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Existence of a Positive Solution for a Right Focal Dynamic Boundary Value Problem

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Abstract: In this work, an application is made of an extension of the Leggett-Williams fixed point theorem to a second-order right focal dynamic boundary value problem which requires neither of the functional boundaries to be invariant. In conclusion, two nontrivial examples are provided.

Keywords: *fixed point theorem; dynamic equation; time scale; functional.*

Mathematics Subject Classification (2010): 34N05.

1 Introduction

For years, fixed point theory has found itself as a center of study for boundary value problems. Many results have provided criteria for the existence of positive solutions or multiple positive solutions using fixed points of operators. Some of these results can be seen in the works of Guo [10], Krosnosel'skii [12], Leggett and Williams [13], and Avery et al. [1, 3, 6].

Applications of the aforementioned fixed point theorems have been seen in works dealing with ordinary differential equations [2,5,9] and finite difference equations [4,7,11], and most relevant to this paper, the theorems have been utilized for results that involve dynamic equations on time scales [8, 14, 15].

In this paper, we show an application of the recent extension of the Leggett-Williams fixed point theorem by Avery et al. [1] to a right-focal dynamic boundary value problem on a time scale.

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Let $\mathbb{T}$ be a time scale with $0, \sigma^2(1) \in \mathbb{T}$. We consider the right focal dynamic boundary value problem

$$x^{\Delta\Delta} + f(x(\sigma(t))) = 0, \quad t \in (0,1) \cap \mathbb{T}, \tag{1}$$

on the time scale $\mathbb{T}$ with boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0, \tag{2}$$

where $f : [0,\infty) \to [0,\infty)$ is continuous.

2 Definitions

In this section, we present definitions and conventions that will be used throughout the rest of the paper.

Definition 2.1 We define the closed interval $[0,1]$ to mean

$$[0,1] = \{t \in \mathbb{T} : 0 \leq t \leq 1\}.$$

All other intervals are defined similarly, except for those specifying the domain or codomain of a function.

Definition 2.2 Let E be a real Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:

(i) $x \in \mathcal{P}$, $\lambda \geq 0$ implies $\lambda x \in \mathcal{P}$;

(ii) $x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.

Definition 2.3 A map α is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space E if $\alpha : \mathcal{P} \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone $\mathcal{P}$ of a real Banach space E if $\beta : \mathcal{P} \to [0,\infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in \mathcal{P}$ and $t \in [0,1]$.

3 The Fixed Point Theorem

We first define sets that are integral to the fixed point theorem. Let α and ψ be nonnegative continuous concave functionals on $\mathcal{P}$ and let δ and β be nonnegative continuous convex functionals on $\mathcal{P}$. We define the sets

$$A = A(\alpha, \beta, a, d) = \{x \in \mathcal{P} \ : \ a \leq \alpha(x) \text{ and } \beta(x) \leq d\},$$

$$B = B(\delta, b) = \{x \in A \ : \ \delta(x) \leq b\},$$

and

$$C = C(\psi, c) = \{x \in A \ : \ c \leq \psi(x)\}.$$

The following fixed point theorem is attributed to Anderson, Avery, and Henderson [1] and is an extension of the original Leggett-Williams fixed point theorem [13].

Theorem 3.1 *Suppose $\mathcal{P}$ is a cone in a real Banach space E, α and ψ are nonnegative continuous concave functionals on $\mathcal{P}$, δ and β are nonnegative continuous convex functionals on $\mathcal{P}$, and for nonnegative real numbers a, b, c, and d, the sets A, B, and C are defined as above. Furthermore, suppose A is a bounded subset of $\mathcal{P}$, $T : A \to \mathcal{P}$ is a completely continuous operator, and that the following conditions hold:*

(A1) $\{x \in A \,:\, c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$, $\{x \in \mathcal{P} \,:\, \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$;

(A2) $\alpha(Tx) \geq a$ *for all* $x \in B$;

(A3) $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(Tx) > b$;

(A4) $\beta(Tx) \leq d$ *for all* $x \in C$; *and*

(A5) $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < C$.

Then T has a fixed point $x^* \in A$.

4 Existence of a Positive Solution of (1), (2)

In this section, we show the existence of at least one positive solution to (1), (2). To that end, we now consider the dynamic equation

$$x^{\Delta\Delta} + f(x(\sigma(t))) = 0,\ t \in (0,1),$$

on a time scale $\mathbb{T}$ with boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0,$$

where $f : [0,\infty) \to [0,\infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^{\sigma(1)} G(t,s) f(x(\sigma(s)))\Delta s,\ t \in [0,\sigma^2(1)],$$

where $G(t,s)$ defined on $[0,\sigma^2(1)] \times [0,\sigma(1)]$ by

$$G(t,s) = \begin{cases} t, & 0 \leq t \leq s \leq \sigma(1), \\ \sigma(s), & \sigma^2(1) \geq t \geq \sigma(s) \geq 0, \end{cases}$$

is the Green's function for the operator L defined by

$$(Lx)(t) := -x^{\Delta\Delta},$$

with right focal boundary conditions

$$x(0) = x^{\Delta}(\sigma(1)) = 0,$$

then it is well known that x is a solution of the boundary value problem (1), (2).

Throughout the remainder of the paper, we will often make use of the following property of the preceeding Green's function. For any $y, w \in [0,\sigma^2(1)]$ with $y \leq w$,

$$yG(w,s) \leq wG(y,s),$$

which implies

$$y\int_0^{\sigma(1)} G(w,s)\Delta s \le w\int_0^{\sigma(1)} G(y,s)\Delta s. \tag{3}$$

Let $E = C_{rd}[0,\sigma^2(1)]$ be the Banach Space composed of right-dense continuous functions from $[0,\sigma^2(1)]$ into $\mathbb{R}$ with the norm

$$||x|| = \max_{t\in[0,\sigma^2(1)]} |x(t)|.$$

Define the cone $\mathcal{P} \subset E$ by

$$\mathcal{P} = \{x \in E : x \text{ is nondecreasing, nonegative, and concave.}\}$$

For fixed $\tau,\mu,\nu \in [0,\sigma^2(1)]$, define the nonnegative concave functionals α and ψ to be

$$\alpha(x) = \min_{t\in[\tau,\sigma^2(1)]} x(t) = x(\tau),$$

$$\psi(x) = \min_{t\in[\mu,\sigma^2(1)]} x(t) = x(\mu),$$

and the nonnegative, convex functionals δ and β to be

$$\delta(x) = \max_{t\in[0,\nu]} x(t) = x(\nu),$$

$$\beta(x) = \max_{t\in[0,\sigma^2(1)]} x(t) = x(\sigma^2(1)).$$

Theorem 4.1 *Let $\tau,\mu,\nu \in (0,\sigma^2(1)]$ with $0 < \tau \le \mu < \nu \le \sigma^2(1)$. Let d and m be positive reals with $0 < m \le \frac{d\mu}{\sigma^2(1)}$ and suppose $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies the following:*

(i) $f(w) \ge \frac{d}{(\nu-\tau)\sigma^2(1)}$ for $\frac{\tau d}{\sigma^2(1)} \le w \le \frac{\nu d}{\sigma^2(1)}$;

(ii) $f(w)$ is decreasing for $0 \le w \le m$ and $f(m) \ge f(w)$ for $m \le w \le d$; and

(iii) $\int_0^\mu \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right)\Delta s \le d - f(m)\sigma^2(1)(\sigma(1)-\mu)$.

Then (1),(2) has at least one positive solution $x^* \in A(\alpha,\beta,\frac{\tau d}{\sigma^2(1)},d)$.

Proof. Let $a = \dfrac{\tau d}{\sigma^2(1)}$, $b = \dfrac{\nu d}{\sigma^2(1)}$, and $c = \dfrac{\mu d}{\sigma^2(1)}$. Define $Tx(t) = \int_0^{\sigma(1)} G(t,s) f(x(\sigma(s)))\Delta s$. Now by definition, $A \subset \mathcal{P}$, and for all $x \in A$, $d \ge \beta(x) = \max_{t\in[0,\sigma(1)]} x(t) = x(\sigma^2(1))$, and so A is bounded.

Now, if $x \in A \subset \mathcal{P}$, then $Tx(t) = \int_0^{\sigma^2(1)} G(t,s)f(x(\sigma(s)))\Delta s$, and so $Tx^{\Delta\Delta}(t) = -f(x(\sigma(s))) \le 0$ for $t \in [0,1]$, and so Tx is concave, and $Tx^{\Delta}(t)$ is nonincreasing on $[0,\sigma(1)]$. Furthermore, $Tx^{\Delta}(\sigma(1)) = 0$, and so $Tx^{\Delta}(t) \ge 0$ on $[0,\sigma(1)]$. So Tx is nondecreasing on $[0,\sigma^2(1)]$. Therefore, $T : A \to \mathcal{P}$.

Now we prove our first enumerated condition (A1). Let $K \in \mathbb{R}$ with $\dfrac{\mu d}{\sigma^2(1)\int_0^{\sigma(1)} G(\mu,s)\Delta s} < K < \dfrac{\nu d}{\sigma^2(1)\int_0^{\sigma(1)} G(\nu,s)\Delta s}$, which is well-defined by (3). Define $x_K(t) = K\int_0^{\sigma(1)} G(t,s)\Delta s$. So $x_K \in \mathcal{P}$,

$$\begin{aligned}
\alpha(x_K) &= K\int_0^{\sigma(1)} G(\tau,s)\Delta s \\
&> \frac{\mu d\int_0^{\sigma(1)} G(\tau,s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\mu,s)\Delta s} \\
&\geq \frac{\tau d\int_0^{\sigma(1)} G(\mu,s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\mu,s)\Delta s} \\
&= \frac{\tau d}{\sigma^2(1)} = a,
\end{aligned}$$

and

$$\begin{aligned}
\beta(x_K) &= K\int_0^{\sigma(1)} G(\sigma^2(1),s)\Delta s \\
&< \frac{\nu d\int_0^{\sigma(1)} G(\sigma^2(1),s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\nu,s)\Delta s} \\
&\leq \frac{\sigma^2(1) d\int_0^{\sigma(1)} G(\nu,s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\nu,s)\Delta s} = d.
\end{aligned}$$

So $x_K \in A$. Now

$$\begin{aligned}
\psi(x_K) &= K\int_0^{\sigma(1)} G(\mu,s)\Delta s \\
&> \frac{\mu d\int_0^{\sigma(1)} G(\mu,s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\mu,s)\Delta s} \\
&= \frac{\mu d}{\sigma^2(1)} = c,
\end{aligned}$$

and

$$\begin{aligned}
\delta(x_K) &= K\int_0^{\sigma(1)} G(\nu,s)\Delta s \\
&< \frac{\nu d\int_0^{\sigma(1)} G(\nu,s)\Delta s}{\sigma^2(1)\int_0^{\sigma(1)} G(\mu,s)\Delta s} \\
&= \frac{\nu d}{\sigma^2(1)} = b.
\end{aligned}$$

So $\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$.

Next, let $x \in \mathcal{P}$ with $\beta(x) > d$. Then since for all $y \leq w$, $wx(y) \geq yx(w)$, $\sigma^2(1)x(\tau) \geq \tau x(\sigma^2(1))$, and so

$$\alpha(x) = x(\tau) \geq \frac{\tau}{\sigma^2(1)} x(\sigma^2(1)) = \frac{\tau\beta(x)}{\sigma^2(1)} > \frac{\tau d}{\sigma^2(1)} = a.$$

Therefore $\{x \in \mathcal{P} : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$.

Next, we prove (A2). Chose $x \in B$. So $\delta(x) \leq b$. Now by (i),

$$\begin{aligned}
\alpha(Tx) &= \int_0^{\sigma(1)} G(\tau, s) f(x(\sigma(s)))\Delta s \\
&\geq \int_\tau^\nu G(\tau, s) f(x(\sigma(s)))\Delta s \\
&= \int_\tau^\nu \tau f(x(\sigma(s)))\Delta s \\
&\geq \int_\tau^\nu \tau \left(\frac{d}{(\nu - \tau)\sigma^2(1)} \right) \Delta s \\
&= \frac{d\tau}{\sigma^2(1)} = a.
\end{aligned}$$

Next, we prove (A3). Let $x \in A$ with $\delta(Tx) > b$. Then, by (3),

$$\begin{aligned}
\alpha(Tx) &= \int_0^{\sigma(1)} G(\tau, s) f(x(\sigma(s)))\Delta s \\
&\geq \frac{\tau}{\nu} \int_0^{\sigma(1)} G(\nu, s) f(x(\sigma(s)))\Delta s \\
&= \frac{\tau}{\nu} \delta(Tx) \\
&> \frac{\tau}{\nu} \cdot \frac{\nu d}{\sigma^2(1)} = \frac{\tau d}{\sigma^2(1)} = a.
\end{aligned}$$

Now we prove (A4). Now, since x is concave and nondecreasing for all $t \in [0, \mu]$,

$$x(\sigma(t)) \geq \frac{x(\sigma(\mu))\sigma(t)}{\sigma(\mu)} \geq \frac{c\sigma(t)}{\sigma(\mu)} \geq \frac{m\sigma(t)}{\sigma(\mu)}.$$

So by conditions (ii) and (iii), we have

$$\begin{aligned}
\beta(Tx) &= \int_0^{\sigma(1)} G(\sigma^2(1), s) f(x(\sigma(s)))\Delta s \\
&= \int_0^{\sigma(1)} \sigma(s) f(x(\sigma(s)))\Delta s \\
&= \int_0^{\mu} \sigma(s) f(x(\sigma(s)))\Delta s + \int_\mu^{\sigma(1)} \sigma(s) f(x(\sigma(s)))\Delta s \\
&\leq \int_0^{\mu} \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)} \right) \Delta s + \int_\mu^{\sigma(1)} \sigma^2(1) f(m)\Delta s \\
&\leq d - f(m)\sigma^2(1)(\sigma(1) - \mu) + f(m)\sigma^2(1)(\sigma(1) - \mu) \\
&= d.
\end{aligned}$$

Finally, we prove our last condition, (A5). Let $x \in A$ with $\psi(Tx) < c$. So, we have

$$\beta(Tx) = \int_0^{\sigma(1)} G(\sigma^2(1), s) f(x(\sigma(s)))\Delta s \leq \frac{\sigma^2(1)}{\mu} \int_0^{\sigma(1)} G(\mu, s) f(x(\sigma(s)))\Delta s$$
$$= \frac{\sigma^2(1)}{\mu}\psi(Tx) \leq \frac{\sigma^2(1)c}{\mu} = d.$$

Thus T has a fixed point $x^* \in A$, and therefore x^* is a positive solution of (1), (2).

5 Two Nontrivial Examples

Example 5.1 Let $\mathbb{T} = [0, \frac{1}{2}] \cup [1, \frac{3}{2}]$ and consider the boundary value problem

$$x^{\Delta\Delta} + \frac{1}{x(\sigma(t)) + 1} = 0,\ t \in (0,1) \cap \mathbb{T}, \quad x(0) = x^{\Delta}(\sigma(1)) = 0.$$

Choose $\tau = \frac{1}{30}$, $\mu = \frac{1}{2}$, $\nu = 1$, $m = \frac{1}{4}$, and $d = \frac{3}{5}$. Note that $0 < \tau \leq \mu < \nu \leq \sigma^2(1) = 1$ and $0 < m < \frac{d\mu}{\sigma^2(1)} = \frac{\frac{3}{5}\cdot\frac{1}{2}}{1} = \frac{3}{10}$. Also, $f(w) = \frac{1}{w+1}$ is continuous from the nonnegative reals to the nonnegative reals. Lastly,

(i) for $\frac{1}{50} \leq w \leq \frac{3}{5}$, $f(w) \geq f(\frac{3}{5}) = \frac{5}{8} > \frac{18}{29} = \frac{d}{(\nu-\tau)\sigma^2(1)}$,

(ii) since $f'(w) < 0$ for $w \geq 0$, $f(w)$ is decreasing for $0 \leq w \leq \frac{1}{4}$ and for $\frac{1}{4} \leq w \leq \frac{3}{5}$, $f(m) = f(\frac{1}{4}) \geq f(w)$, and

(iii) $\int_0^{\mu} \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right)\Delta s = \int_0^{\frac{1}{2}} s f(\frac{1}{4}s)\Delta s = \int_0^{\frac{1}{2}} s \dfrac{1}{\frac{1}{4}s + 1}\Delta s \approx 0.115471 < 0.2 = \frac{3}{5} - \frac{2}{5} = \frac{3}{5} - f(\frac{1}{4})(1)\frac{1}{2} = d - f(m)\sigma^2(1)(\sigma(1) - \mu)$.

Therefore, the boundary value problem has at least one positive solution, x^*, in $A(\alpha, \beta, \frac{1}{50}, \frac{3}{5})$. That is, $x^*(\frac{1}{30}) \geq \frac{1}{50}$ and $x^*(1) \leq \frac{3}{5}$.

Example 5.2 Let $\mathbb{T} = 2^{\mathbb{Z}} = \{2^n \ : \ n \in \mathbb{Z}\} \cup \{0\}$. Consider the boundary value problem

$$x^{\Delta\Delta} + \frac{\cos^2(0.2x(\sigma(t)))}{\sqrt{(x(\sigma(t)))^{1/10} + 1}} = 0,\ t \in (0,1) \cap \mathbb{T}, \quad x(0) = x^{\Delta}(\sigma(1)) = 0.$$

Choose $\tau = \frac{1}{1024}$, $\mu = 2$, $\nu = 4$, $m = \frac{1}{5}$, and $d = \frac{5}{2}$. Note that $0 < \tau \leq \mu < \nu \leq \sigma^2(1) = 4$ and $0 < m < \frac{d\mu}{\sigma^2(1)} = \frac{\frac{5}{2}\cdot 2}{4} = \frac{5}{4}$. Also, $f(w) = \frac{\cos^2(0.2w)}{\sqrt{w^{1/10}+1}}$ is continuous from the nonnegative reals to the nonnegative reals. Now,

(i) for $\frac{5}{8192} \leq w \leq \frac{5}{2}$, $f(w) \geq f(\frac{5}{2}) \approx 0.531967 > \frac{128}{819} = \frac{d}{(\nu-\tau)\sigma^2(1)}$,

(ii) since $f'(w) < 0$ for $0 \leq w \leq \frac{5}{2}$, $f(w)$ is decreasing for $0 \leq w \leq \frac{1}{5}$ and for $\frac{1}{5} \leq w \leq \frac{5}{2}$, $f(m) = f(\frac{1}{5}) \geq f(w)$, and

(iii) $\int_0^{\mu} \sigma(s) f\left(\frac{m\sigma(s)}{\sigma(\mu)}\right)\Delta s = \sum\limits_{k=0}^{\infty} \dfrac{1}{2^{k-1}} f\left(\dfrac{1}{20\cdot 2^{k-1}}\right) \cdot \dfrac{1}{2^k} \approx 2.00009 < \dfrac{5}{2} = \dfrac{5}{2} - f\left(\dfrac{1}{5}\right) \cdot 4(2-2) = d - f(m)\sigma^2(1)(\sigma(1) - \mu)$.

Therefore, the boundary value problem has at least one positive solution, x^*, in $A(\alpha, \beta, \frac{5}{8192}, \frac{5}{2})$. That is, $x^*(\frac{1}{1024}) \geq \frac{5}{8192}$ and $x^*(4) \leq \frac{5}{2}$.

6 Conclusion

Here it was shown how a recent Avery et al. fixed point theorem [1] that was developed as an extension of the original Leggett-Williams fixed point theorem [13] can be applied to show under certain conditions, the existence of a second order right focal dynamic boundary value problem. Two nontrivial examples were then provided to show that these conditions could be applied to specific boundary value problems.

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