 2, \cdots, m$, we have

$$
\begin{aligned}
&\|(\mathcal{F}u)(\tau_2) - (\mathcal{F}u)(\tau_1)\|_\alpha \\
&\leq M\Big\{\|u_0\|_\alpha + \sum_{i=1}^k \|I_i(u(t_i^-))\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\
&\quad + \|A^{\alpha-1}\|(M_f + a_{T_0}M_g)\Big\}(\tau_2 - \tau_1).
\end{aligned} \tag{52}
$$

The right hand side of (52) tends to zero as $\tau_2 \to \tau_1$. Hence, $\mathcal{F}(\Omega)$ is equicontinuous.

Step 4. $\mathcal{F}$ maps Ω into a compact set in H_α.

For this purpose, we decompose $\mathcal{F}$ by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$,

where

$$
\begin{aligned}
(\mathcal{F}_1 u)(t) &= S(t)u_0 + \int_0^t S(t-s)\Big[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\Big]ds, \\
&\quad t \in I \setminus \{t_1, \cdots, t_m\},
\end{aligned}
$$

and

$$
(\mathcal{F}_2 u)(t) = \begin{cases} 0, & t \in [0, t_1], \\ \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)), & t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m. \end{cases}
$$

Since $\mathcal{F}_2$ is a constant map and hence compact.

Finally, we need to prove that $(\mathcal{F}_1 u)(t)$ is relatively compact in Ω for $0 \leq t \leq T_0$. The set $\{S(t)u_0\}$ is precompact in H_α for each $t \in [0, T_0]$, since $\{S(t), t \geq 0\}$ is compact.

For $t \in (0, T_0]$, and $\epsilon > 0$ sufficiently small, we define

$$(\mathcal{F}_1^\epsilon u)(t) = S(\epsilon)\int_0^{t-\epsilon} S(t-\epsilon-s)\Big[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\Big]ds, \quad u \in \Omega.$$

The set $\{(\mathcal{F}_1^\epsilon u)(t) : u \in \Omega\}$ is precompact in H_α since $S(\epsilon)$ is compact. Moreover, for any $u \in \Omega$, we have

$$
\begin{aligned}
\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1^\epsilon u)(t)\|_\alpha & \leq \int_{t-\epsilon}^t \|A^\alpha S(t-s)\| \, \|f(s, u(s), u(w(s, u(s))))\| ds \\
& \quad + \int_{t-\epsilon}^t \|A^\alpha S(t-s)\| \Big\{ \int_0^s |a(s,\tau)| \, \|g(s, u(s))\| d\tau \Big\} ds \\
& \leq M(M_f + a_{T_0} M_g)\epsilon.
\end{aligned}
$$

Therefore, $\{(\mathcal{F}_1^\epsilon u)(t) : u \in \Omega\}$ is arbitrarily close to the set $\{(\mathcal{F}_1 u)(t) : u \in \Omega\}, t > 0$. Hence the set $\{(\mathcal{F}_1 u)(t) : u \in \Omega\}$ is precompact in H_α.

Thus, $\mathcal{F}_1$ is a compact operator by Arzela-Ascoli theorem, and hence $\mathcal{F}$ is a compact operator. Then Schauder fixed point theorem ensures that $\mathcal{F}$ has a fixed point, which gives rise to a $\mathcal{PC}$-mild solution.

5   Application

Consider the following semi-linear heat equation with a deviating argument

$$
\left.
\begin{aligned}
\frac{\partial u}{\partial t} & = \frac{\partial^2 u}{\partial x^2} + \tilde{H}(x, u(x,t)) + G(t, x, u(x,t)), \\
& \quad + \int_0^t a(t,\tau)\frac{\partial}{\partial x}[\xi(x,\tau,u(x,\tau),\frac{\partial}{\partial x}u(x,\tau))]d\tau \\
x \in (0,1) \quad , & \quad t \in (0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1), \\
\Delta u|_{t=\frac{1}{2}} & = \frac{u(\frac{1}{2})^-}{1+u(\frac{1}{2})^-}, \\
u(0,t) & = u(1,t) = 0, \\
u(x,0) & = u_0(x), \quad x \in (0,1),
\end{aligned}
\right\}
\tag{53}
$$

where

$$
\tilde{H}(x, u(x,t)) = \int_0^x K(x,y)u(y, g(t)|u(y,t)|)dy,
$$

and the function $G : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x. Assume that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $\psi(0) = 0$ and $K \in C^1([0,1] \times [0,1]; \mathbb{R})$.

Let $X = L^2((0,1); \mathbb{R})$. We define an operator A as follows,

$$
Au = -\frac{\partial^2 u}{\partial x^2}, \quad D(A) = H_0^1(0,1) \cap H^2(0,1),
\tag{54}
$$

where $X_{1/2} = D(A^{1/2}) = H_0^1(0,1)$ and $X_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) = H^2(0,1)$. Here clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$.

Let us define $g : [0,\infty) \times D(A) \to X$ by

$$
g(t,\phi)(x) = \frac{\partial}{\partial x}[\phi(x,t,\phi(x,t),\frac{\partial}{\partial x}\phi(x,t))],
\tag{55}
$$

and the function $f : \mathbb{R}_+ \times X_{1/2} \times X_{-1/2} \to X$, is given by

$$
f(t,\phi,\psi)(x) = \tilde{H}(x,\psi) + G(t,x,\phi),
\tag{56}
$$

where $\tilde{H} : [0,1] \times X \to H_0^1(0,1)$ is given by

$$\tilde{H}(t, \psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy \tag{57}$$

with $\psi(x,t) = \phi(x, w(t, \phi(x,t)))$ and $w(t, \phi(x,t)) = g(t)|\phi(x,t)|$, $G : \mathbb{R}^+ \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$ satisfies the following

$$\|G(t,x,\phi)\| \le Q(x,t)(1 + \|\psi\|_{H^2(0,1)}) \tag{58}$$

with $Q(.,t) \in X$ and Q is continuous in its second argument. Then, we can easily verify that the assumptions (H1)-(H6) hold. For more details, we refer the reader to [7].

6 Conclusion

The sufficient conditions of the existence and uniqueness of $\mathcal{PC}$-mild solutions to the integro-differential equations with a deviating argument are established.

Acknowledgements

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Existence of a Positive Solution for a Right Focal Dynamic Boundary Value Problem

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Abstract: In this work, an application is made of an extension of the Leggett-Williams fixed point theorem to a second-order right focal dynamic boundary value problem which requires neither of the functional boundaries to be invariant. In conclusion, two nontrivial examples are provided.

Keywords: *fixed point theorem; dynamic equation; time scale; functional.*

Mathematics Subject Classification (2010): 34N05.

1 Introduction

For years, fixed point theory has found itself as a center of study for boundary value problems. Many results have provided criteria for the existence of positive solutions or multiple positive solutions using fixed points of operators. Some of these results can be seen in the works of Guo [10], Krosnosel'skii [12], Leggett and Williams [13], and Avery et al. [1, 3, 6].

Applications of the aforementioned fixed point theorems have been seen in works dealing with ordinary differential equations [2,5,9] and finite difference equations [4,7,11], and most relevant to this paper, the theorems have been utilized for results that involve dynamic equations on time scales [8, 14, 15].

In this paper, we show an application of the recent extension of the Leggett-Williams fixed point theorem by Avery et al. [1] to a right-focal dynamic boundary value problem on a time scale.

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Let $\mathbb{T}$ be a time scale with $0, \sigma^2(1) \in \mathbb{T}$. We consider the right focal dynamic boundary value problem

$$x^{\Delta\Delta} + f(x(\sigma(t))) = 0, \quad t \in (0,1) \cap \mathbb{T}, \tag{1}$$

on the time scale $\mathbb{T}$ with boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0, \tag{2}$$

where $f : [0, \infty) \to [0, \infty)$ is continuous.

2 Definitions

In this section, we present definitions and conventions that will be used throughout the rest of the paper.

Definition 2.1 We define the closed interval $[0,1]$ to mean

$$[0,1] = \{t \in \mathbb{T} : 0 \le t \le 1\}.$$

All other intervals are defined similarly, except for those specifying the domain or codomain of a function.

Definition 2.2 Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

(i) $x \in \mathcal{P}$, $\lambda \ge 0$ implies $\lambda x \in \mathcal{P}$;

(ii) $x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.

Definition 2.3 A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

3 The Fixed Point Theorem

We first define sets that are integral to the fixed point theorem. Let α and ψ be nonnegative continuous concave functionals on $\mathcal{P}$ and let δ and β be nonnegative continuous convex functionals on $\mathcal{P}$. We define the sets

$$A = A(\alpha, \beta, a, d) = \{x \in \mathcal{P} \ : \ a \le \alpha(x) \text{ and } \beta(x) \le d\},$$

$$B = B(\delta, b) = \{x \in A \ : \ \delta(x) \le b\},$$

and

$$C = C(\psi, c) = \{x \in A \ : \ c \le \psi(x)\}.$$

The following fixed point theorem is attributed to Anderson, Avery, and Henderson [1] and is an extension of the original Leggett-Williams fixed point theorem [13].

Theorem 3.1 *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, δ and β are nonnegative continuous convex functionals on $\mathcal{P}$, and for nonnegative real numbers a, b, c, and d, the sets A, B, and C are defined as above. Furthermore, suppose A is a bounded subset of $\mathcal{P}$, $T : A \to \mathcal{P}$ is a completely continuous operator, and that the following conditions hold:*

(A1) $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$, $\{x \in \mathcal{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$;

(A2) $\alpha(Tx) \geq a$ for all $x \in B$;

(A3) $\alpha(Tx) \geq a$ for all $x \in A$ with $\delta(Tx) > b$;

(A4) $\beta(Tx) \leq d$ for all $x \in C$; and

(A5) $\beta(Tx) \leq d$ for all $x \in A$ with $\psi(Tx) < C$.

Then T has a fixed point $x^* \in A$.

4   Existence of a Positive Solution of (1), (2)

In this section, we show the existence of at least one positive solution to (1), (2). To that end, we now consider the dynamic equation

$$x^{\Delta\Delta} + f(x(\sigma(t))) = 0, \ t \in (0, 1),$$

on a time scale $\mathbb{T}$ with boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0,$$

where $f : [0, \infty) \to [0, \infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^{\sigma(1)} G(t, s) f(x(\sigma(s))) \Delta s, \ t \in [0, \sigma^2(1)],$$

where $G(t, s)$ defined on $[0, \sigma^2(1)] \times [0, \sigma(1)]$ by

$$G(t, s) = \begin{cases} t, & 0 \leq t \leq s \leq \sigma(1), \\ \sigma(s), & \sigma^2(1) \geq t \geq \sigma(s) \geq 0, \end{cases}$$

is the Green's function for the operator L defined by

$$(Lx)(t) := -x^{\Delta\Delta},$$

with right focal boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0,$$

then it is well known that x is a solution of the boundary value problem (1), (2).

Throughout the remainder of the paper, we will often make use of the following property of the preceeding Green's function. For any $y, w \in [0, \sigma^2(1)]$ with $y \leq w$,

$$yG(w, s) \leq wG(y, s),$$

which implies

$$y \int_0^{\sigma(1)} G(w,s)\Delta s \le w \int_0^{\sigma(1)} G(y,s)\Delta s. \tag{3}$$

Let $E = C_{rd}[0,\sigma^2(1)]$ be the Banach Space composed of right-dense continuous functions from $[0,\sigma^2(1)]$ into $\mathbb{R}$ with the norm

$$||x|| = \max_{t\in[0,\sigma^2(1)]} |x(t)|.$$

Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} = \{x \in E : x \text{ is nondecreasing, nonnegative, and concave.}\}$$

For fixed $\tau,\mu,\nu \in [0,\sigma^2(1)]$, define the nonnegative concave functionals α and ψ to be

$$\alpha(x) = \min_{t\in[\tau,\sigma^2(1)]} x(t) = x(\tau),$$

$$\psi(x) = \min_{t\in[\mu,\sigma^2(1)]} x(t) = x(\mu),$$

and the nonnegative, convex functionals δ and β to be

$$\delta(x) = \max_{t\in[0,\nu]} x(t) = x(\nu),$$

$$\beta(x) = \max_{t\in[0,\sigma^2(1)]} x(t) = x(\sigma^2(1)).$$

Theorem 4.1 *Let $\tau,\mu,\nu \in (0,\sigma^2(1)]$ with $0 < \tau \le \mu < \nu \le \sigma^2(1)$. Let d and m be positive reals with $0 < m \le \frac{d\mu}{\sigma^2(1)}$ and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the following:*

(i) $f(w) \ge \frac{d}{(\nu-\tau)\sigma^2(1)}$ for $\frac{\tau d}{\sigma^2(1)} \le w \le \frac{\nu d}{\sigma^2(1)}$;

(ii) $f(w)$ is decreasing for $0 \le w \le m$ and $f(m) \ge f(w)$ for $m \le w \le d$; and

(iii) $\int_0^\mu \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right)\Delta s \le d - f(m)\sigma^2(1)(\sigma(1)-\mu).$

Then (1),(2) has at least one positive solution $x^* \in A(\alpha,\beta,\frac{\tau d}{\sigma^2(1)},d)$.

Proof. Let $a = \frac{\tau d}{\sigma^2(1)}$, $b = \frac{\nu d}{\sigma^2(1)}$, and $c = \frac{\mu d}{\sigma^2(1)}$. Define $Tx(t) = \int_0^{\sigma(1)} G(t,s)f(x(\sigma(s)))\Delta s$. Now by definition, $A \subset \mathcal{P}$, and for all $x \in A$, $d \ge \beta(x) = \max_{t\in[0,\sigma(1)]} x(t) = x(\sigma^2(1))$, and so A is bounded.

Now, if $x \in A \subset \mathcal{P}$, then $Tx(t) = \int_0^{\sigma^2(1)} G(t,s)f(x(\sigma(s)))\Delta s$, and so $Tx^{\Delta\Delta}(t) = -f(x(\sigma(s))) \le 0$ for $t \in [0,1]$, and so Tx is concave, and $Tx^{\Delta}(t)$ is nonincreasing on $[0,\sigma(1)]$. Furthermore, $Tx^{\Delta}(\sigma(1)) = 0$, and so $Tx^{\Delta}(t) \ge 0$ on $[0,\sigma(1)]$. So Tx is nondecreasing on $[0,\sigma^2(1)]$. Therefore, $T : A \to \mathcal{P}$.

Now we prove our first enumerated condition (A1). Let $K \in \mathbb{R}$ with $\dfrac{\mu d}{\sigma^2(1) \int_0^{\sigma(1)} G(\mu, s) \Delta s} < K < \dfrac{\nu d}{\sigma^2(1) \int_0^{\sigma(1)} G(\nu, s) \Delta s}$, which is well-defined by (3). Define $x_K(t) = K \int_0^{\sigma(1)} G(t, s) \Delta s$. So $x_K \in \mathcal{P}$,

$$
\begin{aligned}
\alpha(x_K) &= K \int_0^{\sigma(1)} G(\tau, s) \Delta s \\
&> \frac{\mu d \int_0^{\sigma(1)} G(\tau, s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\mu, s) \Delta s} \\
&\geq \frac{\tau d \int_0^{\sigma(1)} G(\mu, s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\mu, s) \Delta s} \\
&= \frac{\tau d}{\sigma^2(1)} = a,
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(x_K) &= K \int_0^{\sigma(1)} G(\sigma^2(1), s) \Delta s \\
&< \frac{\nu d \int_0^{\sigma(1)} G(\sigma^2(1), s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\nu, s) \Delta s} \\
&\leq \frac{\sigma^2(1) d \int_0^{\sigma(1)} G(\nu, s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\nu, s) \Delta s} = d.
\end{aligned}
$$

So $x_K \in A$. Now

$$
\begin{aligned}
\psi(x_K) &= K \int_0^{\sigma(1)} G(\mu, s) \Delta s \\
&> \frac{\mu d \int_0^{\sigma(1)} G(\mu, s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\mu, s) \Delta s} \\
&= \frac{\mu d}{\sigma^2(1)} = c,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta(x_K) &= K \int_0^{\sigma(1)} G(\nu, s) \Delta s \\
&< \frac{\nu d \int_0^{\sigma(1)} G(\nu, s) \Delta s}{\sigma^2(1) \int_0^{\sigma(1)} G(\mu, s) \Delta s} \\
&= \frac{\nu d}{\sigma^2(1)} = b.
\end{aligned}
$$

So $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$.

Next, let $x \in \mathcal{P}$ with $\beta(x) > d$. Then since for all $y \leq w$, $wx(y) \geq yx(w)$, $\sigma^2(1)x(\tau) \geq \tau x(\sigma^2(1))$, and so

$$\alpha(x) = x(\tau) \geq \frac{\tau}{\sigma^2(1)}x(\sigma^2(1)) = \frac{\tau\beta(x)}{\sigma^2(1)} > \frac{\tau d}{\sigma^2(1)} = a.$$

Therefore $\{x \in \mathcal{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$.

Next, we prove (A2). Chose $x \in B$. So $\delta(x) \leq b$. Now by (i),

$$\alpha(Tx) = \int_0^{\sigma(1)} G(\tau, s)f(x(\sigma(s)))\Delta s$$

$$\geq \int_\tau^\nu G(\tau, s)f(x(\sigma(s)))\Delta s$$

$$= \int_\tau^\nu \tau f(x(\sigma(s)))\Delta s$$

$$\geq \int_\tau^\nu \tau \left(\frac{d}{(\nu - \tau)\sigma^2(1)}\right)\Delta s$$

$$= \frac{d\tau}{\sigma^2(1)} = a.$$

Next, we prove (A3). Let $x \in A$ with $\delta(Tx) > b$. Then, by (3),

$$\alpha(Tx) = \int_0^{\sigma(1)} G(\tau, s)f(x(\sigma(s)))\Delta s$$

$$\geq \frac{\tau}{\nu} \int_0^{\sigma(1)} G(\nu, s)f(x(\sigma(s)))\Delta s$$

$$= \frac{\tau}{\nu}\delta(Tx)$$

$$> \frac{\tau}{\nu} \cdot \frac{\nu d}{\sigma^2(1)} = \frac{\tau d}{\sigma^2(1)} = a.$$

Now we prove (A4). Now, since x is concave and nondecreasing for all $t \in [0, \mu]$,

$$x(\sigma(t)) \geq \frac{x(\sigma(\mu)))\sigma(t)}{\sigma(\mu)} \geq \frac{c\sigma(t)}{\sigma(\mu)} \geq \frac{m\sigma(t)}{\sigma(\mu)}.$$

So by conditions (ii) and (iii), we have

$$\beta(Tx) = \int_0^{\sigma(1)} G(\sigma^2(1), s)f(x(\sigma(s)))\Delta s$$

$$= \int_0^{\sigma(1)} \sigma(s)f(x(\sigma(s)))\Delta s$$

$$= \int_0^\mu \sigma(s)f(x(\sigma(s)))\Delta s + \int_\mu^{\sigma(1)} \sigma(s)f(x(\sigma(s)))\Delta s$$

$$\leq \int_0^\mu \sigma(s)f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right)\Delta s + \int_\mu^{\sigma(1)} \sigma^2(1)f(m)\Delta s$$

$$\leq d - f(m)\sigma^2(1)(\sigma(1) - \mu) + f(m)\sigma^2(1)(\sigma(1) - \mu)$$

$$= d.$$

Finally, we prove our last condition, (A5). Let $x \in A$ with $\psi(Tx) < c$. So, we have

$$\beta(Tx) = \int_0^{\sigma(1)} G(\sigma^2(1), s) f(x(\sigma(s))) \Delta s \leq \frac{\sigma^2(1)}{\mu} \int_0^{\sigma(1)} G(\mu, s) f(x(\sigma(s))) \Delta s$$

$$= \frac{\sigma^2(1)}{\mu} \psi(Tx) \leq \frac{\sigma^2(1)c}{\mu} = d.$$

Thus T has a fixed point $x^* \in A$, and therefore x^* is a positive solution of (1), (2).

5 Two Nontrivial Examples

Example 5.1 Let $\mathbb{T} = [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$ and consider the boundary value problem

$$x^{\Delta\Delta} + \frac{1}{x(\sigma(t)) + 1} = 0, \ t \in (0, 1) \cap \mathbb{T}, \quad x(0) = x^{\Delta}(\sigma(1)) = 0.$$

Choose $\tau = \frac{1}{30}$, $\mu = \frac{1}{2}$, $\nu = 1$, $m = \frac{1}{4}$, and $d = \frac{3}{5}$. Note that $0 < \tau \leq \mu < \nu \leq \sigma^2(1) = 1$ and $0 < m < \frac{d\mu}{\sigma^2(1)} = \frac{\frac{3}{5} \cdot \frac{1}{2}}{1} = \frac{3}{10}$. Also, $f(w) = \frac{1}{w+1}$ is continuous from the nonnegative reals to the nonnegative reals. Lastly,

(i) for $\frac{1}{50} \leq w \leq \frac{3}{5}$, $f(w) \geq f(\frac{3}{5}) = \frac{5}{8} > \frac{18}{29} = \frac{d}{(\nu-\tau)\sigma^2(1)}$,

(ii) since $f'(w) < 0$ for $w \geq 0$, $f(w)$ is decreasing for $0 \leq w \leq \frac{1}{4}$ and for $\frac{1}{4} \leq w \leq \frac{3}{5}$, $f(m) = f(\frac{1}{4}) \geq f(w)$, and

(iii) $\int_0^\mu \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right) \Delta s = \int_0^{\frac{1}{2}} sf(\frac{1}{4}s)\Delta s = \int_0^{\frac{1}{2}} s\frac{1}{\frac{1}{4}s+1}\Delta s \approx 0.115471 < 0.2 = \frac{3}{5} - \frac{2}{5} = \frac{3}{5} - f(\frac{1}{4})(1)\frac{1}{2} = d - f(m)\sigma^2(1)(\sigma(1) - \mu)$.

Therefore, the boundary value problem has at least one positive solution, x^*, in $A(\alpha, \beta, \frac{1}{50}, \frac{3}{5})$. That is, $x^*(\frac{1}{30}) \geq \frac{1}{50}$ and $x^*(1) \leq \frac{3}{5}$.

Example 5.2 Let $\mathbb{T} = 2^{\mathbb{Z}} = \{2^n \ : \ n \in \mathbb{Z}\} \cup \{0\}$. Consider the boundary value problem

$$x^{\Delta\Delta} + \frac{\cos^2(0.2x(\sigma(t)))}{\sqrt{(x(\sigma(t)))^{1/10} + 1}} = 0, \ t \in (0, 1) \cap \mathbb{T}, \quad x(0) = x^{\Delta}(\sigma(1)) = 0.$$

Choose $\tau = \frac{1}{1024}$, $\mu = 2$, $\nu = 4$, $m = \frac{1}{5}$, and $d = \frac{5}{2}$. Note that $0 < \tau \leq \mu < \nu \leq \sigma^2(1) = 4$ and $0 < m < \frac{d\mu}{\sigma^2(1)} = \frac{\frac{5}{2} \cdot 2}{4} = \frac{5}{4}$. Also, $f(w) = \frac{\cos^2(0.2w)}{\sqrt{w^{1/10}+1}}$ is continuous from the nonnegative reals to the nonnegative reals. Now,

(i) for $\frac{5}{8192} \leq w \leq \frac{5}{2}$, $f(w) \geq f(\frac{5}{2}) \approx 0.531967 > \frac{128}{819} = \frac{d}{(\nu-\tau)\sigma^2(1)}$,

(ii) since $f'(w) < 0$ for $0 \leq w \leq \frac{5}{2}$, $f(w)$ is decreasing for $0 \leq w \leq \frac{1}{5}$ and for $\frac{1}{5} \leq w \leq \frac{5}{2}$, $f(m) = f(\frac{1}{5}) \geq f(w)$, and

(iii) $\int_0^\mu \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right) \Delta s = \sum_{k=0}^{\infty} \frac{1}{2^{k-1}} f\left(\frac{1}{20 \cdot 2^{k-1}}\right) \cdot \frac{1}{2^k} \approx 2.00009 < \frac{5}{2} = \frac{5}{2} - f\left(\frac{1}{5}\right) \cdot 4(2 - 2) = d - f(m)\sigma^2(1)(\sigma(1) - \mu)$.

Therefore, the boundary value problem has at least one positive solution, x^*, in $A(\alpha, \beta, \frac{5}{8192}, \frac{5}{2})$. That is, $x^*(\frac{1}{1024}) \geq \frac{5}{8192}$ and $x^*(4) \leq \frac{5}{2}$.

6   Conclusion

Here it was shown how a recent Avery et al. fixed point theorem [1] that was developed as an extension of the original Leggett-Williams fixed point theorem [13] can be applied to show under certain conditions, the existence of a second order right focal dynamic boundary value problem. Two nontrivial examples were then provided to show that these conditions could be applied to specific boundary value problems.

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Robust Stabilization of Fractional-Order Uncertain Systems with Multiple Delays in State

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Abstract: In this paper, a sliding mode control law is designed for stabilization of specific class of linear systems of fractional order despite of multi delays in the state system. A fractional order sliding surface is proposed, and using the variable structure control theorem, control law is introduced. A numerical simulation is given to show the effectiveness of the proposed design approach.

Keywords: *sliding mode control (SMC); Lyapunov stability analysis; fractional order system.*

Mathematics Subject Classification (2010): 93C35, 93D05, 93D15.

1 Introduction

Recently, time delays inevitably exist in systems and processes [1, 2] due to poor performance, undesirable system transient responses, and instabilities so that as a result, most systems may include a delay term. In general, the time-delay is believed to have a negative impact on the control system performance. To compensate for this impact, Smith predictor schemes work fine for slow processes [3, 4]. In the last two decades, the theory of fractional calculus has attracted researchers [5–9], because of its wide use in different areas of sciences and engineering, such as viscoelastic systems [12,13], sinusoidal oscillators [14], electromagnetic theory [15,16], and bioengineering [17]. The sliding mode control (SMC) approach is one of the most important methods and this approach can be used in many systems [18, 19] because of its robustness to parameter uncertainties and insensitivity to external disturbances. Sliding mode control (SMC) is based on the theory of variable structure systems [20]. The main feature of SMC is to cause states from initial

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conditions to a sliding surface and then the states are forced to remain on sliding surface because the system on the sliding surface has desirable properties such as stability and disturbance rejection capability [21]. Another approach is the use of fractional order controllers such as the CRONE controller [22, 23], the TID controller [24], the fractional PID controller [25], and the FO adaptive SMC [26] to improve system control function.

The topic of the present work is the stability of fractional-order linear systems with disturbances and multi time-delays have been done using the sliding mode control strategy. In this paper, the sliding mode controller for a class of linear fractional order systems with parameter uncertainties and multi time delay in state and input disturbance is proposed. The paper is presented as follows. In Section 2, basic definitions in fractional calculus are given. In Section 3, problem formulation of fractional-order systems is presented. Section 4 proposes the sliding mode control method. Numerical simulation results are shown in Section 5. Finally, conclusion is made in Section 6.

2 Basic Definition and Preliminaries

There exist many definitions of fractional derivative. Two of the most commonly used definitions are the Riemann-Liouville, and the Grunwald-Letnikov definitions. The Grunwald-Letnikov fractional derivative of order q of a continuous function $f(t)$ is defined by [27]

$$D_t^q f(t) = \lim_{N \to \infty} \left[\frac{t-a}{N} \right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f\left(t - j \left[\frac{t-a}{N} \right]\right).$$

Riemann-Liouville fractional integral and derivative operators of order q are defined as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau.$$

where n is the first integer which is not less than q, i.e., $n - 1 \leq q < n$ and Γ is the Gamma function

$$\Gamma(q) = \int_0^\infty e^{-t} t^{q-1} dt.$$

If $0 < q < 1$, then the Riemann-Liouville fractional derivative and integral operators of order q are defined as

$$D_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\tau)^{-q} f(\tau) d\tau,$$

$$I_t^q f(t) = I^\alpha f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau.$$

3 Stability

Lemma 3.1 *[28] The following autonomous system:*

$$D^q x(t) = A x(t), \qquad x(0) = x_0, \tag{1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is asymptotically stable if and only if $|arg(eig(A))| > \frac{q\pi}{2}$, in this case, each component of the states decays towards origin like t^{-q}. Also, this system is stable if and only if $|arg(eig(A))| \geq \frac{q\pi}{2}$ and those critical eigenvalues that satisfy $|arg(eig(A))| = \frac{q\pi}{2}$ have geometric multiplicity one.

The stable and unstable regions for $0 < q < 1$ are shown in Figure 1.
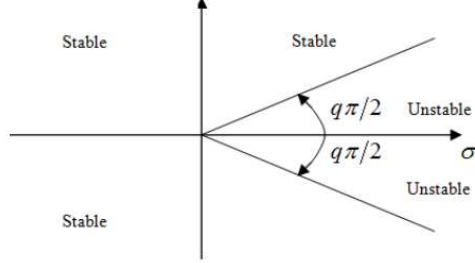
Figure 1: Stability region of LTI fractional order system with order $0 < q < 1$.

4 Problem Formulation

Now consider the linear uncertain system of fractional order with multi delays in state as follows:

$$D_t^q x(t) = \sum_{i=1}^{N} \alpha_i (A_i x(t) + A_{id1} x(t - t_{d1}) + A_{id2} x(t - td2) + \ldots$$

$$+ A_{idl} x(t - t_{dl}) + B_i B(u(t) + w(t))). \tag{2}$$

where and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$ are the state vector, the controller, the exogenous input of the system, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{m \times m}$, $A_{id} \in \mathbb{R}^{n \times n}$ are constant matrices, and q is the fractional derivative, $0 < q < 1$, and α_i are indeterminate parameters which satisfy $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.

Conditions that are necessary mode switching systems starting from any point and move on the switching surface and reach it (to switching level) are called reaching conditions. One of these conditions is as follows. This condition reach is global but does not guarantee limited arrival time:

$$\dot{V}(t) = S\dot{S}, \tag{3}$$

where S is sliding sector. Another requirement in [21] is suggested that including the shown entity,

$$\frac{1}{2} \frac{d}{dt} S^2 \leq -\eta |S|,$$

where η is a positive constant. That fulfilling the above condition causes the switching time reach less than $\frac{|S(t=0)|}{\eta}$.

5 Design of the Controller

In sliding mode control, the system state movement to a desired place, is comprised of two parts, the reaching phase and the sliding phase. The control switching level (reachability phase), should lead the system to the desired level. When all the modes of system were on the surface, sliding mode occurs (sliding phase). In sliding mode, the dynamic behavior

of the system is determined by choosing the switching level. Let the sliding surface S be such that:

$$S(x,t) = I^{1-q}x(t). \tag{4}$$

Theorem 5.1 *The sliding mode control law:*

$$u(t) = \frac{-B^{-1}}{a}k\frac{S(t)}{||S(t)||}, \tag{5}$$

when

$$a = \min\{|B_1|, |B_2|, \ldots, |B_N|\},$$

$$b = \max\{||A_1x(t)||, ||A_2x(t)||, \ldots, ||A_Nx(t)||\},$$

$$d_{delay1} = \max\{||A_{1d1}x(t - t_{d1})||, ||A_{2d1}x(t - t_{d1})||, \ldots, ||A_{Nd1}x(t - t_{d1})||\},$$

$$d_{delay2} = \max\{||A_{1d2}x(t - t_{d2})||, ||A_{2d2}x(t - t_{d2})||, \ldots, ||A_{Nd2}x(t - t_{d2})||\},$$

$$\vdots$$

$$d_{delayl} = \max\{||A_{1dl}x(t - t_{dl})||, ||A_{2dl}x(t - t_{dl})||, \ldots, ||A_{Ndl}x(t - t_{dl})||\},$$

$$k = d + d_{delay1}(x) + d_{delay2}(x) + \cdots + d_{delayN}(x) + b||B||\gamma + \eta e^{-\lambda t}||S(t)||^{1-\delta},$$

and $\eta > 0$, $\lambda > 0$, $0 < \delta \leq 1$.

Proof. The Lyapunov function to be defined in (2) taking the time derivative of S in (3) and substituting by (4), we obtain:

$$\begin{aligned}
\dot{S}(t) &= \sum_{i=1}^{N} \alpha_i A_i x(t) + \sum_{i=1}^{N} \alpha_i A_{id1} x(t - t_{d1}) + \cdots + \sum_{i=1}^{N} \alpha_i A_{idN} x(t - t_{dN}) \\
&+ \sum_{i=1}^{N} \alpha_i B_i Bu(t) + \sum_{i=1}^{N} \alpha_i B_i Bw(t).
\end{aligned} \tag{6}$$

Substituting (4) in (2), we have

$$\begin{aligned}
\dot{V}(t) = S(t)\dot{S}(t) &= S^T(t)\left(\sum_{i=1}^{N} \alpha_i A_i x(t) + \sum_{i=1}^{N} \alpha_i A_{id1} x(t - t_{d1}) + \ldots \right. \\
&\left. + \sum_{i=1}^{N} \alpha_i A_{idN} x(t - t_{dN}) + \sum_{i=1}^{N} \alpha_i B_i Bu(t) + \sum_{i=1}^{N} \alpha_i B_i Bw(t)\right).
\end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned}
\dot{V}(t) = S^T(t)\dot{S}(t) &= S^T(t)\left(\sum_{i=1}^{N} \alpha_i A_i x(t) + \sum_{i=1}^{N} \alpha_i A_{id1} x(t - t_{d1}) + \ldots \right. \\
&\left. + \sum_{i=1}^{N} \alpha_i A_{idN} x(t - t_{dN}) - k\frac{S(t)}{||S(t)||}\frac{\sum_{i=1}^{N} \alpha_i B_i}{a} + \sum_{i=1}^{N} \alpha_i B_i Bw(t)\right),
\end{aligned}$$

hence

$$\dot{V}(t) < \eta e^{-\lambda t}||S(t)||^{2-\delta}.$$

This indicates that the Lyapunov function is positive definite and its derivative is negative definite. By Lyapuonv stability theory and Lemma 1, the closed-loop system (1) with the control law (u) in (4) is asymptotically stable.

We consider, the system states will reach the sliding mode $S = 0$ for a finite time T. We have

$$S^T \dot{S} = \frac{1}{2}\frac{d(S^T S)}{dt} = \frac{1}{2}\frac{dS^2}{dt} = S\frac{dS}{dt}.$$

It follows that

$$\frac{dt}{d||S(T)||} = \frac{1}{\eta e^{-\lambda t}||S(t)||^{1-\delta}},$$

so

$$\frac{d||S(T)||}{dt} = \eta e^{-\lambda t}||S(t)||^{1-\delta}, \tag{8}$$

we can integrate (8) from 0 to T, we have

$$T = -\frac{1}{\lambda}ln(1 - \frac{\lambda}{\delta\eta}||S(0)||^{\delta}).$$

Therefore, $t \geq T$, the system will converge to switching manifold at any initial state. T is positive, it is enough that the selected constants

$$0 \leq \frac{\lambda}{\delta\eta}||S(0)||^{\delta} < 1.$$

$\square$

6   Simulation Results of the Proposed Sliding Mode Controller

The sliding mode controller given by (4) is applied to the fractional order systems given by (1). Now consider this system, for example

$$D_t^q x(t) = \sum_{i=1}^{3} \alpha_i(A_i x(t) + A_{id1} x(t-t_{d1}) + A_{id2} x(t-td2) + A_{id3} x(t-t_{d3}) + B_i B(u(t)+w(t))),$$

$$\begin{aligned} D_t^q x(t) &= \alpha_1(A_1 x(t) + A_{1d1} x(t-t_{d1}) + A_{1d2} x(t-td2) + A_{1d3} x(t-t_{d3}) + B_1 B(u(t)+w(t))) \\ &= \alpha_2(A_2 x(t) + A_{2d1} x(t-t_{d1}) + A_{2d2} x(t-td2) + A_{2d3} x(t-t_{d3}) + B_2 B(u(t)+w(t))) \\ &= \alpha_2(A_2 x(t) + A_{2d1} x(t-t_{d1}) + A_{2d2} x(t-td2) + A_{2d3} x(t-t_{d3}) + B_2 B(u(t)+w(t))). \end{aligned}$$

The initial conditions of system (1) are taken to be $[x_1(0)\ x_2(0)]^T = [2\ -1]^T$. Then, we choose $A_1 = \begin{bmatrix} 13 & -1 \\ 1 & 10 \end{bmatrix}$, $A_2 = \begin{bmatrix} 6 & -8 \\ 12 & 9 \end{bmatrix}$, $A_3 = \begin{bmatrix} 5 & -6 \\ 1 & 2 \end{bmatrix}$, $A_{1d1} = \begin{bmatrix} 1 & 0 \\ -5 & 3 \end{bmatrix}$, $A_{1d2} = \begin{bmatrix} 0 & 1 \\ 2 & 14 \end{bmatrix}$, $A_{1d3} = \begin{bmatrix} 0 & 2 \\ 7 & 4 \end{bmatrix}$, $A_{2d1} = \begin{bmatrix} 0 & 8 \\ 5 & 9 \end{bmatrix}$, $A_{2d2} = \begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix}$, $A_{2d3} = \begin{bmatrix} 11 & 1 \\ 6 & -1 \end{bmatrix}$, $A_{3d1} = \begin{bmatrix} 0 & 10 \\ 10 & 10 \end{bmatrix}$, $A_{3d2} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$, $A_{3d3} = \begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}$,
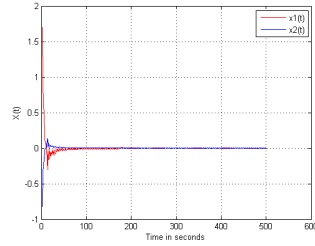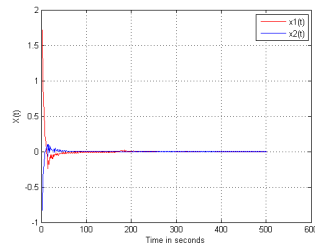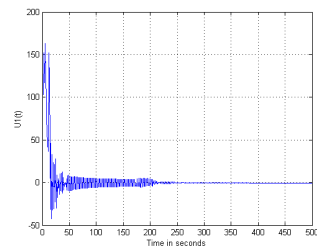
(a) State $X(t)$.
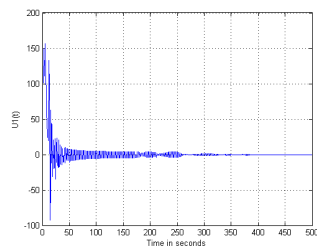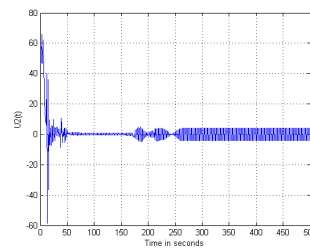
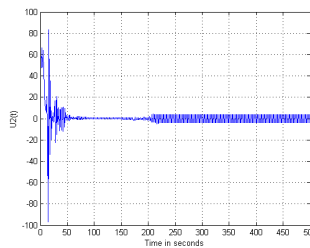(b) Control input $u_1(t)$.   (c) Control input $u_2(t)$.

Figure 2: Sliding mode control $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 0.4$ (sampling interval, $h = 0.005$ s).

(a) State $X(t)$.

(b) Control input $u_1(t)$.   (c) Control input $u_2(t)$.

Figure 3: Sliding mode control $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\alpha_3 = 0$ (sampling interval, $h = 0.005$ s).

$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_1 = 0.4$, $B_2 = 0.6$, $B_3 = 0.2$, $q = 0.5$, $h = 0.005$, and $t_{d1} = 2$, $t_{d2} = 4$, $t_{d3} = 11$, and the disturbance is of the form of $w(t) = \sin(t)$. The parameters of the controller are chosen such that $\eta = 3$, $\delta = 0.4$, $\gamma = 1$, $\lambda = 4$. The performance of the system is simulated. We plot this system for two different categories of parameters α_1, α_2, α_3. The plots of the states of the system are shown in Figures 2(a) and 3(a) for the different parameters α_1, α_2, α_3. Figures 2(b) and 3(b) give the control input $u_1(t)$, and Figures 2(c) and 3(c) give the control input $u_2(t)$. Therefore, it can be concluded that the simulation results indicate that the proposed sliding mode controller works well.

7 Conclusions

In this paper, the sliding mode controller for stabilization of fractional order systems with uncertainties and multiple delay in state and disturbance input is investigated. A switching surface of integral type is proposed such that stability of the closed-loop system in the sliding mode can be guaranteed. An illustrative example shows the effectiveness of the proposed new scheme.

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On Construction and a Class of Non-Volterra Cubic Stochastic Operators

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Abstract: We give a construction of a cubic stochastic operator (CSO) on a finite dimensional simplex. This construction depends on a probability measure μ which is given on a fixed finite graph G. Using the construction of CSO for μ defined as product of measures given on components of G a wide class of non-Volterra CSOs is described. It is shown that the non-Volterra operators can be reduced to N number (where N is the number of components) of Volterra CSOs defined on the components. By such a reduction we describe behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex.

Keywords: *simplex; graph; cubic stochastic operator; Volterra cubic operator.*

Mathematics Subject Classification (2010): 37N25, 92D10.

1 Introduction

There are many systems which are described by nonlinear operators. One of the simplest nonlinear case is quadratic operator (for a recent review on the theory of quadratic stochastic operators see [5]). Quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics [1, 6], physics [11], economy [2], mathematics [10]. In modern scientific investigations non-linear operators of higher order arise. In particular, a cubic stochastic operator (CSO) can be obtained in gene engineering and free population with a ternary production. To study non-linear dynamical systems a method of Lyapunov functions is used (see [5, 9]).

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In [7], [8] and [12] the behavior of trajectories of some CSOs were studied. A CSO arises as follows: consider a population consisting of m species. Let $x^0 = (x_1^0, \ldots, x_m^0)$ be the probability distribution (where $x_i^0 = P(i)$ is the probability of i, $i = 1, 2, \ldots, m$) of species in the initial generation, and $P_{ijk,l}$ be the probability with which individuals in the ith, jth and kth species interbreed to produce an individual l, more precisely $P_{ijk,l}$ is the conditional probability $P(l|i, j, k)$ with which ith, jth and kth species interbred successfully, when they produce an individual l. In this paper we consider models of free population i.e., there is no difference of "sex" and in any generation the "parents" ijk are independent i.e., $P(i, j, k) = P(i)P(j)P(k) = x_i x_j x_k$.

Each CSO W can be uniquely defined by a matrix $\mathbf{P} \equiv \mathbf{P}(W) = \{P_{ijk,l}\}_{i,j,k,l=1}^m$. Usually the matrix $\mathbf{P}$ is known. In this paper we give a constructive description of $\mathbf{P}$. This construction depends on a probability measure μ which is given on a fixed finite graph G and finite set of cells (configurations). Such constructions for quadratic stochastic operators are given in [3] and in the general form in [4].

The main aim of the paper is to show that if μ is the product of the probability measures being defined on the maximal connected subgraphs (components) then corresponding non-Volterra CSO can be reduced to N number (where N is the number of components) of Volterra operators defined on the components.

By such a reduction we describe behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex. These results are a natural generalization of the paper [13] which was devoted to quadratic stochastic operators.

2 Construction of Cubic Stochastic Operators

Recall that a CSO is a mapping of the simplex

$$S^{m-1} = \{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}$$

into itself, of the form

$$W : x_l' = \sum_{i,j,k=1}^m P_{ijk,l} x_i x_j x_k, \quad (l = 1, ..., m), \tag{1}$$

where $P_{ijk,l}$ are coefficients of 'heredity' and

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^m P_{ijk,l} = 1, \quad (i, j, k, l = 1, ..., m). \tag{2}$$

Let $G = (\Lambda, L)$ be a finite graph without loops and multiple edges, where Λ is the set of vertexes and L is the set of edges of the graph.

Furthermore, let Φ be a finite set, called the set of alleles (in problems of statistical mechanics, Φ is called the range of spin). The function $\sigma : \Lambda \to \Phi$ is called a cell (in mechanics it is called configuration). Denote by Ω the set of all cells. Let $S(\Lambda, \Phi)$ be the set of all probability measures defined on the finite set Ω.

Let $\{\Lambda_i, i = 1, ..., N\}$ be the set of maximal connected subgraphs (components) of the graph G. For $\sigma \in \Omega$ denote by $\sigma(M)$ its "projection" (or "restriction") to $M \subset \Lambda$: $\sigma(M) = \{\sigma(x)\}_{x \in M}$. Then any $\sigma \in \Omega$ has the form $\sigma = (\sigma_1, \ldots, \sigma_N)$, where $\sigma_i = \sigma(\Lambda_i)$. We say $\sigma(M)$ is a subcell iff M is a maximal connected subgraph of G.

Fix three cells $\sigma, \varphi, \psi \in \Omega$, and put

$$\Omega(\sigma, \varphi, \psi) = \{\tau = (\tau_1, \dots, \tau_N) \in \Omega : \tau_i \in \{\sigma_i, \varphi_i, \psi_i\}, \forall i = 1, \dots, N\}.$$

Remark 2.1 The set $\Omega(\sigma, \varphi, \psi)$ can be interpreted as the set of all possible 'children' of the 'parents' $\theta = (\sigma, \varphi, \psi)$. A child τ can be born from θ if it only consists the subcells of its parents θ. For quadratic stochastic operators such a set was first considered in [3] and in the general form in [4].

Now let $\mu \in S(\Lambda, \Phi)$ be a probability measure defined on Ω such that $\mu(\sigma) > 0$ for any cell $\sigma \in \Omega$. The heredity coefficients $P_{\sigma\varphi\psi,\tau}$ are defined as

$$P_{\sigma\varphi\psi,\tau} = \begin{cases} \frac{\mu(\tau)}{\mu(\Omega(\sigma,\varphi,\psi))}, & \text{if } \tau \in \Omega(\sigma, \varphi, \psi), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Obviously, $P_{\sigma\varphi\psi,\tau} \geq 0$, and $\sum_{\tau \in \Omega} P_{\sigma\varphi\psi,\tau} = 1$ for all $\sigma, \varphi, \psi \in \Omega$.

The CSO $W \equiv W_\mu$ acting on the simplex $S(\Lambda, \Phi)$ and determined by coefficients (3) is defined as follows: for an arbitrary measure $\lambda \in S(\Lambda, \Phi)$, the measure $W(\lambda) = \lambda' \in S(\Lambda, \Phi)$ is defined by the equality

$$\lambda'(\tau) = \sum_{\sigma, \varphi, \psi \in \Omega} P_{\sigma\varphi\psi,\tau} \lambda(\sigma) \lambda(\varphi) \lambda(\psi) \tag{4}$$

for any cell $\tau \in \Omega$.

The CSO construction is also closely related to the graph structure on the set Λ.

A CSO is called Volterra if the coefficients $P_{ijk,l}$ may be nonzero only when $l \in \{i, j, k\}$ and vanish in all the remaining cases (see [7,8]).

It is easy to see that any Volterra CSO has the following form

$$W : x'_l = x_l \left(x_l^2 + x_l \sum_{\substack{i=1 \\ i \neq l}}^{m} a_{i,l} x_i + \sum_{\substack{i,j=1 \\ i \neq l, \, j \neq l}}^{m} b_{ij,l} x_i x_j \right), \quad (l = 1, \dots, m), \tag{5}$$

where $a_{i,l}$ and $b_{ij,l}$ are some coefficients depending on $P_{ijk,l}$.

Theorem 2.1 *The CSO* (4) *is Volterra if and only if the graph G is connected.*

Proof. Let G be connected then $\Omega(\sigma, \varphi, \psi) = \{\sigma, \varphi, \psi\}$. Consequently, by (3) it follows that the corresponding operator is Volterra. Conversely, if (3) satisfies $P_{\sigma\varphi\psi,\tau} = 0$, for $\tau \notin \{\sigma, \varphi, \psi\}$ then by condition $\mu(\sigma) > 0$ it follows that G is connected.

3 A Class of Non-Volterra CSOs

In this section we describe a condition on measure μ under which the CSO W_μ generated by μ (using the construction described in the previous section) can be studied using the theory of Volterra CSO.

Denote by $\Omega_i = \Phi^{\Lambda_i}$ the set of all cells defined on component Λ_i, $i = 1, \dots, N$. Let μ_i be a probability measure defined on Ω_i, such that $\mu_i(\sigma) > 0$ for any $\sigma \in \Omega_i$, $i = 1, \dots, N$.

Consider probability measure μ on $\Omega = \Omega_1 \times \cdots \times \Omega_N$ defined as

$$\mu(\sigma) = \prod_{i=1}^{N} \mu_i(\sigma_i), \tag{6}$$

where $\sigma = (\sigma_1, ..., \sigma_N)$, with $\sigma_i \in \Omega_i, i = 1, ..., N$.

By Theorem 2.1, if $N = 1$ then QSO constructed on G is Volterra QSO.

Theorem 3.1 *The CSO constructed by* (3) *with measure* (6) *is reducible to N separate Volterra CSOs.*

Proof. For any $\sigma = (\sigma_1, ..., \sigma_N)$, $\varphi = (\varphi_1, ..., \varphi_N)$, $\psi = (\psi_1, ..., \psi_N) \in \Omega$ we have

$$\mu(\Omega(\sigma, \varphi, \psi)) = \sum_{\substack{\tau_1, ..., \tau_N: \\ \tau_i \in \{\sigma_i, \varphi_i, \psi_i\}, i=1,...,N}} \prod_{i=1}^{N} \mu_i(\tau_i) = \prod_{i=1}^{N} \big(\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(\psi_i)\big).$$

Using this equality by (3) we get

$$P_{\sigma\varphi\psi,\tau} = \begin{cases} \prod_{i=1}^{N} \dfrac{\mu_i(\tau_i)}{\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(\psi_i)}, & \text{if } \tau \in \Omega(\sigma, \varphi, \psi), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Thus CSO generated by measure (6) can be written as

$$\lambda'(\tau) = \lambda'(\tau_1, ..., \tau_N) =$$

$$\sum_{\substack{\sigma = (\sigma_1, ..., \sigma_N) : \sigma_i \in \Omega_i \\ \varphi = (\varphi_1, ..., \varphi_N) : \varphi_i \in \Omega_i \\ \psi = (\psi_1, ..., \psi_N) : \psi_i \in \Omega_i}} \prod_{i=1}^{N} \frac{\mu_i(\tau_i)\mathbf{1}_{(\tau_i \in \{\sigma_i, \varphi_i, \psi_i\})}}{\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \lambda(\sigma)\lambda(\varphi)\lambda(\psi). \tag{8}$$

Denote

$$X_{i,w} = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \lambda(\tau) = \sum_{\substack{\tau_1, ..., \tau_{i-1}, \tau_{i+1}, ..., \tau_N \\ \tau_k \in \Omega_k, k \neq i}} \lambda(\tau_1, ..., \tau_{i-1}, w, \tau_{i+1}, ..., \tau_N). \tag{9}$$

From (8) we have

$$X'_{i,w} = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \lambda'(\tau) = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \left[\sum_{\substack{\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N \\ \varphi, \psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(w) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \times \right.$$

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_i, \varphi_j, \psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi)\lambda(\psi) +$$

$$\sum_{\substack{\varphi_1, ..., \varphi_{i-1}, \varphi_{i+1}, ..., \varphi_N \\ \sigma, \psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(\sigma_i) + \mu_i(w) + \mu_i(\psi_i)} \times$$

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_i, \varphi_j, \psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\sigma)\lambda(\varphi_1, ..., \varphi_{i-1}, w, \varphi_{i+1}, ..., \varphi_N)\lambda(\psi) +$$

$$\sum_{\substack{\psi_1, ..., \psi_{i-1}, \psi_{i+1}, ..., \psi_N \\ \sigma, \varphi \in \Omega}} \frac{\mu_i(w)}{\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(w)} \times$$

$$\left. \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_i, \varphi_j, \psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\sigma)\lambda(\varphi)\lambda(\psi_1, ..., \psi_{i-1}, w, \psi_{i+1}, ..., \psi_N) \right] =$$

$$3 \sum_{\substack{\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N \\ \varphi, \psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(w) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \times$$

$$\sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_j, \varphi_j, \psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi)\lambda(\psi). \qquad (10)$$

It is easy to see that

$$\sum_{\tau_1, ..., \tau_{i-1}, \tau_{i+1}, ..., \tau_N} \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_j, \varphi_j, \psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} = 1.$$

Thus from (10) we have

$$\text{RHS of } (10) =$$

$$3 \sum_{\substack{\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N \\ \varphi, \psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(w) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi)\lambda(\psi) =$$

$$\sum_{\substack{\sigma, \varphi, \psi \\ \sigma_i = \varphi_i = \psi_i = w}} \lambda(\sigma)\lambda(\varphi)\lambda(\psi) + 6 \sum_{\psi_i \in \Omega_i \setminus w} \frac{\mu_i(w)}{2\mu_i(w) + \mu_i(\psi_i)} \times$$

$$\sum_{\substack{\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N \\ \varphi_1, ..., \varphi_{i-1}, \varphi_{i+1}, ..., \varphi_N \\ \psi_1, ..., \psi_{i-1}, \psi_{i+1}, ..., \psi_N}} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi_1, ..., \varphi_{i-1}, w, \varphi_{i+1}, ..., \varphi_N)\lambda(\psi) +$$

$$3 \sum_{\varphi_i, \psi_i \in \Omega_i \setminus w} \frac{\mu_i(w)}{\mu_i(w) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \times$$

$$\sum_{\substack{\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N \\ \varphi_1, ..., \varphi_{i-1}, \varphi_{i+1}, ..., \varphi_N \\ \psi_1, ..., \psi_{i-1}, \psi_{i+1}, ..., \psi_N}} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi)\lambda(\psi) =$$

$$X_{i,w}^3 + \sum_{\psi \in \Omega_i \setminus w} \frac{6\mu_i(w)}{2\mu_i(w) + \mu_i(\psi)} X_{i,w}^2 X_{i,\psi} + \sum_{\varphi, \psi \in \Omega_i \setminus w} \frac{3\mu_i(w)}{\mu_i(w) + \mu_i(\varphi) + \mu_i(\psi)} X_{i,w} X_{i,\varphi} X_{i,\psi}.$$

Thus operator (8) can be rewritten as

$$X'_{i,w} = X_{i,w} \left(X_{i,w}^2 + \sum_{\psi \in \Omega_i \setminus w} \frac{6\mu_i(w)}{2\mu_i(w) + \mu_i(\psi)} X_{i,w} X_{i,\psi} + \right.$$

$$\left. \sum_{\varphi, \psi \in \Omega_i \setminus w} \frac{3\mu_i(w)}{\mu_i(w) + \mu_i(\varphi) + \mu_i(\psi)} X_{i,\varphi} X_{i,\psi} \right), \tag{11}$$

where $X_{i,w}$ is defined by (9), $w \in \Omega_i, i = 1, ..., N$.

Note that $\sum_{w \in \Omega_i} X_{i,w} = 1$ for any $i = 1, ..., N$. One can see that for each fixed i ($i = 1, ..., N$) the operator (11) is similar to (5), i.e. is a Volterra CSO $W^{(i)} : S^{|\Omega_i|-1} \to S^{|\Omega_i|-1}$. The theorem is proved.

This theorem allows us to use the theory of Volterra CSO to describe the behavior of trajectories of non-Volterra CSO (8).

If for each $i \in \{1, ..., N\}$ the asymptotical behavior of trajectories of CSO $W^{(i)}$ is known, say $X_{i,w}^{(n)} \to X_{i,w}^*$, $n \to \infty$, then asymptotical behavior of W (i.e. (8)), say $\lambda^{(n)}(\tau) \to \lambda^*(\tau)$, $n \to \infty$, can be found from the following system of linear equations

$$\sum_{\tau \in \Omega : \tau_i = w} \lambda^*(\tau) = X_{i,w}^*, \quad w \in \Omega_i, i = 1, ..., N. \tag{12}$$

In the following section we shall illustrate the restriction of a non-Volterra cubic stochastic operator to two Volterra operators and study the trajectory of the non-Volterra operator by these two Volterra operators.

4 An Example

Consider graph $G = (\Lambda, L)$ with $\Lambda = \{1, 2\}$ and $L = \emptyset$. Take $\Phi = \{1, 2\}$. Then non-Volterra CSO (8) has the form

$$x'_1 = x_1^3 + 3\beta_1(x_1^2 x_2 + x_1 x_2^2) + 3\alpha_1(x_1^2 x_3 + x_1 x_3^2) +$$
$$3\alpha_1\beta_1[x_1^2 x_4 + x_1 x_4^2 + x_2^2 x_3 + x_2 x_3^2 + 2(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)],$$

$$x'_2 = x_2^3 + 3\beta_2(x_1^2 x_2 + x_1 x_2^2) + 3\alpha_1(x_2^2 x_4 + x_2 x_4^2) +$$
$$3\alpha_1\beta_2[x_1^2 x_4 + x_1 x_4^2 + x_2^2 x_3 + x_2 x_3^2 + 2(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)],$$

$$x'_3 = x_3^3 + 3\alpha_2(x_1 x_3^2 + x_1^2 x_3) + 3\beta_1(x_3^2 x_4 + x_3 x_4^2) +$$
$$3\alpha_2\beta_1[x_1^2 x_4 + x_1 x_4^2 + x_2^2 x_3 + x_2 x_3^2 + 2(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)],$$

$$x'_4 = x_4^3 + 3\alpha_2(x_2 x_4^2 + x_2^2 x_4) + 3\beta_2(x_3^2 x_4 + x_3 x_4^2) +$$
$$3\alpha_2\beta_2[x_1^2 x_4 + x_1 x_4^2 + x_2^2 x_3 + x_2 x_3^2 + 2(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4)],$$
$$\tag{13}$$

where $\mu_1 = (\alpha_1, \alpha_2)$, $\alpha_j > 0$, $\alpha_1 + \alpha_2 = 1$; $\mu_2 = (\beta_1, \beta_2)$, $\beta_j \geq 0$, $\beta_1 + \beta_2 = 1$.

Putting $x_1 + x_2 = X_{1,1}$, $x_3 + x_4 = X_{1,2}$ and $x_1 + x_3 = X_{2,1}$, $x_2 + x_4 = X_{2,2}$ we get the Volterra cubic operators:

$$
\begin{aligned}
X'_{1,1} &= X_{1,1}\left(X_{1,1}^2 + 3\alpha_1 X_{1,2}(X_{1,1} + X_{1,2})\right), \\
X'_{1,2} &= X_{1,2}\left(X_{1,2}^2 + 3\alpha_2 X_{1,1}(X_{1,1} + X_{1,2})\right),
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
X'_{2,1} &= X_{2,1}\left(X_{2,1}^2 + 3\beta_1 X_{2,2}(X_{2,1} + X_{2,2})\right), \\
X'_{2,2} &= X_{2,2}\left(X_{2,2}^2 + 3\beta_2 X_{2,1}(X_{2,1} + X_{2,2})\right).
\end{aligned}
\tag{15}
$$

Since $X_{i,1} + X_{i,2} = 1$, $i = 1, 2$, the study of both operators (14) and (15) can be reduced to the study of a dynamical system given by the function $f_\alpha(x) = x(x^2 + 3\alpha(1 - x))$, $x \in [0, 1]$. This is an increasing function of $x \in [0, 1]$ for each parameter $\alpha \in [0, 1]$.

We have

$$
\mathrm{Fix}(f_\alpha) = \{x \in [0, 1] : f_\alpha(x) = x\} =
\begin{cases}
\{0, 1\}, & \text{if } \alpha \in [0, 1/3] \cup [2/3, 1], \\
\{0, 3\alpha - 1, 1\}, & \text{if } \alpha \in (1/3, 2/3).
\end{cases}
$$

Using the above-mentioned properties of the function $f_\alpha(x)$ and checking $|f'_\alpha(a)|$ at $a \in \mathrm{Fix}(f_\alpha)$ one can see that the sequence $x^{(n)} = f_\alpha(x^{(n-1)})$, $n \geq 1$ for $x^{(0)} \in [0, 1]$ has the following limits

$$
\lim_{n \to \infty} x^{(n)} =
\begin{cases}
0, & \text{for any } x^{(0)} \in [0, 1), \ \alpha \in [0, 1/3], \\
3\alpha - 1, & \text{for any } x^{(0)} \in (0, 1), \ \alpha \in (1/3, 2/3), \\
1, & \text{for any } x^{(0)} \in (0, 1], \ \alpha \in [2/3, 1].
\end{cases}
\tag{16}
$$

By equalities (16) for operators (14) we get the following

$$
\lim_{n \to \infty} (X_{1,1}^{(n)}, X_{1,2}^{(n)}) =
\begin{cases}
(0, 1), & \text{for any } X_{1,1}^{(0)} \in [0, 1), \ \alpha_1 \in [0, 1/3], \\
(3\alpha_1 - 1, 2 - 3\alpha_1), & \text{for any } X_{1,1}^{(0)} \in (0, 1), \ \alpha_1 \in (1/3, 2/3), \\
(1, 0), & \text{for any } X_{1,1}^{(0)} \in (0, 1], \ \alpha_1 \in [2/3, 1].
\end{cases}
\tag{17}
$$

A similar formula is true for the operator (15), where α_1 is replaced by β_1. Combining these formulas and using formula (12) one proves the following.

Proposition 4.1 *The trajectory of the non-Volterra CSO* (13) *has the following limit*

$$
\lim_{n\to\infty} x^{(n)} =
\begin{cases}
(1,0,0,0), & \text{if } \alpha_1,\beta_1 \in [2/3,1], \\[4pt]
(0,1,0,0), & \text{if } \alpha_1 \in [2/3,1],\ \beta_1 \in [0,1/3], \\[4pt]
(0,0,1,0), & \text{if } \alpha_1 \in [0,1/3],\ \beta_1 \in [2/3,1], \\[4pt]
(0,0,0,1), & \text{if } \alpha_1,\beta_1 \in [0,1/3], \\[4pt]
(0,0,3\beta_1-1,2-3\beta_1), & \text{if } \alpha_1 \in [0,1/3],\ \beta_1 \in (1/3,2/3), \\[4pt]
(3\beta_1-1,2-3\beta_1,0,0), & \text{if } \alpha_1 \in [2/3,1],\ \beta_1 \in (1/3,2/3), \\[4pt]
(0,3\alpha_1-1,0,2-3\alpha_1), & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in [0,1/3], \\[4pt]
(3\alpha_1-1,0,2-3\alpha_1,0), & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in [2/3,1], \\[4pt]
\in U, & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in (1/3,2/3),
\end{cases}
$$

where

$$U = \{x \in S^3 : x_1+x_2 = 3\alpha_1-1,\ x_3+x_4 = 2-3\alpha_1,\ x_1+x_3 = 3\beta_1-1,\ x_2+x_4 = 2-3\beta_1\}.$$

5   Concluding Remarks

In mathematical biology, the nonlinear operator W is called an evolution operator. The fixed points of W are interpreted as equilibrium states of the population, $\lambda \in S^{m-1}$ is called a state of the population, and $W(\lambda), W^2(\lambda), \ldots$ are called states of the population in subsequent generations (offsprings). Since W is a non-linear operator, the investigation of the sequence $W^n(\lambda)$ is a difficult problem in general. So one has to consider a particular case of W, for which the problem is respectively simple. In this paper to define such an operator, a construction of CSO on a finite dimensional simplex is given. Using the construction of CSO a wide class of non-Volterra CSOs is described. Then we have showed that the non-Volterra operators can be reduced to a finitely many of Volterra CSOs. By such a reduction we described behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex.

Here we shall give a biological interpretation of Proposition 4.1. Assume that the evolution of a certain biological system consisting of 4 types of individuals is described by operator (13). Using Proposition 4.1, we can conclude the following:

1. The biological system has up to 5 equilibrium states.

2. After a certain period of time, some types will be at the vanishing point.

3. If a system is in an equilibrium state, then, depending on the state, it can have only one of $1, 2, 3, 4$ types.

Acknowledgment

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