Nonlinear Dynamics and Systems Theory, **14** (1) (2014) 92–100

On Construction and a Class of Non-Volterra Cubic Stochastic Operators

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Received: May 13, 2013; Revised: January 20, 2014

Abstract: We give a construction of a cubic stochastic operator (CSO) on a finite dimensional simplex. This construction depends on a probability measure μ which is given on a fixed finite graph G. Using the construction of CSO for μ defined as product of measures given on components of G a wide class of non-Volterra CSOs is described. It is shown that the non-Volterra operators can be reduced to N number (where N is the number of components) of Volterra CSOs defined on the components. By such a reduction we describe behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex.

Keywords: *simplex; graph; cubic stochastic operator; Volterra cubic operator.*

Mathematics Subject Classification (2010): 37N25, 92D10.

1 Introduction

There are many systems which are described by nonlinear operators. One of the simplest nonlinear case is quadratic operator (for a recent review on the theory of quadratic stochastic operators see [5]). Quadratic dynamical systems have been proved to be a rich source of analysis for the investigation of dynamical properties and modeling in different domains, such as population dynamics [1, 6], physics [11], economy [2], mathematics [10]. In modern scientific investigations non-linear operators of higher order arise. In particular, a cubic stochastic operator (CSO) can be obtained in gene engineering and free population with a ternary production. To study non-linear dynamical systems a method of Lyapunov functions is used (see [5, 9]).

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In [7], [8] and [12] the behavior of trajectories of some CSOs were studied. A CSO arises as follows: consider a population consisting of m species. Let $x^0 = (x_1^0, \dots, x_m^0)$ be the probability distribution (where $x_i^0 = P(i)$ is the probability of i, $i = 1, 2, \dots, m$) of species in the initial generation, and $P_{ijk,l}$ be the probability with which individuals in the ith, jth and kth species interbreed to produce an individual l, more precisely $P_{ijk,l}$ is the conditional probability $P(l|i, j, k)$ with which ith, jth and kth species interbred successfully, when they produce an individual l. In this paper we consider models of free population i.e., there is no difference of "sex" and in any generation the "parents" ijk are independent i.e., $P(i, j, k) = P(i)P(j)P(k) = x_i x_j x_k$.

Each CSO W can be uniquely defined by a matrix $\mathbf{P} \equiv \mathbf{P}(W) = \{P_{ijk,l}\}_{i,j,k,l=1}^m$. Usually the matrix $\mathbf{P}$ is known. In this paper we give a constructive description of $\mathbf{P}$. This construction depends on a probability measure μ which is given on a fixed finite graph G and finite set of cells (configurations). Such constructions for quadratic stochastic operators are given in [3] and in the general form in [4].

The main aim of the paper is to show that if μ is the product of the probability measures being defined on the maximal connected subgraphs (components) then corresponding non-Volterra CSO can be reduced to N number (where N is the number of components) of Volterra operators defined on the components.

By such a reduction we describe behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex. These results are a natural generalization of the paper [13] which was devoted to quadratic stochastic operators.

2 Construction of Cubic Stochastic Operators

Recall that a CSO is a mapping of the simplex

$$S^{m-1} = \{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}$$

into itself, of the form

$$W : x'_l = \sum_{i,j,k=1}^m P_{ijk,l} x_i x_j x_k, \quad (l = 1, ..., m), \tag{1}$$

where $P_{ijk,l}$ are coefficients of 'heredity' and

$$P_{ijk,l} \geq 0, \quad \sum_{l=1}^m P_{ijk,l} = 1, \quad (i, j, k, l = 1, ..., m). \tag{2}$$

Let $G = (\Lambda, L)$ be a finite graph without loops and multiple edges, where Λ is the set of vertexes and L is the set of edges of the graph.

Furthermore, let Φ be a finite set, called the set of alleles (in problems of statistical mechanics, Φ is called the range of spin). The function $\sigma : \Lambda \to \Phi$ is called a cell (in mechanics it is called configuration). Denote by Ω the set of all cells. Let $S(\Lambda, \Phi)$ be the set of all probability measures defined on the finite set Ω.

Let $\{\Lambda_i, i = 1, ..., N\}$ be the set of maximal connected subgraphs (components) of the graph G. For $\sigma \in \Omega$ denote by $\sigma(M)$ its "projection" (or "restriction") to $M \subset \Lambda$: $\sigma(M) = \{\sigma(x)\}_{x \in M}$. Then any $\sigma \in \Omega$ has the form $\sigma = (\sigma_1, \dots, \sigma_N)$, where $\sigma_i = \sigma(\Lambda_i)$. We say $\sigma(M)$ is a subcell iff M is a maximal connected subgraph of G.

Fix three cells $\sigma, \varphi, \psi \in \Omega$, and put

$$\Omega(\sigma, \varphi, \psi) = \{\tau = (\tau_1, \dots, \tau_N) \in \Omega : \tau_i \in \{\sigma_i, \varphi_i, \psi_i\}, \forall i = 1, \dots, N\}.$$

Remark 2.1 The set $\Omega(\sigma, \varphi, \psi)$ can be interpreted as the set of all possible 'children' of the 'parents' $\theta = (\sigma, \varphi, \psi)$. A child τ can be born from θ if it only consists the subcells of its parents θ. For quadratic stochastic operators such a set was first considered in [3] and in the general form in [4].

Now let $\mu \in S(\Lambda, \Phi)$ be a probability measure defined on Ω such that $\mu(\sigma) > 0$ for any cell $\sigma \in \Omega$. The heredity coefficients $P_{\sigma\varphi\psi,\tau}$ are defined as

$$P_{\sigma\varphi\psi,\tau} = \begin{cases} \frac{\mu(\tau)}{\mu(\Omega(\sigma,\varphi,\psi))}, & \text{if } \tau \in \Omega(\sigma, \varphi, \psi), \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Obviously, $P_{\sigma\varphi\psi,\tau} \geq 0$, and $\sum_{\tau\in\Omega} P_{\sigma\varphi\psi,\tau} = 1$ for all $\sigma, \varphi, \psi \in \Omega$.

The CSO $W \equiv W_\mu$ acting on the simplex $S(\Lambda, \Phi)$ and determined by coefficients (3) is defined as follows: for an arbitrary measure $\lambda \in S(\Lambda, \Phi)$, the measure $W(\lambda) = \lambda' \in S(\Lambda, \Phi)$ is defined by the equality

$$\lambda'(\tau) = \sum_{\sigma,\varphi,\psi\in\Omega} P_{\sigma\varphi\psi,\tau}\lambda(\sigma)\lambda(\varphi)\lambda(\psi) \quad (4)$$

for any cell $\tau \in \Omega$.

The CSO construction is also closely related to the graph structure on the set Λ.

A CSO is called Volterra if the coefficients $P_{ijk,l}$ may be nonzero only when $l \in \{i, j, k\}$ and vanish in all the remaining cases (see [7,8]).

It is easy to see that any Volterra CSO has the following form

$$W : x'_l = x_l \left(x_l^2 + x_l \sum_{\substack{i=1 \\ i\neq l}}^{m} a_{i,l}x_i + \sum_{\substack{i,j=1 \\ i\neq l,\, j\neq l}}^{m} b_{ij,l}x_ix_j \right), \quad (l = 1, ..., m), \quad (5)$$

where $a_{i,l}$ and $b_{ij,l}$ are some coefficients depending on $P_{ijk,l}$.

Theorem 2.1 *The CSO* (4) *is Volterra if and only if the graph G is connected.*

Proof. Let G be connected then $\Omega(\sigma, \varphi, \psi) = \{\sigma, \varphi, \psi\}$. Consequently, by (3) it follows that the corresponding operator is Volterra. Conversely, if (3) satisfies $P_{\sigma\varphi\psi,\tau} = 0$, for $\tau \notin \{\sigma, \varphi, \psi\}$ then by condition $\mu(\sigma) > 0$ it follows that G is connected.

3 A Class of Non-Volterra CSOs

In this section we describe a condition on measure μ under which the CSO W_μ generated by μ (using the construction described in the previous section) can be studied using the theory of Volterra CSO.

Denote by $\Omega_i = \Phi^{\Lambda_i}$ the set of all cells defined on component Λ_i, $i = 1, ..., N$. Let μ_i be a probability measure defined on Ω_i, such that $\mu_i(\sigma) > 0$ for any $\sigma \in \Omega_i$, $i = 1, ..., N$.

Consider probability measure μ on $\Omega = \Omega_1 \times \cdots \times \Omega_N$ defined as

$$\mu(\sigma) = \prod_{i=1}^{N} \mu_i(\sigma_i), \tag{6}$$

where $\sigma = (\sigma_1, ..., \sigma_N)$, with $\sigma_i \in \Omega_i, i = 1, ..., N$.

By Theorem 2.1, if $N = 1$ then QSO constructed on G is Volterra QSO.

Theorem 3.1 *The CSO constructed by* (3) *with measure* (6) *is reducible to* N *separate Volterra CSOs.*

Proof. For any $\sigma = (\sigma_1, ..., \sigma_N)$, $\varphi = (\varphi_1, ..., \varphi_N)$, $\psi = (\psi_1, ..., \psi_N) \in \Omega$ we have

$$\mu(\Omega(\sigma, \varphi, \psi)) = \sum_{\substack{\tau_1,\dots,\tau_N: \\ \tau_i \in \{\sigma_i,\varphi_i,\psi_i\}, i=1,\dots,N}} \prod_{i=1}^{N} \mu_i(\tau_i) = \prod_{i=1}^{N} \big(\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(\psi_i)\big).$$

Using this equality by (3) we get

$$P_{\sigma\varphi\psi,\tau} = \begin{cases} \prod_{i=1}^{N} \frac{\mu_i(\tau_i)}{\mu_i(\sigma_i)+\mu_i(\varphi_i)+\mu_i(\psi_i)}, & \text{if } \tau \in \Omega(\sigma, \varphi, \psi), \\ 0 \ \text{ otherwise.} \end{cases} \tag{7}$$

Thus CSO generated by measure (6) can be written as

$$\lambda'(\tau) = \lambda'(\tau_1, ..., \tau_N) =$$

$$\sum_{\substack{\sigma = (\sigma_1,...,\sigma_N) : \sigma_i \in \Omega_i \\ \varphi = (\varphi_1,...,\varphi_N) : \varphi_i \in \Omega_i \\ \psi = (\psi_1,...,\psi_N) : \psi_i \in \Omega_i}} \prod_{i=1}^{N} \frac{\mu_i(\tau_i)\mathbf{1}_{(\tau_i \in \{\sigma_i,\varphi_i,\psi_i\})}}{\mu_i(\sigma_i) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \lambda(\sigma)\lambda(\varphi)\lambda(\psi). \tag{8}$$

Denote

$$X_{i,w} = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \lambda(\tau) = \sum_{\substack{\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_N \\ \tau_k \in \Omega_k, k \neq i}} \lambda(\tau_1, ..., \tau_{i-1}, w, \tau_{i+1}, ..., \tau_N). \tag{9}$$

From (8) we have

$$X'_{i,w} = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \lambda'(\tau) = \sum_{\substack{\tau \in \Omega: \\ \tau_i = w}} \Bigg[\sum_{\substack{\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_N \\ \varphi,\psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(w) + \mu_i(\varphi_i) + \mu_i(\psi_i)} \times$$

$$\prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j \in \{\sigma_j,\varphi_j,\psi_j\})}}{\mu_j(\sigma_j) + \mu_j(\varphi_j) + \mu_j(\psi_j)} \lambda(\sigma_1, ..., \sigma_{i-1}, w, \sigma_{i+1}, ..., \sigma_N)\lambda(\varphi)\lambda(\psi) +$$

$$\sum_{\substack{\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_N \\ \sigma,\psi \in \Omega}} \frac{\mu_i(w)}{\mu_i(\sigma_i) + \mu_i(w) + \mu_i(\psi_i)} \times$$

$$\prod_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j\in\{\sigma_i,\varphi_j,\psi_j\})}}{\mu_j(\sigma_j)+\mu_j(\varphi_j)+\mu_j(\psi_j)}\lambda(\sigma)\lambda(\varphi_1,...,\varphi_{i-1},w,\varphi_{i+1},...,\varphi_N)\lambda(\psi)+$$

$$\sum_{\substack{\psi_1,...,\psi_{i-1},\psi_{i+1},...,\psi_N\\ \sigma,\varphi\in\Omega}}\frac{\mu_i(w)}{\mu_i(\sigma_i)+\mu_i(\varphi_i)+\mu_i(w)}\times$$

$$\left.\prod_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j\in\{\sigma_i,\varphi_j,\psi_j\})}}{\mu_j(\sigma_j)+\mu_j(\varphi_j)+\mu_j(\psi_j)}\lambda(\sigma)\lambda(\varphi)\lambda(\psi_1,...,\psi_{i-1},w,\psi_{i+1},...,\psi_N)\right]=$$

$$3\sum_{\substack{\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_N\\ \varphi,\psi\in\Omega}}\frac{\mu_i(w)}{\mu_i(w)+\mu_i(\varphi_i)+\mu_i(\psi_i)}\times$$

$$\sum_{\substack{\tau\in\Omega:\\ \tau_i=w}}\prod_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j\in\{\sigma_j,\varphi_j,\psi_j\})}}{\mu_j(\sigma_j)+\mu_j(\varphi_j)+\mu_j(\psi_j)}\lambda(\sigma_1,...,\sigma_{i-1},w,\sigma_{i+1},...,\sigma_N)\lambda(\varphi)\lambda(\psi). \qquad (10)$$

It is easy to see that

$$\sum_{\tau_1,...,\tau_{i-1},\tau_{i+1},...,\tau_N}\prod_{\substack{j=1\\j\neq i}}^{N}\frac{\mu_j(\tau_j)\mathbf{1}_{(\tau_j\in\{\sigma_j,\varphi_j,\psi_j\})}}{\mu_j(\sigma_j)+\mu_j(\varphi_j)+\mu_j(\psi_j)}=1.$$

Thus from (10) we have

$$\text{RHS of (10)}=$$

$$3\sum_{\substack{\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_N\\ \varphi,\psi\in\Omega}}\frac{\mu_i(w)}{\mu_i(w)+\mu_i(\varphi_i)+\mu_i(\psi_i)}\lambda(\sigma_1,...,\sigma_{i-1},w,\sigma_{i+1},...,\sigma_N)\lambda(\varphi)\lambda(\psi)=$$

$$\sum_{\substack{\sigma,\varphi,\psi\\ \sigma_i=\varphi_i=\psi_i=w}}\lambda(\sigma)\lambda(\varphi)\lambda(\psi)+6\sum_{\psi_i\in\Omega_i\setminus w}\frac{\mu_i(w)}{2\mu_i(w)+\mu_i(\psi_i)}\times$$

$$\sum_{\substack{\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_N\\ \varphi_1,...,\varphi_{i-1},\varphi_{i+1},...,\varphi_N\\ \psi_1,...,\psi_{i-1},\psi_{i+1},...,\psi_N}}\lambda(\sigma_1,...,\sigma_{i-1},w,\sigma_{i+1},...,\sigma_N)\lambda(\varphi_1,...,\varphi_{i-1},w,\varphi_{i+1},...,\varphi_N)\lambda(\psi)+$$

$$3\sum_{\varphi_i,\psi_i\in\Omega_i\setminus w}\frac{\mu_i(w)}{\mu_i(w)+\mu_i(\varphi_i)+\mu_i(\psi_i)}\times$$

$$\sum_{\substack{\sigma_1,...,\sigma_{i-1},\sigma_{i+1},...,\sigma_N\\ \varphi_1,...,\varphi_{i-1},\varphi_{i+1},...,\varphi_N\\ \psi_1,...,\psi_{i-1},\psi_{i+1},...,\psi_N}}\lambda(\sigma_1,...,\sigma_{i-1},w,\sigma_{i+1},...,\sigma_N)\lambda(\varphi)\lambda(\psi)=$$

$$X_{i,w}^3+\sum_{\psi\in\Omega_i\setminus w}\frac{6\mu_i(w)}{2\mu_i(w)+\mu_i(\psi)}X_{i,w}^2X_{i,\psi}+\sum_{\varphi,\psi\in\Omega_i\setminus w}\frac{3\mu_i(w)}{\mu_i(w)+\mu_i(\varphi)+\mu_i(\psi)}X_{i,w}X_{i,\varphi}X_{i,\psi}.$$

Thus operator (8) can be rewritten as

$$X'_{i,w} = X_{i,w}\left(X_{i,w}^2 + \sum_{\psi\in\Omega_i\setminus w}\frac{6\mu_i(w)}{2\mu_i(w)+\mu_i(\psi)}X_{i,w}X_{i,\psi}+\right.$$

$$\left.\sum_{\varphi,\psi\in\Omega_i\setminus w}\frac{3\mu_i(w)}{\mu_i(w)+\mu_i(\varphi)+\mu_i(\psi)}X_{i,\varphi}X_{i,\psi}\right), \qquad (11)$$

where $X_{i,w}$ is defined by (9), $w\in\Omega_i, i=1,...,N$.

Note that $\sum_{w\in\Omega_i}X_{i,w}=1$ for any $i=1,...,N$. One can see that for each fixed i ($i=1,...,N$) the operator (11) is similar to (5), i.e. is a Volterra CSO $W^{(i)}: S^{|\Omega_i|-1}\to S^{|\Omega_i|-1}$. The theorem is proved.

This theorem allows us to use the theory of Volterra CSO to describe the behavior of trajectories of non-Volterra CSO (8).

If for each $i\in\{1,...,N\}$ the asymptotical behavior of trajectories of CSO $W^{(i)}$ is known, say $X_{i,w}^{(n)}\to X_{i,w}^*$, $n\to\infty$, then asymptotical behavior of W (i.e. (8)), say $\lambda^{(n)}(\tau)\to\lambda^*(\tau)$, $n\to\infty$, can be found from the following system of linear equations

$$\sum_{\tau\in\Omega:\tau_i=w}\lambda^*(\tau) = X_{i,w}^*, \quad w\in\Omega_i, i=1,...,N. \qquad (12)$$

In the following section we shall illustrate the restriction of a non-Volterra cubic stochastic operator to two Volterra operators and study the trajectory of the non-Volterra operator by these two Volterra operators.

4 An Example

Consider graph $G=(\Lambda,L)$ with $\Lambda=\{1,2\}$ and $L=\emptyset$. Take $\Phi=\{1,2\}$. Then non-Volterra CSO (8) has the form

$$\begin{aligned}
x'_1 &= x_1^3 + 3\beta_1(x_1^2x_2 + x_1x_2^2) + 3\alpha_1(x_1^2x_3 + x_1x_3^2) + \\
&\quad 3\alpha_1\beta_1[x_1^2x_4 + x_1x_4^2 + x_2^2x_3 + x_2x_3^2 + 2(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)],\\
x'_2 &= x_2^3 + 3\beta_2(x_1^2x_2 + x_1x_2^2) + 3\alpha_1(x_2^2x_4 + x_2x_4^2) + \\
&\quad 3\alpha_1\beta_2[x_1^2x_4 + x_1x_4^2 + x_2^2x_3 + x_2x_3^2 + 2(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)],\\
x'_3 &= x_3^3 + 3\alpha_2(x_1x_3^2 + x_1^2x_3) + 3\beta_1(x_3^2x_4 + x_3x_4^2) + \\
&\quad 3\alpha_2\beta_1[x_1^2x_4 + x_1x_4^2 + x_2^2x_3 + x_2x_3^2 + 2(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)],\\
x'_4 &= x_4^3 + 3\alpha_2(x_2x_4^2 + x_2^2x_4) + 3\beta_2(x_3^2x_4 + x_3x_4^2) + \\
&\quad 3\alpha_2\beta_2[x_1^2x_4 + x_1x_4^2 + x_2^2x_3 + x_2x_3^2 + 2(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)],
\end{aligned} \qquad (13)$$

where $\mu_1=(\alpha_1,\alpha_2)$, $\alpha_j>0$, $\alpha_1+\alpha_2=1$; $\mu_2=(\beta_1,\beta_2)$, $\beta_j\geq 0$, $\beta_1+\beta_2=1$.

Putting $x_1 + x_2 = X_{1,1}$, $x_3 + x_4 = X_{1,2}$ and $x_1 + x_3 = X_{2,1}$, $x_2 + x_4 = X_{2,2}$ we get the Volterra cubic operators:

$$\begin{aligned} X'_{1,1} &= X_{1,1}\left(X^2_{1,1} + 3\alpha_1 X_{1,2}(X_{1,1} + X_{1,2})\right), \\ X'_{1,2} &= X_{1,2}\left(X^2_{1,2} + 3\alpha_2 X_{1,1}(X_{1,1} + X_{1,2})\right), \end{aligned} \tag{14}$$

and

$$\begin{aligned} X'_{2,1} &= X_{2,1}\left(X^2_{2,1} + 3\beta_1 X_{2,2}(X_{2,1} + X_{2,2})\right), \\ X'_{2,2} &= X_{2,2}\left(X^2_{2,2} + 3\beta_2 X_{2,1}(X_{2,1} + X_{2,2})\right). \end{aligned} \tag{15}$$

Since $X_{i,1} + X_{i,2} = 1$, $i = 1, 2$, the study of both operators (14) and (15) can be reduced to the study of a dynamical system given by the function $f_\alpha(x) = x(x^2 + 3\alpha(1-x))$, $x \in [0, 1]$. This is an increasing function of $x \in [0, 1]$ for each parameter $\alpha \in [0, 1]$.

We have

$$\mathrm{Fix}(f_\alpha) = \{x \in [0,1] : f_\alpha(x) = x\} = \begin{cases} \{0, 1\}, & \text{if } \ \alpha \in [0, 1/3] \cup [2/3, 1], \\ \{0, 3\alpha - 1, 1\}, & \text{if } \ \alpha \in (1/3, 2/3). \end{cases}$$

Using the above-mentioned properties of the function $f_\alpha(x)$ and checking $|f'_\alpha(a)|$ at $a \in \mathrm{Fix}(f_\alpha)$ one can see that the sequence $x^{(n)} = f_\alpha(x^{(n-1)})$, $n \geq 1$ for $x^{(0)} \in [0, 1]$ has the following limits

$$\lim_{n\to\infty} x^{(n)} = \begin{cases} 0, & \text{for any } \ x^{(0)} \in [0, 1), \quad \alpha \in [0, 1/3], \\ 3\alpha - 1, & \text{for any } \ x^{(0)} \in (0, 1), \quad \alpha \in (1/3, 2/3), \\ 1, & \text{for any } \ x^{(0)} \in (0, 1], \quad \alpha \in [2/3, 1]. \end{cases} \tag{16}$$

By equalities (16) for operators (14) we get the following

$$\lim_{n\to\infty} (X^{(n)}_{1,1}, X^{(n)}_{1,2}) = \begin{cases} (0, 1), & \text{for any } \ X^{(0)}_{1,1} \in [0, 1), \quad \alpha_1 \in [0, 1/3], \\ (3\alpha_1 - 1, 2 - 3\alpha_1), & \text{for any } \ X^{(0)}_{1,1} \in (0, 1), \quad \alpha_1 \in (1/3, 2/3), \\ (1, 0), & \text{for any } \ X^{(0)}_{1,1} \in (0, 1], \quad \alpha_1 \in [2/3, 1]. \end{cases} \tag{17}$$

A similar formula is true for the operator (15), where α_1 is replaced by β_1. Combining these formulas and using formula (12) one proves the following.

Proposition 4.1 *The trajectory of the non-Volterra CSO* (13) *has the following limit*

$$\lim_{n\to\infty} x^{(n)} = \begin{cases} (1,0,0,0), & \text{if } \alpha_1, \beta_1 \in [2/3,1], \\ (0,1,0,0), & \text{if } \alpha_1 \in [2/3,1],\ \beta_1 \in [0,1/3], \\ (0,0,1,0), & \text{if } \alpha_1 \in [0,1/3],\ \beta_1 \in [2/3,1], \\ (0,0,0,1), & \text{if } \alpha_1, \beta_1 \in [0,1/3], \\ (0,0,3\beta_1-1,2-3\beta_1), & \text{if } \alpha_1 \in [0,1/3],\ \beta_1 \in (1/3,2/3), \\ (3\beta_1-1,2-3\beta_1,0,0), & \text{if } \alpha_1 \in [2/3,1],\ \beta_1 \in (1/3,2/3), \\ (0,3\alpha_1-1,0,2-3\alpha_1), & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in [0,1/3], \\ (3\alpha_1-1,0,2-3\alpha_1,0), & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in [2/3,1], \\ \in U, & \text{if } \alpha_1 \in (1/3,2/3),\ \beta_1 \in (1/3,2/3), \end{cases}$$

where

$$U = \{x \in S^3 : x_1+x_2 = 3\alpha_1-1,\ x_3+x_4 = 2-3\alpha_1,\ x_1+x_3 = 3\beta_1-1,\ x_2+x_4 = 2-3\beta_1\}.$$

5 Concluding Remarks

In mathematical biology, the nonlinear operator W is called an evolution operator. The fixed points of W are interpreted as equilibrium states of the population, $\lambda \in S^{m-1}$ is called a state of the population, and $W(\lambda), W^2(\lambda), \dots$ are called states of the population in subsequent generations (offsprings). Since W is a non-linear operator, the investigation of the sequence $W^n(\lambda)$ is a difficult problem in general. So one has to consider a particular case of W, for which the problem is respectively simple. In this paper to define such an operator, a construction of CSO on a finite dimensional simplex is given. Using the construction of CSO a wide class of non-Volterra CSOs is described. Then we have showed that the non-Volterra operators can be reduced to a finitely many of Volterra CSOs. By such a reduction we described behavior of trajectories of a non-Volterra CSO defined on the three dimensional simplex.

Here we shall give a biological interpretation of Proposition 4.1. Assume that the evolution of a certain biological system consisting of 4 types of individuals is described by operator (13). Using Proposition 4.1, we can conclude the following:

1. The biological system has up to 5 equilibrium states.
2. After a certain period of time, some types will be at the vanishing point.
3. If a system is in an equilibrium state, then, depending on the state, it can have only one of $1, 2, 3, 4$ types.

Acknowledgment

U. Rozikov thanks the Université du Sud Toulon Var for financial support of his visits to the University and the Centre de Physique Théorique–Marseille for kind hospitality. He also is supported by the Grant No.0251/GF3 of Education and Science Ministry of Republic of Kazakhstan.

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