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On the New Concepts of Solutions and Existence Results for Impulsive Integro-Differential Equations with a Deviating Argument

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Abstract: In this paper, we prove the existence of $\mathcal{PC}$-mild solutions for impulsive integro-differential equations with a deviating argument in a Banach space H. The results are obtained by using the analytic semigroup theory and the fixed point methods.

Keywords: *impulsive integro-differential equation; deviating argument; analytic semigroup; fixed point theorems.*

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1 Introduction

In the theory of differential equations with deviating arguments, we study the differential equations involving variables (arguments) as well as unknown functions and its derivative, generally speaking, under different values of the variables (arguments). It is a very important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences, and many other areas of science and technology. The book [3] by El'sgol'ts and Norkin provides a comprehensive study of differential equations with deviated arguments. The existence, uniqueness, almost automorphic solutions and asymptotic behaviors of differential equations with deviating arguments have been studied by many authors like Driver [4], Obreg [5], Grimm [6], Gal [7], Haloi [8, 10, 11] (see [12–16] and references cited therein).

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Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process, such phenomena may also be called impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations since such equations are mathematical approaches for simulation of process and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, economics and so on. Chang et al. [27] have studied the existence of $\mathcal{PC}$-mild solutions for first order impulsive neutral integro-differential inclusions with nonlocal initial conditions. Ding et al. [17] discussed a class of second-order impulsive differential equations with integral boundary values. By using Krasnoselskii's fixed point theorem, the existence of solutions for the system is obtained. For more details, one can see ([18, 20, 21, 24–26, 28]) and references cited therein.

On the other hand, due to theoretical and practical difficulties, the study of impulsive differential equations with deviating arguments has been developed rather slowly. Recently, the study of impulsive differential equations with deviating arguments has been found in some papers. For example, in [32], Jankowski discussed the existence of solutions for second order impulsive differential equations with deviating arguments. Guobing et al. [29] established the existence solution of periodic boundary value problems for a class of impulsive neutral differential equations with multi-deviation arguments (see also [30–35] and the references therein).

The existence and uniqueness of abstract integro-differential equations have been discussed by many authors (see [9, 10, 19, 22, 23] and references cited therein). Bahuguna [2] proved the existence, uniqueness, regularity and continuation of solutions to the following integro-differential equations in an arbitrary Banach space H:

$$\left.\begin{aligned} \frac{du(t)}{dt} + Au(t) &= f(t, u(t)) + K(u)(t), \quad t > t_0, \\ u(t_0) &= u_0, \end{aligned}\right\} \tag{1}$$

where

$$K(u)(t) = \int_{t_0}^{t} a(t-s) g(s, u(s)) ds.$$

Under the assumptions that $-A$ generates an analytic semigroup $S(t),\ \ t \geq 0$ on H, the function a is real-valued and locally integrable on $[0, \infty)$, the nonlinear maps f and g are defined on $[0, \infty) \times H$ into H.

Gal [7] proved the global existence and uniqueness to the following differential equation with deviated argument in a Banach space $(X, \|.\|)$:

$$\left.\begin{aligned} &\frac{du}{dt} = Au(t) + f(t, u(t), u([h(u(t), t)])), \quad t > 0, \\ &u(0) = u_0, \end{aligned}\right\} \tag{2}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. He proved the results under the following assumptions on f and h:

1. $f : [0, \infty) \times X_\alpha \times X_{\alpha-1} \to X$ satisfies

   $$\|f(t, x, x') - f(s, y, y')\| \leq L_f\{|t-s|^{\theta_1} + \|x-y\|_\alpha + \|x'-y'\|_{\alpha-1}\} \tag{3}$$

   for all $x, y \in X_\alpha,\ \ x', y' \in X_{\alpha-1},\ s, t \in [0, \infty)$, for some constants $L_f > 0$ and $0 < \theta_1 \leq 1$.

2. $h : X_\alpha \times [0,\infty) \to [0,\infty)$ satisfies

$$|h(x,t) - h(y,s)| \leq L_h\{\|x-y\|_\alpha + |t-s|^{\theta_2}\} \tag{4}$$

for all $x, y \in X_\alpha,\quad s,t \in [0,\infty)$, for some constants $L_h > 0$ and $0 < \theta_2 \leq 1$.

Here $\|x\|_\alpha = \|(A)^\alpha x\|$, denotes the norm on X_α, the domain of A^α, for $0 < \alpha \leq 1$.

In this paper, we extend the Cauchy problem (1) for integro-differential equations to the Cauchy problems for the impulsive integro-differential equations with a deviated argument in a Banach space $(H, \|.\|)$:

$$\left.\begin{aligned} \frac{d}{dt}u(t) + Au(t) &= f(t, u(t), u[w(t,u(t))]) + \int_0^t a(t,\tau)g(\tau,u(\tau))d\tau, \\ & t \in I = [0, T_0], \quad t \neq t_k, \\ u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \cdots, m, \\ u(0) &= u_0, \end{aligned}\right\} \tag{5}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $S(t),\ t \geq 0$ on H. Functions $f,\ a,\ g$ and w are suitably defined and satisfying certain conditions to be stated later. $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T_0,\quad I_k \in C(H,H)(k = 1, 2, ..., m)$, are bounded functions. $I_k(u(t_k)) = u(t_k^+) - u(t_k^-),\quad u(t_k^-)$ and $u(t_k^+)$ represent the left and right limits of $u(t)$ at $t = t_k$, respectively.

The paper is organized as follows. In "Preliminaries and Assumptions" we provide some basic definitions, notations, lemmas and proposition which are used throughout the paper. In "Local existence of mild solution" we will prove some existence and uniqueness results concerning the $\mathcal{PC}$-mild solutions. At last (i.e., in "Application"), we give an example to demonstrate the application of the main results.

2 Preliminaries and Assumptions

In this section, we will introduce some basic definitions, notations, lemmas and proposition which are used throughout this paper.

It is assume that $-A$ generates an analytic semigroup of bounded operators, denoted by $\{S(t)\}_{t\geq 0}$. It is known that there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq \tilde{M}e^{\omega t}, \quad t \geq 0.$$

If necessary, we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ for $t \geq 0$, and $0 \in \rho(-A)$, i.e., $-A$ is invertible. In this case, it is possible to define the fractional power A^α for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$. Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|$$

defines a norm on $D(A^\alpha)$. Henceforth, we denote the space $D(A^\alpha)$ by H_α endowed with the norm $\|\cdot\|_\alpha$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_\alpha)^*$, the dual space of H_α with the norm

$$\|x\|_{-\alpha} = \|A^{-\alpha}x\|.$$

Then $H_{-\alpha}$ is a Banach space endowed with this norm. For more details, we refer to the book by Pazy [1].

Lemma 2.1 *[1, pp. 72,74,195-196] Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following:*

(i) H_α is a Banach space for $0 \leq \alpha \leq 1$;

(ii) For any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;

(iii) The operator $A^\alpha S(t)$ is bounded for every $t > 0$ and

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}.$$

We define the following space

$$\begin{aligned} X = \mathcal{PC}(H_\alpha) = \{u : [0, T_0] \to H_\alpha : u \in C((t_k, t_{k+1}], H_\alpha), k = 0, 1, \cdots, m, \\ \text{and there exists } u(t_k^-), u(t_k^+) \text{ and } u(t_k^-) = u(t_k)\}. \end{aligned}$$

X is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in I} \|u(t)\|_\alpha.$$

We shall use the following conditions on f and w in its arguments:

(H1) Let $W \subset \mathrm{Dom}(f)$ be an open subset of $\mathbb{R}_+ \times H_\alpha \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u, v) \in W$, there is a neighborhood $V_1 \subset W$ of (t, u, v), such that the nonlinear map satisfies the following condition,

$$\|f(t, u, v) - f(s, u_1, v_1)\| \leq L_f\{|t - s|^{\theta_1} + \|u - u_1\|_\alpha + \|v - v_1\|_{\alpha-1}\},$$

for all (t, u, v), $(s, u_1, v_1) \in V_1$, $L_f = L_f(t, u, v, V_1) > 0$ and $0 < \theta_1 \leq 1$ are constants.

(H2) Let $U \subset \mathrm{Dom}(w)$ be a open subsets of $\mathbb{R}_+ \times H_{\alpha-1}$, where $0 \leq \alpha < 1$. For each $(t, u) \in U$, there is a neighborhood $V_2 \subset U$ of (t, u), $w(\cdot, 0) = 0$ such that

$$|w(t, u) - w(s, v)| \leq L_w\ \{\|u - v\|_{\alpha-1} + |t - s|^{\theta_2}\},$$

for all (t, u), $(s, v) \in V_2$, $L_w = L_w(u, t, U) > 0$ and $0 < \theta_2 \leq 1$ are constants.

(H3) Let W_1 be an open subset of $\mathbb{R}_+ \times H_\alpha$. For each $(t, x) \in W_1$ there exists a neighborhood $V_3 \subset W_1$ of (t, x) and a positive constant $L_g = L_g(t, x, V_3)$ such that

$$\|g(t, x) - g(s, y)\| \leq L_g \|x - y\|_\alpha,$$

for all $(t, x), (s, y) \in V_3$.

(H4) Let $a : [0, T_0] \times [0, T_0] \to [0, T_0]$ be a continuous function that satisfies the Holder condition uniformly in the first variable, i.e., there exist positive constants $L_a > 0$ and $0 < \theta_3 \leq 1$, such that

$$|a(t, s) - a(\tau, s)| \leq L_a |t - \tau|^{\theta_3},$$

for all $t, \tau, s \in [0, T_0]$.

(H5) The functions $I_k : H_\alpha \to H_\alpha$ are continuous and there exists D_k such that $\|I_k(u)\|_\alpha \le D_k,\ k = 0, 1, \cdots, m.$

(H6) There exists continuous nondecreasing $d_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|I_k(u) - I_k(v)\|_\alpha \le d_k\|u - v\|_\alpha, k = 1, 2, \cdots, m.$$

New concept of solutions. Here, we prove a new concept of solutions for the following problem (6)

$$\begin{cases} u'(t) + Au(t) &= r(t) + \int_0^t a(t,\tau)g(\tau, u(\tau))d\tau, \quad t \in [0, T_0],\ t \neq t_k, \\ u(0) &= u_0, \\ u(t_k) &= I_k(u(t_k^-)), \quad k = 1, 2 \cdots, m, \end{cases} \tag{6}$$

where $r \in \mathcal{PC}(I, H)$.

Let

$$\begin{cases} v'(t) + Av(t) &= r(t) + \int_0^t a(t,\tau)g(\tau, u(\tau))d\tau, \quad t \in [0, T_0], \\ v(0) &= v_0, \end{cases} \tag{7}$$

and

$$\begin{cases} w'(t) + Aw(t) &= 0, \quad t \in [0, T_0], \quad t \neq t_k, \\ w(0) &= 0, \\ w(t_k) &= I_k(u(t_k^-)), \quad k = 1, 2, \cdots, m, \end{cases} \tag{8}$$

be the decomposition of $u(.) = v(.) + w(.)$, where v is the continuous mild solution of (7) and w is the $\mathcal{PC}$ mild solution of (8).

By a mild solution for (7), we mean a continuous function $v : [0, T_0] \to H$ satisfying the following integral equation (For more details we refer to [2] and [10])

$$v(t) = S(t)v_0 + \int_0^t S(t-s)\Big[r(s) + \Upsilon v(s)\Big]ds, \quad t \in [0, T_0], \tag{9}$$

where

$$\Upsilon v(t) = \int_0^t a(t,\tau)g(\tau, u(\tau))d\tau.$$

and by a $\mathcal{PC}$ mild solution for (8), we mean a function $w \in \mathcal{PC}([0, T_0], D(A))$ satisfying the following integral equation (see [20, Lemma 2.3])

$$w(t) = \begin{cases} -\int_0^t Aw(s)ds, & t \in [0, t_1], \\ I_1(u(t_1^-)) - \int_0^t Aw(s)ds, & t \in (t_1, t_2], \\ \vdots & \\ \sum_{i=1}^k I_i(u(t_i^-)) - \int_0^t Aw(s)ds, & t \in (t_k, t_{k+1}], \\ \qquad k = 1, 2, \cdots, m. & \end{cases} \tag{10}$$

The above equation (10) can be expressed as

$$w(t) = \sum_{i=1}^{k} \chi_i(t) I_i(w(t_i^-)) - \int_0^t Aw(s)ds, \tag{11}$$

for $t \in [0, T_0]$, where

$$\chi_i(t) = \begin{cases} 0, & \text{for} \quad t \in [0, t_1], \\ 1, & \text{for} \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, 3, \cdots, m. \end{cases} \tag{12}$$

Taking Laplace transform of (11), we obtain

$$w(p) = \sum_{i=1}^{k} \frac{e^{-t_i p}}{p} I_i - \frac{Aw(p)}{p},$$

this gives

$$w(p) = \sum_{i=1}^{k} e^{-t_i p}(pI + A)^{-1} I_i, \tag{13}$$

Also, we note that $(pI + A)^{-1} = \int_0^\infty e^{-pt} S(t)dt$. Thus we can derive the mild solution for (8)

$$w(t) = \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(w(t_i^-)).$$

Hence, the mild solution for the problem (6) is given by

$$u(t) = S(t)u_0 + \sum_{i=1}^{k} \chi_i(t) S(t - t_i) I_i(u(t_i^-)) + \int_0^t S(t - s)\Big[r(s) + \Upsilon u(s)\Big]ds. \tag{14}$$

We can rewrite (14) as

$$u(t) = \begin{cases} S(t)u_0 + \int_0^t S(t - s)\Big[r(s) + \Upsilon u(s)\Big]ds, \quad t \in [0, t_1], \\ \\ S(t)u_0 + S(t - t_1) I_1(u(t_1^-)) \\ + \int_0^t S(t - s)\Big[r(s) + \Upsilon u(s)\Big]ds, \qquad t \in (t_1, t_2], \\ \vdots \\ S(t)u_0 + \sum_{i=1}^{k} S(t - t_i) I_i(u(t_i^-)) \\ + \int_0^t S(t - s)\Big[r(s) + \Upsilon u(s)\Big]ds, \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, \cdots, m. \end{cases} \tag{15}$$

3 Local Existence of Mild Solutions

In this section, we will prove the existence and uniqueness results concerning $\mathcal{PC}$-mild solutions for system (5). For $0 \le \alpha < 1$, we define

$$X_1 = \{u \in X : \|u(t) - u(s)\|_{\alpha-1} \le L|t-s|, \forall\ t, s \in (t_k, t_{k+1}], k = 0, 1, \cdots, m\},$$

where L is a suitable positive constant to be specified later.

Definition 3.1 A continuous function $u : [0, T_0] \to H$ solution of problem (5)

$$u(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in [0, t_1], \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1, t_2], \\ \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k, t_{k+1}], \\ \qquad k = 1, 2, \cdots, m. \end{cases} \tag{16}$$

is said to be a mild solution.

For a fixed $R > 0$, we define

$$\mathcal{W} = \{u \in X \cap X_1 : u(0) = u_0, \ \ \|u - u_0\|_{\mathcal{PC}} \le R\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of X_1 and is a Banach space.

Let

$$N_1 = \sup_{0 \le t \le T_0} \|f(0, u_0, u_0)\|, \tag{17}$$

$$N_2 = \sup_{0 \le t \le T_0} \|g(0, u_0)\| \tag{18}$$

and

$$a_{T_0} = \int_0^{T_0} |a(s)|ds. \tag{19}$$

Now we define a map $\mathcal{G} : \mathcal{W} \to \mathcal{W}$ by

$$(\mathcal{G}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \ t \in [0, t_1], \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1, t_2], \\ \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ + \int_0^t S(t-s)[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k, t_{k+1}], \\ \qquad k = 1, 2, \cdots, m. \end{cases} \tag{20}$$

Theorem 3.1 *Let $u_0 \in H_\alpha$ and the assumptions (H1) – (H4) hold. Then the problem (5) has a mild solution provided that*

$$C_\alpha[(N_f + a_{T_0}N_g)]\frac{T_0^{1-\alpha}}{1-\alpha} + M \sum_{i=1}^{k} D_i \leq \frac{R}{2} \tag{21}$$

and

$$C_\alpha\{L_f(2+LL_w) + a_{T_0}L_g\}\frac{T_0^{1-\alpha}}{(1-\alpha)} + M\sum_{0}^{m} d_i < 1, \tag{22}$$

Proof. We begin with showing that $\mathcal{G}u \in X_1$ for each $u \in X_1$. Clearly, $\mathcal{G} : X \to X$. Let $u \in X_1$, then for each $\tau_1, \tau_2 \in [0, t_1]$, $\tau_1 < \tau_2$ and $0 \leq \alpha < 1$, we have

$$\begin{aligned}
&\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\
&\quad \leq \|[S(\tau_2) - S(\tau_1)]u_0\|_{\alpha-1} \\
&\qquad + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\| \ \|f(s, u(s), u(w(s, u(s))))\|ds \\
&\qquad + \int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\|\Big\{\int_0^s |a(s,\tau)| \ \|g(\tau, u(\tau))\|d\tau\Big\}ds \\
&\qquad + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2 - s)\| \ \|f(s, u(s), u(w(s, u(s))))\|ds \\
&\qquad + \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2 - s)\|\Big\{\int_0^s |a(s,\tau)| \ \|g(\tau, u(\tau))\|d\tau\Big\}ds.
\end{aligned} \tag{23}$$

Since $f(t, u(t), u(w(u(t), t)))$ and $g(t, u(t))$ are continuous, together with the assumptions (H1), (H2) and (H3), there exist constants N_f and N_g, such that

$$\left.\begin{aligned} \|f(t, u(t), u(w(t, u(t))))\| \leq N_f, \\ \|g(t, u(t))\| \leq N_g \end{aligned}\right\}, u \in X, \ t \in [0, T_0], \tag{24}$$

where $N_f = L_f\Big\{T_0^{\theta_1} + R(1 + LL_w) + LL_wT_0^{\theta_2}\Big\} + N_1$ and $N_f = L_gR + N_2$.

For the first term on the right hand side of (23), we have

$$\begin{aligned}
\|A^{\alpha-1}[S(\tau_2) - S(\tau_1)]u_0\| &\leq \int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S'(s)u_0\|ds \\
&= \int_{\tau_1}^{\tau_2} \|A^{\alpha}S(s)u_0\|ds \\
&= \int_{\tau_1}^{\tau_2} \|S(s)\| \ \|u_0\|_\alpha ds \\
&\leq M\|u_0\|_\alpha(\tau_2 - \tau_1).
\end{aligned} \tag{25}$$

For the second and third term on the right hand side of (23), we have the following estimate

$$\begin{aligned} \|(S(\tau_2 - s) - S(\tau_1 - s))\|_{\alpha-1} &\le \int_0^{\tau_2-\tau_1} \|A^{\alpha-1}S'(l)S(\tau_1 - s)\|dl \\ &= \int_0^{\tau_2-\tau_1} \|S(l)A^{\alpha}S(\tau_1 - s)\|dl \\ &\le MC_\alpha(\tau_2 - \tau_1)(\tau_1 - s)^{-\alpha}. \end{aligned} \tag{26}$$

Then using the inequality (26), we get the following bounds for the second and third term on the right hand side of (23) as

$$\begin{aligned} &\int_0^{\tau_1} \|(S(\tau_2 - s) - S(\tau_1 - s))A^{\alpha-1}\|\|f(s, u(s), u(w(s, u(s))))\|ds \\ &\le N_f MC_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}(\tau_2 - \tau_1). \end{aligned} \tag{27}$$

$$\begin{aligned} &\int_0^{\tau_1} \|(S(\tau_2 - s) - S(\tau_1 - s))A^{\alpha-1}\|\Big\{\int_0^s |a(s,\tau)|\ \|g(\tau, u(\tau))\|d\tau\Big\}ds \\ &\le MN_g C_\alpha a_{T_0}\frac{T_0^{1-\alpha}}{1-\alpha}(\tau_2 - \tau_1). \end{aligned} \tag{28}$$

The fourth and fifth term on the right side of (23) are estimated as

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)A^{\alpha-1}\|\|f(s, u(s), u(w(s, u(s))))\|ds \\ &\qquad \le \|A^{\alpha-1}\|MN_f(\tau_2 - \tau_1). \end{aligned} \tag{29}$$

$$\begin{aligned} &\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)A^{\alpha-1}\|\Big\{\int_0^s |a(s,\tau)|\ \|g(\tau, u(\tau))\|d\tau\Big\}ds \\ &\qquad \le \|A^{\alpha-1}\|a_{T_0}MN_g(\tau_2 - \tau_1). \end{aligned} \tag{30}$$

Thus from the inequalities (25) and (27)-(30), we see that

$$\begin{aligned} \|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \le M\Big\{&\|u_0\|_\alpha + C_\alpha(N_f + a_{T_0}N_g)\ \frac{T_0^{1-\alpha}}{1-\alpha} \\ &+(N_f + a_{T_0}N_g)\ \|A^{\alpha-1}\|\Big\}(\tau_2 - \tau_1). \end{aligned} \tag{31}$$

For $\tau_1, \tau_2 \in (t_1, t_2],\ \ \tau_1 < \tau_2$ and $0 \le \alpha < 1$, we have

$$\begin{aligned} &\|(\mathcal{G}u)(\tau_2) - (\mathcal{G}u)(\tau_1)\|_{\alpha-1} \\ \le\ &\|[S(\tau_2) - S(\tau_1)]u_0\|_{\alpha-1} + \|A^{\alpha-1}[S(\tau_2 - t_1) - S(\tau_1 - t_1)]I_1(u(t_1^-))\| \\ &+\int_0^{\tau_1} \|A^{\alpha-1}[S(\tau_2 - s) - S(\tau_1 - s)]\|\Big\{\|f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\|\Big\}ds \\ &+\int_{\tau_1}^{\tau_2} \|A^{\alpha-1}S(\tau_2 - s)\|\Big\{\|f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\|\Big\}ds. \end{aligned} \tag{32}$$

The second term on the right side of (32) is estimated as

$$\begin{aligned}\|A^{\alpha-1}[S(\tau_2-t_1)-S(\tau_1-t_1)]I_1(u(t_1^-))\| &\leq \int_{\tau_1}^{\tau_2}\|A^{\alpha-1}S'(t-t_1)\|\|I_1(u(t_1^-))\|ds\\ &= \int_{\tau_1}^{\tau_2}\|A^{\alpha}S(t-t_1)\| \ \ \|I_1(u(t_1^-))\|ds\\ &\leq M \ \|I_1(u(t_1^-))\|_\alpha(\tau_2-\tau_1). \qquad (33)\end{aligned}$$

Thus, from the inequalities (25), (27)-(30) and (33), we see that

$$\begin{aligned}&\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\\ &\leq M\Big\{\|u_0\|_\alpha+\|I_1(u(t_1^-))\|_\alpha+C_\alpha(N_f+a_{T_0}N_g)\ \frac{T_0^{1-\alpha}}{1-\alpha}\\ &\qquad +(N_f+a_{T_0}N_g)\ \|A^{\alpha-1}\|\Big\}(\tau_2-\tau_1). \qquad (34)\end{aligned}$$

Similarly, for $\tau_1,\tau_2\in(t_k,t_{k+1}], \quad \tau_1<\tau_2, k=1,2,\cdots,m$ and $0\leq\alpha<1$, we have

$$\begin{aligned}&\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\\ &\leq M\Big\{\|u_0\|_\alpha+\sum_{i=1}^{k}\|I_i(u(t_i^-))\|_\alpha+C_\alpha(N_f+a_{T_0}N_g)\ \frac{T_0^{1-\alpha}}{1-\alpha}\\ &\qquad +(N_f+a_{T_0}N_g)\ \|A^{\alpha-1}\|\Big\}(\tau_2-\tau_1). \qquad (35)\end{aligned}$$

Thus, for each $\tau_1,\tau_2\in[0,T_0],\ \tau_1<\tau_2$ and $0\leq\alpha<1$, we have

$$\|(\mathcal{G}u)(\tau_2)-(\mathcal{G}u)(\tau_1)\|_{\alpha-1}\leq L(\tau_2-\tau_1), \qquad (36)$$

where $L = \max\{M\|u_0\|_\alpha, M \ \sum_{i=1}^{m}\|I_i(u(t_i^-))\|_\alpha, (N_f + a_{T_0}N_g)MC_\alpha\frac{T_0^{1-\alpha}}{1-\alpha}, (N_f + a_{T_0}N_g)M\|A^{1-\alpha}\|\}$.

Therefore, $\mathcal{G}$ is piecewise Lipschitz continuous on $[0,T_0]$ and so $\mathcal{G}: X_1\to X_1$.

Next we will show that $\mathcal{G}:\mathcal{W}\to\mathcal{W}$.

Let $u\in X\cap X_1$ and $t\in[0,t_1]$, we have

$$\begin{aligned}\|(\mathcal{G}u)(t)-u_0\|_\alpha &\leq \|(S(t)-I)A^\alpha u_0\|\\ &\quad+\int_0^t\|S(t-s)A^\alpha\|\|f(s,u(s),u(w(s,u(s))))\|ds\\ &\quad+\int_0^t\|S(t-s)A^\alpha\|\Big[\int_0^s|a(s,\tau)|\ \|g(s,u(s))\|d\tau\Big]ds\\ &\leq \frac{R}{2}+C_\alpha[(N_f+a_{T_0}N_g)]\frac{T_0^{1-\alpha}}{1-\alpha}\\ &\leq R. \qquad (37)\end{aligned}$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1 \cdots, m$, we have

$$\begin{aligned}
\|(\mathcal{G}u)(t) - u_0\|_\alpha &\le \|(S(t) - I)A^\alpha u_0\| \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \|f(s, u(s), u(w(s, u(s))))\| ds \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \Big[\int_0^s |a(s,\tau)| \; \|g(s, u(s))\| d\tau \Big] ds \\
&\quad + \sum_{i=1}^{k} \|A^\alpha S(t - t_i) I_i(u(t_i^-))\| \\
&\le \frac{R}{2} + C_\alpha[(N_f + a_{T_0} N_g)] \frac{T_0^{1-\alpha}}{1-\alpha} + M \sum_{i=1}^{k} \|I_i(u(t_i^-))\|_\alpha \\
&\le R. \qquad (38)
\end{aligned}$$

Thus, from (37), (38) and (21), it is clear that

$$\|\mathcal{G}u - u_0\|_{\mathcal{PC}} \le R.$$

Therefore, $\mathcal{G} : \mathcal{W} \to \mathcal{W}$ is well defined.

Finally, we will claim that $\mathcal{G}$ is a contraction on $\mathcal{W}$. If $[0, t_1]$, $u, v \in \mathcal{W}$, then we have

$$\begin{aligned}
\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \le \int_0^t \|S(t-s)A^\alpha\| \; &\|f(s, u(s), u(w(s, u(s)))) \\
&-f(s, v(s), u(v(s, v(s))))\| ds + \int_0^t \|S(t-s)A^\alpha\| \\
&\Big[\int_0^s |a(s,\tau)| \; \|g(\tau, u(\tau)) - g(\tau, v(\tau))\| d\tau \Big] ds. \qquad (39)
\end{aligned}$$

We also note that

$$\begin{aligned}
&\|f(s, u(s), u(w(s, u(s)))) - f(s, v(s), u(v(s, v(s))))\| \\
&\le L_f \Big\{ \|u(s) - v(s)\|_\alpha + \|u(w(s, u(s))) - u(w(s, v(s)))\|_{\alpha-1} \\
&\qquad + \|u(w(s, v(s))) - v(w(s, v(s)))\|_{\alpha-1} \Big\} \\
&\le L_f(2 + LL_w)\|u - v\|_{\mathcal{PC}}. \qquad (40)
\end{aligned}$$

and

$$\|g(\tau, u(\tau)) - g(\tau, v(\tau))\|_\alpha \le L_g \|u - v\|_{\mathcal{PC}}. \qquad (41)$$

We use (40) and (41) into (39), we get

$$\begin{aligned}
&\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \\
&\le \frac{C_\alpha}{(1-\alpha)} \Big\{ L_f(2 + LL_w) + a_{T_0} L_g \Big\} T_0^{1-\alpha} \|u - v\|_{\mathcal{PC}}.
\end{aligned}$$

For $t \in (t_1, t_2]$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha\{L_f(2 + LL_w) + a_{T_0}L_g\}\frac{T_0^{1-\alpha}}{(1-\alpha)} + M\|I_1(u(t_1^-))\|_\alpha\Big]\|u - v\|_{\mathcal{PC}}.$$

For $t \in (t_k, t_{k+1}],\ k = 1, 2, 3, \cdots, m$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha\{L_f(2 + LL_w) + a_{T_0}L_g\}\frac{T_0^{1-\alpha}}{(1-\alpha)} + M\sum_{i=1}^{k}\|I_i(u(t_i^-))\|_\alpha\Big]\|u - v\|_{\mathcal{PC}}.$$

Thus, for each $t \in [0, T_0]$, we have

$$\|(\mathcal{G}u)(t) - (\mathcal{G}v)(t)\|_\alpha \leq \Big[C_\alpha\{L_f(2 + LL_w) + a_{T_0}L_g\}\frac{T_0^{1-\alpha}}{(1-\alpha)} + M\sum_{i=1}^{m}d_i\Big]\|u - v\|_{\mathcal{PC}}. \tag{42}$$

Therefore, the map $\mathcal{G}$ is a contraction map, hence G has a unique fixed point $u \in \mathcal{W}$. That is, problem (5) has a unique mild solution.

4 Further Existence Results

Theorem 3.1 can be proved if we drop the hypothesis (H1),(H2) and (H3). In that case the proof is based on the idea of Wang et al. [21].

Theorem 4.1 *Assume the conditions (H4)-(H6) hold. The semigroup $\{S(t)\}_{t\geq 0}$ is compact, $f : I \times H \times H \to H$ and $g : I \times H \to H$ are continuous. Let $u_0 \in H_\alpha$ there exists a constant $r > 0$ such that*

$$M\Big\{\|u_0\|_\alpha + \sum_{i=1}^{k}\|I_i(u(t_i^-))\|_\alpha\Big\} + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha} \leq r, \tag{43}$$

where

$$M_f = \sup_{s\in I, u\in\Omega}\|f(s, u(s), u(w(s, u(s))))\|, \quad M_g = \sup_{s\in I, u\in\Omega}\|g(s, u(s))\| \tag{44}$$

and

$$\Omega = \{v \in \mathcal{PC}(H_\alpha) : \|v\|_{\mathcal{PC}} \leq r\}.$$

Then there exists a mild solution $u \in \mathcal{PC}(H_\alpha)$ of the problem (5).

Proof. Let us define a map $\mathcal{F} : \mathcal{PC}(H_\alpha) \to \mathcal{PC}(H_\alpha)$, by

$$(\mathcal{F}u)(t) = \begin{cases} S(t)u_0 + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \ t \in [0,t_1], \\ S(t)u_0 + S(t-t_1)I_1(u(t_1^-)) \\ + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \qquad t \in (t_1,t_2], \\ \quad \vdots \\ S(t)u_0 + \sum_{i=1}^k S(t-t_i)I_i(u(t_i^-)) \\ + \int_0^t S(t-s)[f(s,u(s),u(w(s,u(s)))) + \Upsilon u(s)]ds, \quad t \in (t_k,t_{k+1}], \\ \qquad k = 1,2,\cdots,m. \end{cases}$$

Step 1. First we show that $\mathcal{F}$ is continuous. It follows from the continuity of f and g that

$$\begin{aligned} \|f(s,u_n(s),u_n(w(s,u_n(s)))) - f(s,u(s),u(w(s,u(s))))\| &\leq \epsilon, \quad \text{as} \quad n \to \infty, \\ \|g(s,u_n(s)) - g(s,u(s))\| &\leq \epsilon, \quad \text{as} \quad n \to \infty, \end{aligned}$$

for $s \in [0,t], \quad t \in [0,T_0]$.

Now, for each $t \in [0,t_1]$, we have

$$\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha \leq C_\alpha(1 + a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{as} \quad n \to \infty. \tag{45}$$

For, $t \in (t_1,t_2]$, we have

$$\begin{aligned} &\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha \\ &\leq M\|I_1(u_n(t_1^-)) - I_1(u(t_1^-))\|_\alpha + C_\alpha(1 + a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{as} \quad n \to \infty. \end{aligned} \tag{46}$$

Similarly, for each $t \in (t_k,t_{k+1}], k = 1,2,\cdots,m$,

$$\begin{aligned} &\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\|_\alpha \\ &\leq M\sum_{i=1}^k \|I_i(u_n(t_i^-)) - I_i(u(t_i^-))\|_\alpha + C_\alpha(1 + a_{T_0})\frac{T_0^{1-\alpha}}{1-\alpha}\epsilon \to 0, \text{as} \quad n \to \infty. \end{aligned} \tag{47}$$

Thus, from the inequalities (45)-(47), we see that $\mathcal{F}$ is continuous.

Step 2. Next we show that $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{PC}(H_\alpha)$.

Let $u \in \Omega$, then for $t \in [0,t_1]$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \leq M\|u_0\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{48}$$

For each $t \in (t_1,t_2]$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \leq M\Big\{\|u_0\|_\alpha + \|I_1(u(t_1^-))\|_\alpha\Big\} + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{49}$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m$, we have

$$\|(\mathcal{F}u)(t)\|_\alpha \le M\Big\{\|u_0\|_\alpha + \sum_{i=1}^{k} \|I_i(u(t_i^-))\|_\alpha\Big\} + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha}. \tag{50}$$

Thus, from inequalities (43) and (48)-(50), we see that $\mathcal{F} : \Omega \to \Omega$.

Step 3. In this step, we show that $\mathcal{F}$ maps bounded sets into equicontinuous sets in $\mathcal{PC}(H_\alpha)$. Let $\tau_1, \tau_2 \in [0, t_1], \tau_1 < \tau_2$, we have

$$\begin{aligned}&\|(\mathcal{F}u)(\tau_2) - (\mathcal{F}u)(\tau_1)\|_\alpha \\ &\le M\Big\{\|u_0\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\ &\quad + \|A^{\alpha-1}\|(M_f + a_{T_0}M_g)\Big\}(\tau_2 - \tau_1).\end{aligned} \tag{51}$$

Similarly, for each $\tau_1, \tau_2 \in (t_k, t_{k+1}], \tau_1 < \tau_2$, $k = 1, 2, \cdots, m$, we have

$$\begin{aligned}&\|(\mathcal{F}u)(\tau_2) - (\mathcal{F}u)(\tau_1)\|_\alpha \\ &\le M\Big\{\|u_0\|_\alpha + \sum_{i=1}^{k} \|I_i(u(t_i^-))\|_\alpha + C_\alpha(M_f + a_{T_0}M_g)\frac{T_0^{1-\alpha}}{1-\alpha} \\ &\quad + \|A^{\alpha-1}\|(M_f + a_{T_0}M_g)\Big\}(\tau_2 - \tau_1).\end{aligned} \tag{52}$$

The right hand side of (52) tends to zero as $\tau_2 \to \tau_1$. Hence, $\mathcal{F}(\Omega)$ is equicontinuous.

Step 4. $\mathcal{F}$ maps Ω into a compact set in H_α.

For this purpose, we decompose $\mathcal{F}$ by $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$,

where

$$\begin{aligned}(\mathcal{F}_1u)(t) = S(t)u_0 + \int_0^t S(t-s)\Big[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\Big]ds,\\ t \in I \backslash\ \{t_1, \cdots, t_m\},\end{aligned}$$

and

$$(\mathcal{F}_2u)(t) = \begin{cases} 0, \quad t \in [0, t_1], \\ \sum_{i=1}^{k} S(t - t_i)I_i(u(t_i^-)), \qquad t \in (t_k, t_{k+1}], k = 1, 2, \cdots, m. \end{cases}$$

Since $\mathcal{F}_2$ is a constant map and hence compact.

Finally, we need to prove that $(\mathcal{F}_1u)(t)$ is relatively compact in Ω for $0 \le t \le T_0$. The set $\{S(t)u_0\}$ is precompact in H_α for each $t \in [0, T_0]$, since $\{S(t), t \ge 0\}$ is compact.

For $t \in (0, T_0]$, and $\epsilon > 0$ sufficiently small, we define

$$(\mathcal{F}_1^\epsilon u)(t) = S(\epsilon)\int_0^{t-\epsilon} S(t-\epsilon-s)\Big[f(s, u(s), u(w(s, u(s)))) + \Upsilon u(s)\Big]ds, \quad u \in \Omega.$$

The set $\{(\mathcal{F}_1^\epsilon u)(t) : u \in \Omega\}$ is precompact in H_α since $S(\epsilon)$ is compact. Moreover, for any $u \in \Omega$, we have

$$\begin{aligned}\|(\mathcal{F}_1 u)(t) - (\mathcal{F}_1^\epsilon u)(t)\|_\alpha &\leq \int_{t-\epsilon}^{t} \|A^\alpha S(t-s)\| \, \|f(s, u(s), u(w(s, u(s))))\| ds \\ &\quad + \int_{t-\epsilon}^{t} \|A^\alpha S(t-s)\| \Big\{ \int_0^s |a(s,\tau)| \, \|g(s, u(s))\| d\tau \Big\} ds \\ &\leq M(M_f + a_{T_0} M_g)\epsilon.\end{aligned}$$

Therefore, $\{(\mathcal{F}_1^\epsilon u)(t) : u \in \Omega\}$ is arbitrarily close to the set $\{(\mathcal{F}_1 u)(t) : u \in \Omega\}, t > 0$. Hence the set $\{(\mathcal{F}_1 u)(t) : u \in \Omega\}$ is precompact in H_α.

Thus, $\mathcal{F}_1$ is a compact operator by Arzela-Ascoli theorem, and hence $\mathcal{F}$ is a compact operator. Then Schauder fixed point theorem ensures that $\mathcal{F}$ has a fixed point, which gives rise to a $\mathcal{PC}$-mild solution.

5 Application

Consider the following semi-linear heat equation with a deviating argument

$$\left.\begin{array}{rcl} \dfrac{\partial u}{\partial t} &=& \dfrac{\partial^2 u}{\partial x^2} + \tilde{H}(x, u(x,t)) + G(t, x, u(x,t)), \\ && + \int_0^t a(t,\tau) \frac{\partial}{\partial x}[\xi(x, \tau, u(x,\tau), \frac{\partial}{\partial x} u(x,\tau))] d\tau \\ x \in (0,1) &,& t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \Delta u|_{t=\frac{1}{2}} &=& \frac{u(\frac{1}{2})^-}{1+u(\frac{1}{2})^-}, \\ u(0,t) &=& u(1,t) = 0, \\ u(x,0) &=& u_0(x), \quad x \in (0,1), \end{array}\right\} \tag{53}$$

where

$$\tilde{H}(x, u(x,t)) = \int_0^x K(x,y) u(y, g(t)|u(y,t)|) dy,$$

and the function $G : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x. Assume that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $\psi(0) = 0$ and $K \in C^1([0,1] \times [0,1]; \mathbb{R})$.

Let $X = L^2((0,1); \mathbb{R})$. We define an operator A as follows,

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad D(A) = H_0^1(0,1) \cap H^2(0,1), \tag{54}$$

where $X_{1/2} = D(A^{1/2}) = H_0^1(0,1)$ and $X_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) = H^2(0,1)$. Here clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$.

Let us define $g : [0,\infty) \times D(A) \to X$ by

$$g(t,\phi)(x) = \frac{\partial}{\partial x}[\phi(x, t, \phi(x,t), \frac{\partial}{\partial x}\phi(x,t))], \tag{55}$$

and the function $f : \mathbb{R}_+ \times X_{1/2} \times X_{-1/2} \to X$, is given by

$$f(t, \phi, \psi)(x) = \tilde{H}(x, \psi) + G(t, x, \phi), \tag{56}$$

where $\tilde{H} : [0,1] \times X \to H_0^1(0,1)$ is given by

$$\tilde{H}(t, \psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy \tag{57}$$

with $\psi(x,t) = \phi(x, w(t, \phi(x,t)))$ and $w(t, \phi(x,t)) = g(t)|\phi(x,t)|$, $G : \mathbb{R}^+ \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$ satisfies the following

$$\|G(t,x,\phi)\| \leq Q(x,t)(1 + \|\psi\|_{H^2(0,1)}) \tag{58}$$

with $Q(.,t) \in X$ and Q is continuous in its second argument. Then, we can easily verify that the assumptions (H1)-(H6) hold. For more details, we refer the reader to [7].

6 Conclusion

The sufficient conditions of the existence and uniqueness of $\mathcal{PC}$-mild solutions to the integro-differential equations with a deviating argument are established.

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