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Robust Stabilization of Fractional-Order Uncertain Systems with Multiple Delays in State

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Abstract: In this paper, a sliding mode control law is designed for stabilization of specific class of linear systems of fractional order despite of multi delays in the state system. A fractional order sliding surface is proposed, and using the variable structure control theorem, control law is introduced. A numerical simulation is given to show the effectiveness of the proposed design approach.

Keywords: *sliding mode control (SMC); Lyapunov stability analysis; fractional order system.*

Mathematics Subject Classification (2010): 93C35, 93D05, 93D15.

1 Introduction

Recently, time delays inevitably exist in systems and processes [1, 2] due to poor performance, undesirable system transient responses, and instabilities so that as a result, most systems may include a delay term. In general, the time-delay is believed to have a negative impact on the control system performance. To compensate for this impact, Smith predictor schemes work fine for slow processes [3, 4]. In the last two decades, the theory of fractional calculus has attracted researchers [5–9], because of its wide use in different areas of sciences and engineering, such as viscoelastic systems [12,13], sinusoidal oscillators [14], electromagnetic theory [15,16], and bioengineering [17]. The sliding mode control (SMC) approach is one of the most important methods and this approach can be used in many systems [18, 19] because of its robustness to parameter uncertainties and insensitivity to external disturbances. Sliding mode control (SMC) is based on the theory of variable structure systems [20]. The main feature of SMC is to cause states from initial

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conditions to a sliding surface and then the states are forced to remain on sliding surface because the system on the sliding surface has desirable properties such as stability and disturbance rejection capability [21]. Another approach is the use of fractional order controllers such as the CRONE controller [22,23], the TID controller [24], the fractional PID controller [25], and the FO adaptive SMC [26] to improve system control function.

The topic of the present work is the stability of fractional-order linear systems with disturbances and multi time-delays have been done using the sliding mode control strategy. In this paper, the sliding mode controller for a class of linear fractional order systems with parameter uncertainties and multi time delay in state and input disturbance is proposed. The paper is presented as follows. In Section 2, basic definitions in fractional calculus are given. In Section 3, problem formulation of fractional-order systems is presented. Section 4 proposes the sliding mode control method. Numerical simulation results are shown in Section 5. Finally, conclusion is made in Section 6.

2 Basic Definition and Preliminaries

There exist many definitions of fractional derivative. Two of the most commonly used definitions are the Riemann-Liouville, and the Grunwald-Letnikov definitions. The Grunwald-Letnikov fractional derivative of order q of a continuous function $f(t)$ is defined by [27]

$$D_t^q f(t) = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f(t - j\left[\frac{t-a}{N}\right]).$$

Riemann-Liouville fractional integral and derivative operators of order q are defined as

$$D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau.$$

where n is the first integer which is not less than q, i.e., $n-1 \le q < n$ and Γ is the Gamma function

$$\Gamma(q) = \int_0^\infty e^{-t} t^{q-1} dt.$$

If $0 < q < 1$, then the Riemann-Liouville fractional derivative and integral operators of order q are defined as

$$D_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-\tau)^{-q} f(\tau) d\tau,$$

$$I_t^q f(t) = I^\alpha f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau.$$

3 Stability

Lemma 3.1 *[28] The following autonomous system:*

$$D^q x(t) = Ax(t), \qquad x(0) = x_0, \tag{1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is asymptotically stable if and only if $|arg(eig(A))| > \frac{q\pi}{2}$, in this case, each component of the states decays towards origin like t^{-q}. Also, this system is stable if and only if $|arg(eig(A))| \ge \frac{q\pi}{2}$ and those critical eigenvalues that satisfy $|arg(eig(A))| = \frac{q\pi}{2}$ have geometric multiplicity one.

The stable and unstable regions for $0 < q < 1$ are shown in Figure 1.

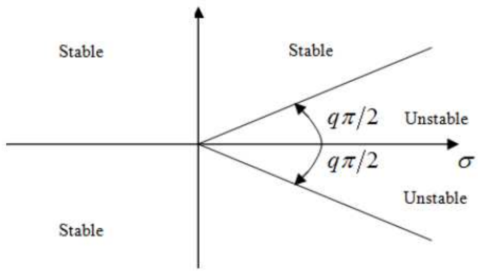


Figure 1: Stability region of LTI fractional order system with order $0 < q < 1$.

4 Problem Formulation

Now consider the linear uncertain system of fractional order with multi delays in state as follows:

$$D_t^q x(t) = \sum_{i=1}^{N} \alpha_i (A_i x(t) + A_{id1} x(t - t_{d1}) + A_{id2} x(t - td2) + \dots$$
$$+ A_{idl} x(t - t_{dl}) + B_i B(u(t) + w(t))). \tag{2}$$

where and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$ are the state vector, the controller, the exogenous input of the system, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times m}$, $A_{id} \in \mathbb{R}^{n\times n}$ are constant matrices, and q is the fractional derivative, $0 < q < 1$, and α_i are indeterminate parameters which satisfy $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.

Conditions that are necessary mode switching systems starting from any point and move on the switching surface and reach it (to switching level) are called reaching conditions. One of these conditions is as follows. This condition reach is global but does not guarantee limited arrival time:

$$\dot{V}(t) = S\dot{S}, \tag{3}$$

where S is sliding sector. Another requirement in [21] is suggested that including the shown entity,

$$\frac{1}{2}\frac{d}{dt}S^2 \leq -\eta |S|,$$

where η is a positive constant. That fulfilling the above condition causes the switching time reach less than $\frac{|S(t=0)|}{\eta}$.

5 Design of the Controller

In sliding mode control, the system state movement to a desired place, is comprised of two parts, the reaching phase and the sliding phase. The control switching level (reachability phase), should lead the system to the desired level. When all the modes of system were on the surface, sliding mode occurs (sliding phase). In sliding mode, the dynamic behavior of the system is determined by choosing the switching level. Let the sliding surface S be such that:

$$S(x,t) = I^{1-q}x(t). \tag{4}$$

Theorem 5.1 *The sliding mode control law:*

$$u(t) = \frac{-B^{-1}}{a}k\frac{S(t)}{||S(t)||}, \tag{5}$$

when

$$a = \min\{|B_1|, |B_2|, \dots, |B_N|\},$$
$$b = \max\{||A_1x(t)||, ||A_2x(t)||, \dots, ||A_Nx(t)||\},$$
$$d_{delay1} = \max\{||A_{1d1}x(t-t_{d1})||, ||A_{2d1}x(t-t_{d1})||, \dots, ||A_{Nd1}x(t-t_{d1})||\},$$
$$d_{delay2} = \max\{||A_{1d2}x(t-t_{d2})||, ||A_{2d2}x(t-t_{d2})||, \dots, ||A_{Nd2}x(t-t_{d2})||\},$$
$$\vdots$$
$$d_{delayl} = \max\{||A_{1dl}x(t-t_{dl})||, ||A_{2dl}x(t-t_{dl})||, \dots, ||A_{Ndl}x(t-t_{dl})||\},$$
$$k = d + d_{delay1}(x) + d_{delay2}(x) + \dots + d_{delayN}(x) + b||B||\gamma + \eta e^{-\lambda t}||S(t)||^{1-\delta},$$

and $\eta > 0,\ \lambda > 0,\ 0 < \delta \leq 1$.

Proof. The Lyapunov function to be defined in (2) taking the time derivative of S in (3) and substituting by (4), we obtain:

$$\begin{aligned}\dot{S}(t) = &\sum_{i=1}^{N}\alpha_i A_i x(t) + \sum_{i=1}^{N}\alpha_i A_{id1}x(t-t_{d1}) + \dots + \sum_{i=1}^{N}\alpha_i A_{idN}x(t-t_{dN}) \\ &+ \sum_{i=1}^{N}\alpha_i B_i B u(t) + \sum_{i=1}^{N}\alpha_i B_i B w(t).\end{aligned} \tag{6}$$

Substituting (4) in (2), we have

$$\begin{aligned}\dot{V}(t) = S(t)\dot{S}(t) = S^T(t)(&\sum_{i=1}^{N}\alpha_i A_i x(t) + \sum_{i=1}^{N}\alpha_i A_{id1}x(t-t_{d1}) + \dots \\ &+ \sum_{i=1}^{N}\alpha_i A_{idN}x(t-t_{dN}) + \sum_{i=1}^{N}\alpha_i B_i B u(t) + \sum_{i=1}^{N}\alpha_i B_i B w(t)).\end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned}\dot{V}(t) = S^T(t)\dot{S}(t) = S^T(t)(&\sum_{i=1}^{N}\alpha_i A_i x(t) + \sum_{i=1}^{N}\alpha_i A_{id1}x(t-t_{d1}) + \dots \\ &+ \sum_{i=1}^{N}\alpha_i A_{idN}x(t-t_{dN}) - k\frac{S(t)}{||S(t)||}\frac{\sum_{i=1}^{N}\alpha_i B_i}{a} + \sum_{i=1}^{N}\alpha_i B_i B w(t)),\end{aligned}$$

hence

$$\dot{V}(t) < \eta e^{-\lambda t}||S(t)||^{2-\delta}.$$

This indicates that the Lyapunov function is positive definite and its derivative is negative definite. By Lyapuonv stability theory and Lemma 1, the closed-loop system (1) with the control law (u) in (4) is asymptotically stable.

We consider, the system states will reach the sliding mode $S = 0$ for a finite time T. We have

$$S^T\dot{S} = \frac{1}{2}\frac{d(S^TS)}{dt} = \frac{1}{2}\frac{dS^2}{dt} = S\frac{dS}{dt}.$$

It follows that

$$\frac{dt}{d||S(T)||} = \frac{1}{\eta e^{-\lambda t}||S(t)||^{1-\delta}},$$

so

$$\frac{d||S(T)||}{dt} = \eta e^{-\lambda t}||S(t)||^{1-\delta}, \tag{8}$$

we can integrate (8) from 0 to T, we have

$$T = -\frac{1}{\lambda}ln(1 - \frac{\lambda}{\delta\eta}||S(0)||^{\delta}).$$

Therefore, $t \geq T$, the system will converge to switching manifold at any initial state. T is positive, it is enough that the selected constants

$$0 \leq \frac{\lambda}{\delta\eta}||S(0)||^{\delta} < 1.$$

□

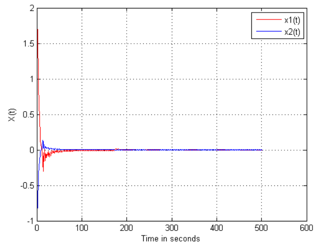
6 Simulation Results of the Proposed Sliding Mode Controller

The sliding mode controller given by (4) is applied to the fractional order systems given by (1). Now consider this system, for example

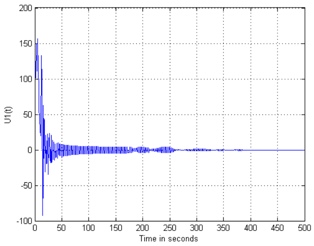
$$D_t^q x(t) = \sum_{i=1}^{3}\alpha_i(A_ix(t)+A_{id1}x(t-t_{d1})+A_{id2}x(t-td2)+A_{id3}x(t-t_{d3})+B_iB(u(t)+w(t))),$$

$$\begin{aligned}D_t^q x(t) &= \alpha_1(A_1x(t)+A_{1d1}x(t-t_{d1})+A_{1d2}x(t-td2)+A_{1d3}x(t-t_{d3})+B_1B(u(t)+w(t)))\\ &= \alpha_2(A_2x(t)+A_{2d1}x(t-t_{d1})+A_{2d2}x(t-td2)+A_{2d3}x(t-t_{d3})+B_2B(u(t)+w(t)))\\ &= \alpha_2(A_2x(t)+A_{2d1}x(t-t_{d1})+A_{2d2}x(t-td2)+A_{2d3}x(t-t_{d3})+B_2B(u(t)+w(t))).\end{aligned}$$

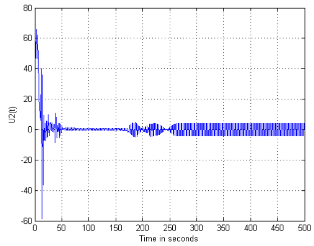
The initial conditions of system (1) are taken to be $[x_1(0)\ x_2(0)]^T = [2\ -1]^T$. Then, we choose $A_1 = \begin{bmatrix} 13 & -1 \\ 1 & 10 \end{bmatrix}$, $A_2 = \begin{bmatrix} 6 & -8 \\ 12 & 9 \end{bmatrix}$, $A_3 = \begin{bmatrix} 5 & -6 \\ 1 & 2 \end{bmatrix}$, $A_{1d1} = \begin{bmatrix} 1 & 0 \\ -5 & 3 \end{bmatrix}$, $A_{1d2} = \begin{bmatrix} 0 & 1 \\ 2 & 14 \end{bmatrix}$, $A_{1d3} = \begin{bmatrix} 0 & 2 \\ 7 & 4 \end{bmatrix}$, $A_{2d1} = \begin{bmatrix} 0 & 8 \\ 5 & 9 \end{bmatrix}$, $A_{2d2} = \begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix}$, $A_{2d3} = \begin{bmatrix} 11 & 1 \\ 6 & -1 \end{bmatrix}$, $A_{3d1} = \begin{bmatrix} 0 & 10 \\ 10 & 10 \end{bmatrix}$, $A_{3d2} = \begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$, $A_{3d3} = \begin{bmatrix} 0 & 5 \\ 1 & 4 \end{bmatrix}$,



(a) State $X(t)$.

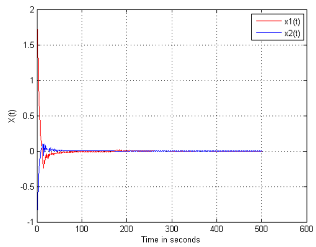


(b) Control input $u_1(t)$.

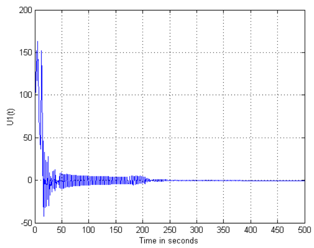


(c) Control input $u_2(t)$.

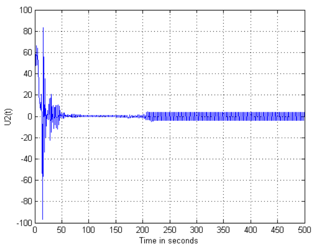
Figure 2: Sliding mode control $\alpha_1 = 0.1,\ \alpha_2 = 0.5,\ \alpha_3 = 0.4$ (sampling interval, $h = 0.005$ s).



(a) State $X(t)$.



(b) Control input $u_1(t)$.



(c) Control input $u_2(t)$.

Figure 3: Sliding mode control $\alpha_1 = 0.5,\ \alpha_2 = 0.5,\ \alpha_3 = 0$ (sampling interval, $h = 0.005$ s).

$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $B_1 = 0.4$, $B_2 = 0.6$, $B_3 = 0.2$, $q = 0.5$, $h = 0.005$, and $t_{d1} = 2$, $t_{d2} = 4$, $t_{d3} = 11$, and the disturbance is of the form of $w(t) = \sin(t)$. The parameters of the controller are chosen such that $\eta = 3$, $\delta = 0.4$, $\gamma = 1$, $\lambda = 4$. The performance of the system is simulated. We plot this system for two different categories of parameters α_1, α_2, α_3. The plots of the states of the system are shown in Figures 2(a) and 3(a) for the different parameters α_1, α_2, α_3. Figures 2(b) and 3(b) give the control input $u_1(t)$, and Figures 2(c) and 3(c) give the control input $u_2(t)$. Therefore, it can be concluded that the simulation results indicate that the proposed sliding mode controller works well.

7 Conclusions

In this paper, the sliding mode controller for stabilization of fractional order systems with uncertainties and multiple delay in state and disturbance input is investigated. A switching surface of integral type is proposed such that stability of the closed-loop system in the sliding mode can be guaranteed. An illustrative example shows the effectiveness of the proposed new scheme.

References

[1] Jos Paulo Carvalho dos Santos, M. Mallika Arjunan and Claudio Cuevas. Existence results for fractional neutral integro-differential equations with state-dependent delay. *Computers and Mathematics with Applications* **62** (2011) 1275–1283.

[2] Jiang Wei. The constant variation formulae for singular fractional differential systems with delay. *Computers and Mathematics with Applications* **59** (2010) 1184–1190.

[3] Krol, K. Asymptotic properties of fractional delay differential equations. *Applied Mathematics and Computation* **218** (2011) 1515–1532.

[4] Smith, O. J. M. A controller to overcome dead time. *Instrumentation, Systems and Automation (ISA) Society Journal* **6** (1959) 28–33.

[5] Moornani, K. A. and Haeri, M. On robust stability of LTI fractional-order delay systems of retarded and neutral type. *Automatica* **46** (2010) 362–368.

[6] Kovrnek, T. Podlubny, I. and Petras, I. Modeling of the national economies in state-space: A fractional calculus approach. *Economic Modelling* **29** (2012) 1322–1327.

[7] Kilbas, A., Srivastava, H. and Trujillo, J. *Theory and Appliccations of Fractional Differential Equations.* Elsevier, Amsterdam, 2006.

[8] Anastassiou, G.A. On right fractional calculus. *Chaos, Solitons and Fractals*, 2009.

[9] Xiuyun Zhang. Some results of linear fractional order time-delay system. *Applied Mathematics and Computation* **197** (2008) 407–411.

[10] Erjaee, G.H. and Taghvafard, H. Stability analysis of phase synchronization in coupled chaotic systems presented by fractional differential equations. *Nonlinear Dynamics and Systems Theory* **11** (2) (2011) 147–154.

[11] Yu, J.M., Luo, Y.W., Zhou, S.B. and Lin, X.R. Existence and uniqueness for nonlinear multi-variables fractional differential equations. *Nonlinear Dynamics and Systems Theory* **11** (2) (2011) 213–221.

[12] Liu, H., Oliphant, T. E. and Taylor, L. General fractional derivative viscoelastic models applied to vibration elastography. In: *Proc. IEEE Ultrason. Symp.*, 2003, 933–936.

[13] Adolfsson, K., Enelund, M. and Olsson, P. On the fractional order model of viscoelasticity. *Mechanic of Time-Dependent Mater* **9** (2005) 15–34.

[14] Chai Hok Eab and Lim, S.C. Path integral representation of fractional harmonic oscillator. *Physica A* **371** (2006) 303–316.

[15] Tenreiro, J.A., Machado, I. Jesus, A. Galhano. A fractional calculus perspective in electromagnetics. In: *2005 ASME International Design Engineering Technical Conferences and Computers and Information In Engineering Conference*. Long Beach, California, 24–28 September, 2005.

[16] Baleanu, D. and Golmankhaneh, A.K. On electromagnetic field in fractional space. *Nonlinear Analysis: Real World Applications* **11** (1) (2010) 288–292.

[17] Magin, R.L. *Fractional Calculus in Bioengineering*. Begell House, Connecticut, 2006.

[18] Fraguela, L., Fridman, L. and Alexandrov, V.V. Output integral sliding mode control to stabilize position of a Stewart platform. *Journal of the Franklin Institute* **349** (2012) 1526–1542.

[19] Gong-You Tang, Rui Dong and Hong-Wei Gao. Optimal sliding mode control for nonlinear systems with time-delay. *Nonlinear Analysis: Hybrid Systems* **2** (2008) 891–899.

[20] Hailong Xing and C.C. Gao. Sliding mode variable structure control for parameter uncertain stochastic systems with time-varying delay. *J. Math. Anal. Appl.* **355** (2009) 689–699.

[21] Hnng, J.Y., Gao, W. and Hung, J.C. Variable Structure Control: A Survey. *IEEE Transactions on industrial electronics* **40** (1) (1993) 2–22.

[22] Cerver, J. and Banos, A. Automatic loop shaping in QFT by using CRONE structures. In: *Proceedings of the 2nd IFAC Workshop on Fractional Differentiation and its Applications*, $FDA'06, 2006$.

[23] Oustaloup, A., Sabatier, J. and Lanusse, P. From fractal robustness to CRONE control. *Fract. Calc. Appl. Anal.* **2** (1) (1999) 1–30.

[24] Lurie, B. J. *Three-Parameter Tunable Tiltintegral-Derivative (TID) Controller*. US Patent US5371670, 1994.

[25] Zbay, H., Bonnet, C. and Fioravanti, A. R. PID controller design for fractional-order systems with time delays. *Systems and Control Letters* **61** (2012) 18–23.

[26] Chun Yin, Dadras S, Shou-ming Zhong and YangQuan Chen. Control of a novel class of fractional-order chaotic systems via adaptive sliding mode control approach. *Applied Mathematical Modelling* **37** (4) (2013) 2469–2483.

[27] Podlubny, I. *Fractional Differential Equations*. Academic Press, New York, 1999.

[28] Matignon, D. Stability results for fractional differential equations with applications to control processing. *Computational Engineering in System Application* (1996) 963–968.