Nonlinear Dynamics and Systems Theory, **14** (1) (2014) 44–57

Adaptive Hybrid Function Projective Synchronization of Chaotic Space-Tether System

A. Khan [1] and R. Pal [2*]

[1] *Department of Mathematics, Jamia Millia Islamia University, New Delhi, India*
[2] *Department of Mathematics, University of Delhi, New Delhi, India*

Received: April 25, 2013; Revised: January 30, 2014

Abstract: In this paper, we have achieved adaptive hybrid function projective synchronization between two identical chaotic space-tether systems with uncertain time-varying parameters and with each system evolving from different initial conditions by applying adaptive control technique. Based on Lyapunov stability theory, adaptive control laws and parameter update laws for estimating the uncertain, time-varying parameters are derived to make the states of the two identical chaotic systems asymptotically synchronized. Complete synchronization, antisynchronization, hybrid projective synchronization are obtained as special cases from the above synchronization method. The control techniques and the proposed update laws are verified by numerical simulation results.

Keywords: *adaptive control; parameter estimation; hybrid function projective synchronization; Lyapunov stability theory; space-tether system, celestial mechanics.*

Mathematics Subject Classification (2010): 93C40, 70F15, 37N05, 93D20.

1 Introduction

Two identical chaotic systems with different initial conditions were first made to synchronize in 1990 by Pecora and Carroll [25]. Since then, chaos synchronization has attracted a great deal of attention from various scientific fields. The idea of synchronization is to use the output of the master system to control the slave system so that the output of the response system follows the output of the master system asymptotically. Many methods and techniques for handling chaos control and synchronization of various chaotic systems have been developed such as PC method [25], OGY method [19], time-delay feedback approach [24], feedback approach [9, 14], backstepping design technique [29],

* Corresponding author: mailto:rimpipal@yahoo.co.in

adaptive method [5, 7, 15, 21, 27, 28], linear control method [16, 22], nonlinear control scheme [21, 23].

Till now, different types of synchronization phenomenon have been presented such as complete synchronization (CS) [11], generalized synchronization (GS) [8], lag synchronization [26], anticipated synchronization [18], phase synchronization [2], hybrid synchronization (HS) [6] and antiphase synchronization [13], etc. Among all kinds of chaos synchronization schemes, projective synchronization characterized by a scaling factor that two systems synchronize proportionally has been of recent interest as it can be used to obtain faster communication with its proportional feature. Recently, a new kind of synchronization, Function Projective Synchronization (FPS) was introduced [4]. FPS is a more general definition of Projective Synchronization where the drive system and the response system can be synchronized upto a scaling function which is not a constant. Another synchronization phenomenon called a Hybrid Projective Synchronization (HPS) has also been investigated where the different state variables of the two systems synchronize up to different state factors [10]. Combining these two, we have a new kind of synchronization phenomenon called a Hybrid Function Projective Synchronization (HFPS) which is of latest interest [12, 20, 30]. Here, the different state vectors of the drive and response system synchronize up to different scaling functions which are not scalars. Thus, it is the most modified and generalised form of Projective Synchronization.

Motivated by the aforementioned research, we have formulated Hybrid Function Projective Synchronization (HFPS) of two identical chaotic systems with different initial conditions using adaptive control scheme where the response system has uncertain time-varying parameters. Based on Lyapunov stability theory, adaptive control law and the parameter update law are derived using which HFPS between the two systems is achieved.

Application of chaos synchronization is varied. We consider its application in the field of celestial mechanics. In the recent decades, this field has slowly gained interest and some work has followed [1, 3, 17, 31]. The model we choose in this manuscript as identical chaotic systems is that of a space-tether system. The dynamics of space-tether system has recently been of great interest due to its vast applicability in the field of celestial mechanics. A tether is a long cable used to couple spacecrafts to each other or to other masses such as rocket, space station etc., so that their dynamics can be connected. So, a space-craft together with a tether forms a space-tether system and depending upon the objective and mission, there always arise problems of synchronizing its motion with other spacecrafts using a tether itself or with another space-tether system altogether. Here, in this manuscript, we consider the problem where there is a need to synchronize two identical space-tether systems. A space-tether system can have numerous applications like creation of artificial gravitation on board of the spacecraft, maintainance of spacecraft with electric power, study of upper atmosphere, in research of distant space and many more. Thus, the study of dynamics of a space-tether system is an important topic in celestial mechanics.

Consequently, the paper is organized as follows. In Section 2, model of the space-tether system is explained, in Section 3, adaptive HFPS (AHFPS) between the aforementioned two systems is studied in details. In Section 4, numerical simulations are presented following which observations are made. Finally, in Section 5, conclusion is drawn.

2 Model Explaination

The dynamics of a space-tether system can be developed using different kinds of mathematical models which describe its motion. In this paper, we have chosen the model where tether is considered as massless rod. It is given by equation (1).

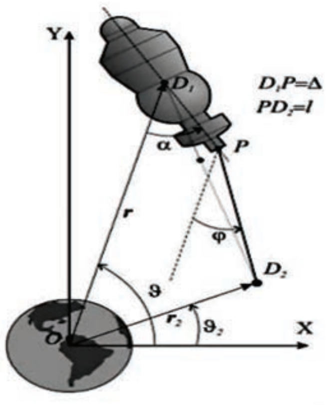


Figure 1: The space-tether problem where tether is considered a massless rod.

$$
\begin{aligned}
\frac{d^2\alpha}{dt^2} &= \frac{3\omega^2}{2}\frac{A-B}{C}\sin 2\alpha - \frac{\Delta c}{C}(l-l_o)\sin(\alpha-\varphi),\\
\frac{d^2 l}{dt^2} &= -\frac{c(l-l_o)}{2C}[\Delta^2 + \frac{2C}{m} - \Delta^2\cos(2\alpha-2\varphi)]\\
&\quad +3\omega^2\cos\varphi(l\cos\varphi+\Delta\cos\alpha) + (\frac{d\varphi}{dt})^2 l + 2l\omega\frac{d\varphi}{dt}\\
&\quad +\frac{d\alpha}{dt}\Delta(\frac{d\alpha}{dt}+2\omega)\cos(\alpha-\varphi) +\\
&\quad \frac{3\Delta\omega^2\sin 2\alpha\sin(\alpha-\varphi)}{2C},\\
\frac{d^2\varphi}{dt^2} &= \frac{\Delta^2 c}{2lC}(l-l_o)\sin(2\alpha-2\varphi) +\\
&\quad \frac{\frac{d\alpha}{dt}\Delta(\frac{d\alpha}{dt}+2\omega)\sin(\alpha-\varphi)}{l} -\\
&\quad \frac{3\omega^2\sin\varphi(l\cos\varphi+\Delta\cos\alpha)}{l} -\\
&\quad \frac{2\frac{dl}{dt}(\omega+\frac{d\varphi}{dt})}{l} - \frac{3\Delta\omega^2}{2l}\frac{A-B}{C}\sin 2\alpha,
\end{aligned}
\tag{1}
$$

where the parameters are defined as follows:
$A, B, C=$ principal of moments of inertia of the spacecraft;
l_o = length of unstrained tether;
$\alpha=$ angle which the line joining the centres of mass of earth and spacecraft makes with a fixed axis through the center of mass of earth;
$l=$ variable length of the strained tether;

φ= inclination of the oscillating plane of the orbit of the center of mass of the system with the plane of ecliptic;
α= angle which the line joining centers of mass of earth and spacecraft makes with the tether;
Δ= distance between the center of mass of the spacecraft and the position on the spacecraft to which the tether is attached;
m= mass of the spacecraft;
ω= angular velocity of the carrying spacecraft in circular orbit.

3 Adaptive Control Scheme for AHFPS

For the applicability of the adaptive control scheme, the system is identified in the form of first order differential equations. For this, we make the following substitution:

$$\alpha(t) = x_1(t), \frac{d\alpha}{dt} = x_2(t), l(t) = x_3(t), \frac{dl}{dt} = x_4(t), \varphi(t) = x_5(t), \frac{d\varphi}{dt} = x_6(t).$$

Also, we rename the parameters in the following manner:

$$\frac{3\omega^2}{2}\frac{A-B}{C} = a, \frac{\Delta c}{C} = b, \frac{\Delta c l_o}{C} = d, \frac{c}{2C}[\Delta^2 + \frac{2C}{m}] = e,$$

$$\frac{\Delta^2 c}{2C} = f, \frac{\Delta^2 c l_o}{2C} = g, 3\omega^2 = h, 3\omega^2\Delta = j, 2\omega\Delta = k,$$

$$\frac{3\Delta\omega^2}{2C} = n, \frac{3\Delta\omega^2}{2}\frac{A-B}{C} = p, \frac{c l_o}{2C}[\Delta^2 + \frac{2C}{m}] = q.$$

Based on these substitutions, the system of equations is given as:

$$\begin{aligned}
\frac{dx_1}{dt} &= x_2, \\
\frac{dx_2}{dt} &= a\sin 2x_1 - bx_3\sin(x_1 - x_5) + d\sin(x_1 - x_5), \\
\frac{dx_3}{dt} &= x_4, \\
\frac{dx_4}{dt} &= -ex_3 + fx_3\cos(2x_1 - 2x_5) - g\cos(2x_1 - 2x_5) + hx_3\cos^2 x_5 + \\
& \quad j\cos x_1\cos x_5 + x_3x_6^2 + 2\omega x_3x_6 + \Delta x_2^2\cos(x_1 - x_5) + \\
& \quad kx_2\cos(x_1 - x_5) + n\sin 2x_1\sin(x_1 - x_5) + q, \\
\frac{dx_5}{dt} &= x_6, \\
\frac{dx_6}{dt} &= f\sin(2x_1 - 2x_5) - g\frac{\sin(2x_1 - 2x_5)}{x_3} + \frac{\Delta x_2^2\sin(x_1 - x_5)}{x_3} + \\
& \quad k\frac{x_2\sin(x_1 - x_5)}{x_3} - h\sin x_5\cos x_5 - j\frac{\cos x_1\sin x_5}{x_3} - \\
& \quad \frac{2\omega x_4}{x_3} - \frac{2x_4x_6}{x_3} - \frac{p\sin 2x_1}{x_3}.
\end{aligned} \tag{2}$$

The system of equations (2) is considered as our master system. Then the identical slave system is given by:

$$\begin{aligned}
\frac{dy_1}{dt} &= y_2 + u_1,\\
\frac{dy_2}{dt} &= a_1 \sin 2y_1 - b_1 x_3 \sin(y_1 - y_5) + d_1 \sin(y_1 - y_5) + u_2,\\
\frac{dy_3}{dt} &= y_4 + u_3,\\
\frac{dy_4}{dt} &= -e_0 y_3 + f_1 x_3 \cos(2y_1 - 2y_5) - g_1 \cos(2y_1 - 2y_5) +\\
&\quad h_1 y_3 \cos^2 y_5 + j_1 \cos y_1 \cos y_5 + y_3 y_6^2 + 2\omega_1 y_3 y_6 +\\
&\quad \Delta_1 y_2^2 \cos(y_1 - y_5) + k_1 y_2 \cos(y_1 - y_5) +\\
&\quad n_1 \sin 2y_1 \sin(y_1 - y_5) + u_4 + q_1,\\
\frac{dy_5}{dt} &= y_6 + u_5,\\
\frac{dy_6}{dt} &= f_1 \sin(2y_1 - 2y_5) - g_1 \frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1 y_2^2 \sin(y_1 - y_5)}{y_3} +\\
&\quad k_1 \frac{y_2 \sin(y_1 - y_5)}{y_3} - h_1 \sin y_5 \cos y_5 - j_1 \frac{\cos y_1 \sin y_5}{y_3} -\\
&\quad \frac{2\omega_1 y_4}{y_3} - \frac{2y_4 y_6}{y_3} - \frac{p_1 \sin 2y_1}{y_3} + u_6,
\end{aligned} \tag{3}$$

where x_i, y_i stand for the state variables of the master system and slave system respectively, $a_1, b_1, d_1, e_0, f_1, g_1, h_1, j_1, k_1, n_1, p_1, q_1, \Delta_1, \omega_1$ are the uncertain time-varying parameters of the slave system which are to be estimated and $u_1, u_2, u_3, u_4, u_5, u_6$ are the time-dependent non-linear controls which are also to be determined.

Let us now suppose that that the time-varying scaling function matrix be given by $A(t) = \text{diag}\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t))$ where $\alpha_i(t) \neq 0; i = \overline{1,6}$. The synchronization errors are defined by

$$e_r(t) = x_r(t) - \alpha_r(t) y_r(t), \quad r = \overline{1,6}. \tag{4}$$

AHFPS between the two systems (2) and (3) will be achieved up to the desired scaling function matrix $A(t)$ if $\lim_{t\to\infty} ||e_r(t)|| = 0, r = \overline{1,6}$. Following these, the error dynamics is given by:

$$\begin{aligned}
\frac{de_1}{dt} &= x_2 - \alpha_1 y_2 - \alpha_1 u_1 - \frac{d\alpha_1}{dt} y_1,\\
\frac{de_2}{dt} &= a \sin 2x_1 - b x_3 \sin(x_1 - x_5) + d \sin(x_1 - x_5) -\\
&\quad \alpha_2 [a_1 \sin 2y_1 - b_1 x_3 \sin(y_1 - y_5) + d_1 \sin(y_1 - y_5)] -\\
&\quad \alpha_2 u_2 - \frac{d\alpha_2}{dt} y_2,\\
\frac{de_3}{dt} &= x_4 - \alpha_3 y_4 - \alpha_3 u_3 - \frac{d\alpha_3}{dt} y_3,
\end{aligned}$$

$$
\begin{aligned}
\frac{de_4}{dt} &= -ex_3 + fx_3\cos(2x_1 - 2x_5) - g\cos(2x_1 - 2x_5) + q + \\
&\quad hx_3\cos^2 x_5 + j\cos x_1\cos x_5 + x_3x_6^2 + 2\omega x_3x_6 + \\
&\quad \Delta x_2^2\cos(x_1 - x_5) + kx_2\cos(x_1 - x_5) + \\
&\quad n\sin 2x_1\sin(x_1 - x_5) - \alpha_4[-e_0y_3 + f_1x_3\cos(2y_1 - 2y_5) - \\
&\quad g_1\cos(2y_1 - 2y_5) + q_1 + h_1y_3\cos^2 y_5 + j_1\cos y_1\cos y_5 + \\
&\quad y_3y_6^2 + 2\omega_1y_3y_6 + \Delta_1y_2^2\cos(y_1 - y_5) + k_1y_2\cos(y_1 - y_5) + \\
&\quad n_1\sin 2y_1\sin(y_1 - y_5)] - \alpha_4u_4 - \frac{d\alpha_4}{dt}y_4, \\
\frac{de_5}{dt} &= x_6 - \alpha_5y_6 - \alpha_5u_5 - \frac{d\alpha_5}{dt}y_5, \\
\frac{de_6}{dt} &= f\sin(2x_1 - 2x_5) - g\frac{\sin(2x_1 - 2x_5)}{x_3} + \frac{\Delta x_2^2\sin(x_1 - x_5)}{x_3} + \\
&\quad k\frac{x_2\sin(x_1 - x_5)}{x_3} - h\sin x_5\cos x_5 - j\frac{\cos x_1\sin x_5}{x_3} - \\
&\quad \frac{2\omega x_4}{x_3} - \frac{2x_4x_6}{x_3} - \frac{p\sin 2x_1}{x_3} - \alpha_6[f_1\sin(2y_1 - 2y_5) - \\
&\quad g_1\frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1y_2^2\sin(y_1 - y_5)}{y_3} + k_1\frac{y_2\sin(y_1 - y_5)}{y_3} - \\
&\quad h_1\sin y_5\cos y_5 - j_1\frac{\cos y_1\sin y_5}{y_3} - \frac{2\omega_1y_4}{y_3} - \frac{2y_4y_6}{y_3} - \\
&\quad \frac{p_1\sin 2y_1}{y_3}] - \alpha_6u_6 - \frac{d\alpha_6}{dt}y_6.
\end{aligned} \tag{5}
$$

When we have two identical chaotic systems without controls (i.e. $u_i = 0$), if they evolve from different initial conditions, the trajectories of the two systems eventually separate from each other and become unindentifiable and irrelevant. But when we have two controlled chaotic systems, the two systems will approach synchronization for any initial condition by appropriate control gain and update laws for uncertain time-varying parameters. So, taking $[k_i; i = \overline{1,20}]$ as control gains which are positive constants and letting $e_a = a_1 - a, e_b = b_1 - b, e_d = d_1 - d, e_e = e_0 - e, e_f = f_1 - f, e_g = g_1 - g, e_h = h_1 - h, e_j = j_1 - j, e_k = k_1 - k, e_n = n_1 - n, e_p = p_1 - p, e_q = q_1 - q, e_\Delta = \Delta_1 - \Delta, e_\omega = \omega_1 - \omega$,the following adaptive control laws and parameter update laws are proposed:

Adaptive control laws:

$$
\begin{aligned}
-\alpha_1u_1 &= -x_2 + \alpha_1y_2 + \frac{d\alpha_1}{dt}y_1 - k_1e_1, \\
-\alpha_2u_2 &= -[a_1\sin 2x_1 - b_1x_3\sin(x_1 - x_5) + d_1\sin(x_1 - x_5)] + \\
&\quad \alpha_2[a_1\sin 2y_1 - b_1x_3\sin(y_1 - y_5) + d_1\sin(y_1 - y_5)] + \\
&\quad \frac{d\alpha_2}{dt}y_2 - k_2e_2, \\
-\alpha_3u_3 &= -x_4 + \alpha_3y_4 + \frac{d\alpha_3}{dt}y_3 - k_3e_3,
\end{aligned}
$$

$$
\begin{aligned}
-\alpha_4 u_4 &= -[-e_0 x_3 + f_1 x_3 \cos(2x_1 - 2x_5) - g_1 \cos(2x_1 - 2x_5) + q_1 + \\
&\quad h_1 x_3 \cos^2 x_5 + j_1 \cos x_1 \cos x_5 + x_3 x_6^2 + 2\omega_1 x_3 x_6 + \\
&\quad \Delta_1 x_2^2 \cos(x_1 - x_5) + k_1 x_2 \cos(x_1 - x_5) + \\
&\quad n_1 \sin 2x_1 \sin(x_1 - x_5)] + \alpha_4[-e_0 y_3 + f_1 x_3 \cos(2y_1 - 2y_5) - \\
&\quad g_1 \cos(2y_1 - 2y_5) + q_1 + h_1 y_3 \cos^2 y_5 + j_1 \cos y_1 \cos y_5 + \\
&\quad y_3 y_6^2 + 2\omega_1 y_3 y_6 + \Delta_1 y_2^2 \cos(y_1 - y_5) + \\
&\quad k_1 y_2 \cos(y_1 - y_5) + n_1 \sin 2y_1 \sin(y_1 - y_5)] + \\
&\quad \frac{d\alpha_4}{dt} y_4 - k_4 e_4, \\
-\alpha_5 u_5 &= -x_6 + \alpha_5 y_6 + \frac{d\alpha_5}{dt} y_5 - k_5 e_5, \\
-\alpha_6 u_6 &= -[-f_1 \sin(2x_1 - 2x_5) - g_1 \frac{\sin(2x_1 - 2x_5)}{x_3} + \\
&\quad \frac{\Delta_1 x_2^2 \sin(x_1 - x_5)}{x_3} + k \frac{x_2 \sin(x_1 - x_5)}{x_3} - h \sin x_5 \cos x_5 - \\
&\quad j \frac{\cos x_1 \sin x_5}{x_3} - \frac{2\omega x_4}{x_3} - \frac{2x_4 x_6}{x_3} - \frac{p_1 \sin 2x_1}{x_3}] + \\
&\quad \alpha_6[f_1 \sin(2y_1 - 2y_5) - g_1 \frac{\sin(2y_1 - 2y_5)}{y_3} + \frac{\Delta_1 y_2^2 \sin(y_1 - y_5)}{y_3} + \\
&\quad k_1 \frac{y_2 \sin(y_1 - y_5)}{y_3} - h_1 \sin y_5 \cos y_5 - j_1 \frac{\cos y_1 \sin y_5}{y_3} - \frac{2\omega_1 y_4}{y_3} - \\
&\quad \frac{2y_4 y_6}{y_3} - \frac{p_1 \sin 2y_1}{y_3}] + \frac{d\alpha_6}{dt} y_6 - k_6 e_6.
\end{aligned} \tag{6}
$$

While, parameter update laws are:

$$
\begin{aligned}
\frac{da_1}{dt} &= \sin 2x_1 e_2 - k_7 e_a, \\
\frac{db_1}{dt} &= -x_3 \sin(x_1 - x_5) e_2 - k_8 e_b, \\
\frac{dd_1}{dt} &= \sin(x_1 - x_5) e_2 - k_9 e_d, \\
\frac{de_0}{dt} &= -x_3 e_4 - k_{10} e_e, \\
\frac{df_1}{dt} &= x_3 \cos(2x_1 - 2x_5) e_4 + \sin(2x_1 - 2x_5) e_6 - k_{11} e_f, \\
\frac{dg_1}{dt} &= -\cos(2x_1 - 2x_5) e_4 - \frac{\sin(2x_1 - 2x_5)}{x_3} e_6 - k_{12} e_g, \\
\frac{dh_1}{dt} &= x_3 \cos^2 x_5 e_4 - \sin x_5 \cos x_5 e_6 - k_{13} e_h, \\
\frac{dj_1}{dt} &= \cos x_1 \cos x_5 e_4 - \frac{\cos x_1 \sin x_5}{x_3} e_6 - k_{14} e_j, \\
\frac{dk_1}{dt} &= x_2 \cos(x_1 - x_5) e_4 + \frac{x_2 \sin(x_1 - x_5)}{x_3} e_6 - k_{15} e_k,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dn_1}{dt} &= \sin 2x_1 \sin(x_1 - x_5)e_4 - k_{16}e_f, \\
\frac{dp_1}{dt} &= -\frac{\sin 2x_1}{x_3}e_6 - k_{17}e_p, \\
\frac{dq_1}{dt} &= e_4 - k_{18}e_q, \\
\frac{d\Delta_1}{dt} &= x_2^2 \cos(x_1 - x_5)e_4 + \frac{x_2^2 \sin(x_1 - x_5)}{x_3}e_6 - k_{19}e_\Delta, \\
\frac{d\omega_1}{dt} &= 2x_3x_6e_4 - \frac{2x_4}{x_3} - k_{20}e_\omega. \qquad (7)
\end{aligned}
$$

Now we have the following theorem which shows the stability and control performance of the adaptive control scheme:

Theorem 3.1 *For a given scaling function matrix*

$$A(t) = diag\ (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)),$$

where $\alpha_i(t) \neq 0$*,* $i = \overline{1,6}$*, and any initial conditions* $x_i(0), y_i(0)$*,* $i = \overline{1,6}$*, the adaptive control law (6) and parameter update law (7) warrant that the error functions* $e_i(t)$ *are asymptotically convergent to zero, i.e.* $\lim_{t\to\infty} ||e_i(t)|| = 0$*,* $i = \overline{1,6}$*.*

Proof. We choose a Lyapunov function as follows:

$$
\begin{aligned}
V &= \frac{1}{2}[e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_a^2 + e_b^2 + e_d^2 + e_e^2 + e_f^2 + \\
&\quad e_g^2 + e_h^2 + e_j^2 + e_k^2 + e_n^2 + e_p^2 + e_q^2 + e_\Delta^2 + e_\omega^2].
\end{aligned}
$$

We substitute the values of the controls u_i using adaptive control laws (6) into error dynamical system (5) and also note that for each uncertain parameter say, $a_1, \dot{e}_a = \dot{a}_1$ (where $(\cdot)$ represents differentiation with respect to t) and its value is given by the first equation of parameter update laws (7). Similarly, it follows for the other parameters. Using all these values, it can be shown that the time derivative of the Lyapunov function along the trajectory of the error system (5) is given by:

$$\frac{dV}{dt} = e^T\frac{de}{dt} = -e^TQe. \qquad (8)$$

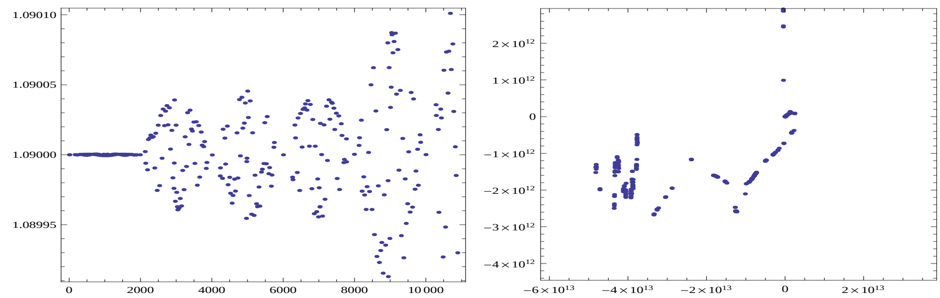
where $e = (e_1, e_2, e_3, e_4, e_5, e_6, e_a, e_b, e_d, e_e, e_f, e_g, e_h, e_j, e_k, e_n, e_p, e_q, e_\Delta, e_\omega)^T$ and $Q = \text{diag}\ (k_i; i = \overline{1,20})$.

Clearly, Q is a positive definite matrix and hence, $V(t)$ is negative definite. Based on the Lyapunov stability theory, the error dynamical system (5) is globally and asymptotically stable at the origin and we have $\lim_{t\to\infty} ||e_r(t)|| = 0; r = \overline{1,6}$. Thus, AHFPS between the master system (2) and slave system (3) is achieved. This proves the theorem. □

4 Numerical Simulation Results and Discussions

In this section, we verify and demonstrate the effectiveness of the proposed method by displaying and discussing the simulation results. We find by simulating that the system given by (2) shows chaotic behavior for the following sets of values : $a = 0, b =$

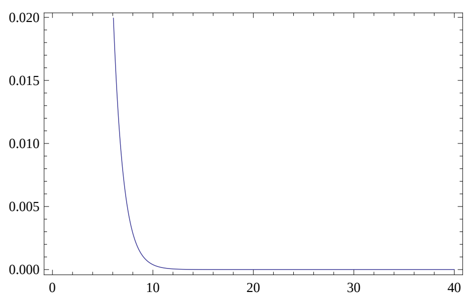
$10^{-10}, d = 10^{-11}, e = 10^{-6}, f = 5 \times 10^{-20}, g = 5 \times 10^{-21}, h = 0.03, j = 3. \times 10^{-9}, k = 2. \times 10^{-8}, n = 1.5 \times 10^{-10}, p = 0, q = 10^{-7}, \Delta = 0.0000001, \Omega = 0.1$ with initial conditions chosen as $x_1(0) = 0.8, x_2(0) = 1.09, x_3(0) = 0.8, x_4(0) = 1.9, x_5(0) = 0.8, x_6(0) = 1.9$. With these values, we take the resulting system as the master system (2) (see Figure 2(a)). Now, we take the initial values of the unknown estimated parameters as



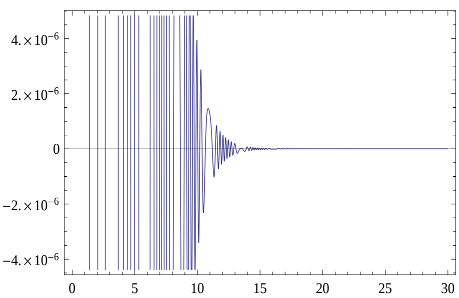
(a) Master system. (b) Slave system (without controls).

Figure 2: Poincare map showing chaotic master and slave systems.

$a_1(0) = -0.00136336, b_1(0) = 9. \times 10^{-7}, d_1(0) = 4.5 \times 10^{-6}, e_1(0) = 0.00130435, f_1(0) = 4.5 \times 10^{-11}, g_1(0) = 2.25 \times 10^{-10}, h_1(0) = 0.030603, j_1(0) = 3.0603 \times 10^{-6}, k_1(0) = 0.0000202, n_1(0) = 1.53015 \times 10^{-6}, p_1(0) = -1.36336 \times 10^{-7}, q_1(0) = 0.00652174, \Delta_1(0) = 0.0001, \Omega_1(0) = 0.101$ with initial conditions chosen as $y_1(0) = 1.3, y_2(0) = 0.5, y_3(0) = 0.8, y_4(0) = 3.01, y_5(0) = -0.8, y_6(0) = 1.1$. We find that when the system is considered with these values, without the controls, then the system again is chaotic. Thus, this is chosen as our slave system (3) which is to be controlled using the adaptive controllers $u_i(t); i = \overline{1,6}$ (see Figure 2(b)). Also, we choose the control gains as $k_i = 1; i = \overline{1,20}$. With these values, we now test AHFPS between systems (2) and (3). We can have numerous cases of AHFPS, to test, let us as an example, choose the scaling function matrix as $A(t) = diag\ (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (5\sin t - 6, 2, 5, 0.9e^{-t}, 1, 10)$. Clearly, $\alpha_i(t) \neq 0; i = \overline{1,6}$; for all t. Accordingly, the initial values of the error variables are: $e_1(0) = 8.6, e_2(0) = 0.09, e_3(0) = -3.2, e_4(0) = -0.809, e_5(0) = 1.6, e_6(0) = -9.1$.

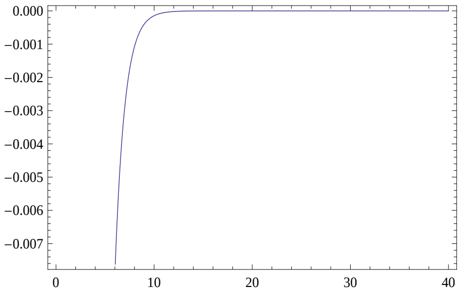


(a) **Figure 3**: Time Series Analysis of $e_1(t)$.

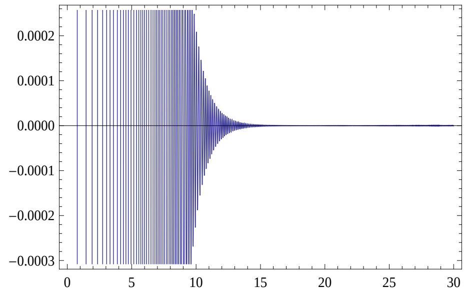


(b) **Figure 4**: Time Series Analysis of $e_2(t)$.

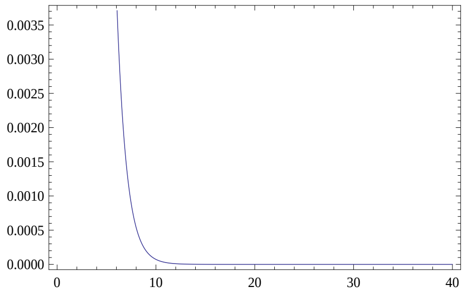
The time-evolution graphs of the error variables $e_i(t), i = \overline{1,6}$, are plotted in Figures 3 to 8 while time-evolution graphs of the estimated parameters $a_1, b_1, d_1, e_0, f_1, g_1, h_1, j_1, k_1, n_1, p_1, q_1, \Delta_1, \omega_1$ are presented in Figures 9 to 22. It is clear



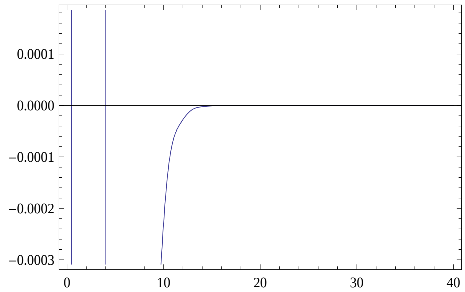
(c) **Figure 5**: Time Series Analysis of $e_3(t)$.



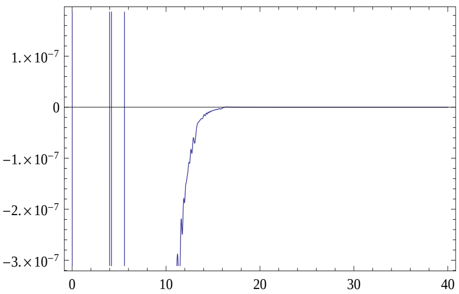
(d) **Figure 6**: Time Series Analysis of $e_4(t)$.



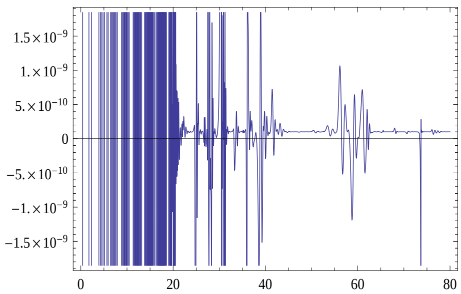
(e) **Figure 7**: Time Series Analysis of $e_5(t)$.



(f) **Figure 8**: Time Series Analysis of $e_6(t)$.



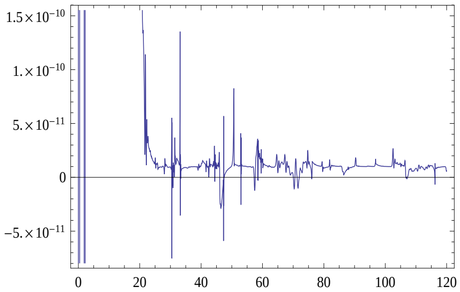
(g) **Figure 9**: Time Series Analysis of $a_1(t)(a_1 \to a = 0)$.



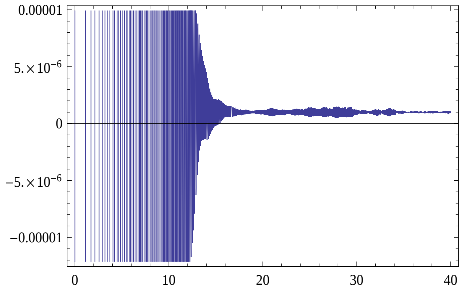
(h) **Figure 10**: Time Series Analysis of $b_1(t)(b_1 \to b = 10^{-10})$.

from time-evolution graphs of all error variables in Figures 3 to 8 that they converge to zero asymptotically while Figures 9 to 22 show that $a_1 \to a, b_1 \to b, d_1 \to d, e_0 \to e, f_1 \to f, g_1 \to g, h_1 \to h, j_1 \to j, k_1 \to k, n_1 \to n, p_1 \to p, q_1 \to q, \Delta_1 \to \Delta, \omega_1 \to \omega$, respectively. Hence parameter update law is verified. All these graphs together indicate the achievement of AHFPS between systems (2) and (3).

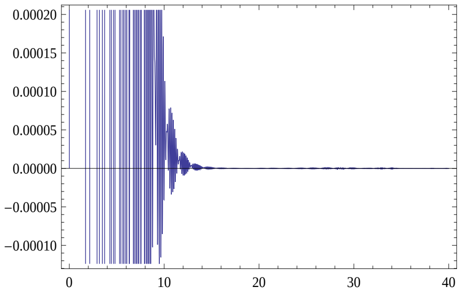
By choosing different scaling function matrices $A(t)$, we can obtain different synchronization phenomenon between the systems (2) and (3) as special cases:



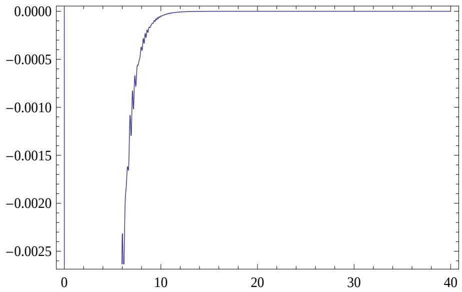
(i) **Figure 11**: Time Series Analysis of $d_1(t)(d_1 \to d = 10^{-11})$.



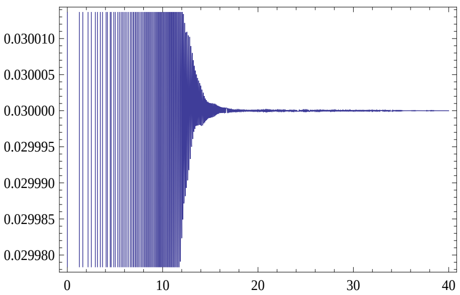
(j) **Figure 12**: Time Series Analysis of $e_0(t)(e_0 \to e = 10^{-6})$.



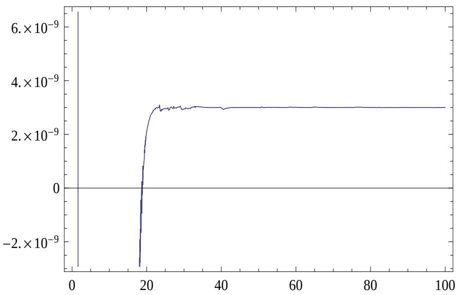
(k) **Figure 13**: Time Series Analysis of $f_1(t)(f_1 \to f = 5 \times 10^{-20})$.



(l) **Figure 14**: Time Series Analysis of $g_1(t)(g_1 \to g = 5 \times 10^{-21})$.



(m) **Figure 15**: Time Series Analysis of $h_1(t)(h_1 \to h = 0.03)$.



(n) **Figure 16**: Time Series Analysis of $j_1(t)(j_1 \to j = 3. \times 10^{-9})$.

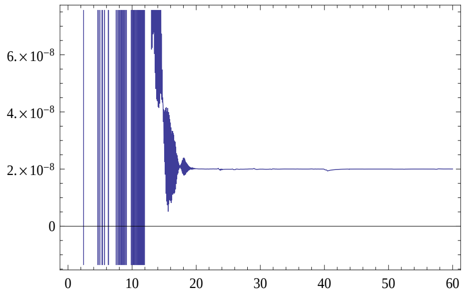
4.1 Complete Synchronization

We choose $A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (1, 1, 1, 1, 1, 1)$. Accordingly, the initial values of the error variables are: $e_1(0) = -0.5, e_2(0) = 0.59, e_3(0) = 0, e_4(0) = -1.11, e_5(0) = 1.6, e_6(0) = 0.8$.

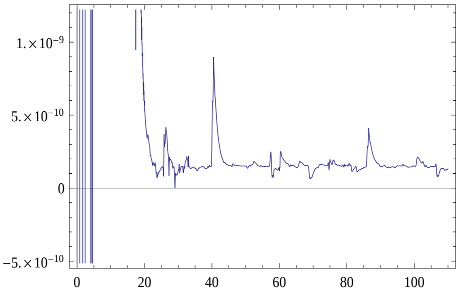
4.2 Antisynchronization

We choose

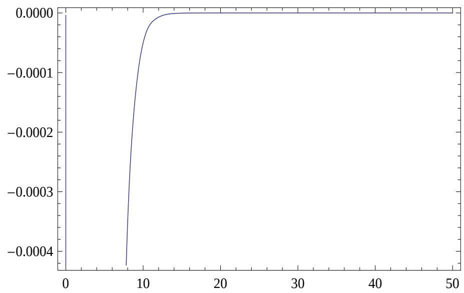
$$A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (-1, -1, -1, -1, -1, -1).$$



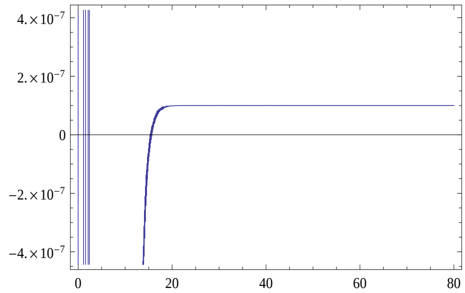
(o) **Figure 17**: Time Series Analysis of $k_1(t)(k_1 \to k = 2. \times 10^{-8})$.



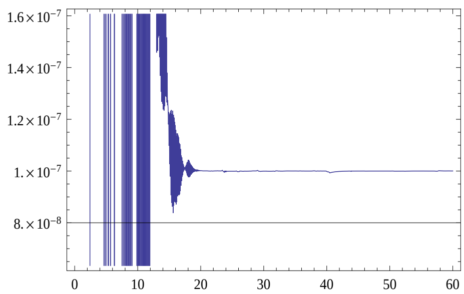
(p) **Figure 18**: Time Series Analysis of $n_1(t)(n_1 \to n = 1.5 \times 10^{-10})$.



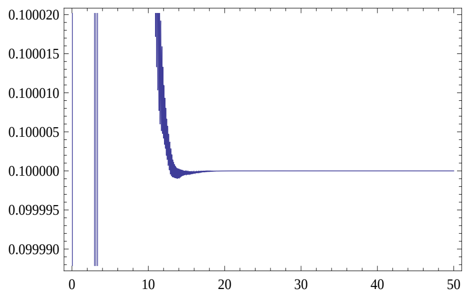
(q) **Figure 19**: Time Series Analysis of $p_1(t)(p_1 \to p = 0)$.



(r) **Figure 20**: Time Series Analysis of $q_1(t)(q_1 \to q = 10^{-7})$.



(s) **Figure 21**: Time Series Analysis of $\Delta_1(t)(\Delta_1 \to \Delta = 0.0000001)$.



(t) **Figure 22**: Time Series Analysis of $\omega_1(t)(\omega_1 \to \omega = 0.1)$.

Accordingly, the initial values of the error variables are: $e_1(0) = 2.1, e_2(0) = 1.59, e_3(0) = 1.6, e_4(0) = 4.91, e_5(0) = 0, e_6(0) = 3.0$.

4.3 Hybrid Projective Synchronization (HPS)

We can have numerous cases of HPS, as an example let us choose

$$A(t) = diag\,(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t), \alpha_5(t), \alpha_6(t)) = (1, 2, 5, 90, 10, 0.1).$$

Accordingly, the initial values of the error variables are: $e_1(0) = 0.7935, e_2(0) = 1.0875, e_3(0) = 0.796, e_4(0) = 1.88495, e_5(0) = 0.804, e_6(0) = 1.8945$. When the time-evolution graphs of $e_i(t); i = \overline{1,6}$ and the uncertain parameters a_1, b_1, d_1, $e_0, f_1, g_1, h_1, j_1, k_1, n_1$, $p_1, q_1, \Delta_1, \omega_1$ are plotted in each of the above cases, we find they are similar to those plotted in Figures 3 to 22. Clearly, then, complete synchronization, antisynchronization, hybrid projective synchronization, all can be achieved as special cases of AHFPS.

5 Conclusion

In this paper, we have presented an application of adaptive control technique in the field of celestial mechanics. The control method has been applied to two identical chaotic space-tether systems, where each system starts from different initial conditions and the response system contains uncertain parameters so that AHFPS is achieved between them. Based on Lyapunov stability theory, adaptive control laws and parameter update laws are designed to make the states between the drive and response systems synchronized asymptotically and they have also been used to estimate the uncertain time-varying parameters. Both theoretical analysis and numerical simulation confirm the effectiveness of our proposed method.

References

[1] Bajodah, Abdulrahman H. State Dependent Generalized Inversion-Based Liapunov Equation for Spacecraft Attitude Control. *Nonlinear Dynamics and Systems Theory* **9** (2) (2009) 147–160.

[2] Banerjee, S., Saha, P. and Chowdhury A.R. On the application of adaptive control and phase synchronization in non-linear fluid dynamics. *International Journal of Non-Linear Mechanics* **39** (1) (2004) 25–31.

[3] Bevilacqua, R., Romano, M. and Curti, F. Decoupled-natural-dynamics Model for the Relative Motion of two Spacecraft without and with $J2$ Perturbation. *Nonlinear Dynamics and Systems Theory* **10** (1) (2010) 11–20.

[4] Du, H.Y., Zeng, Q.S. and Wang, C.G. Function projective synchronization of different chaotic systems with uncertain parameters. *Physics Letters A* **372** (33) (2008) 5402–5410.

[5] Elabbasy, E.M., Agiza, H.N. and El-Dessoky, M.M. Adaptive synchronization of Lu system with uncertain parameters. *Chaos, Solitons, Fractals* **21** (2004) 657–667.

[6] Filali, R.L., Hammami, S., Benrejeb, M. and Borne, P. On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized H'enon Map. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 81–95.

[7] Han, X., Lu, J.A. and Wu, X. Adaptive feedback synchronization of Lu systems. *Chaos, Solitons, Fractals* **22** (2004) 221–227.

[8] Hramov, A.E. and Koronovskii A.A. Generalized synchronization: a modified system approach. *Physical Review E* **71** (6) (2005) Article ID 067201.

[9] Hwang, C.C., Hsieh, J.Y. and Lin, R.S. A linear continuous feedback control of Chuas circuit. *Chaos, Solitons, Fractals* **8** (1997) 1507–1515.

[10] Hu, M.F., Yang, Y.Q., Xu, Z.Y. and Guo, L.X. Hybrid projective synchronization in a chaotic complex nonlinear system. *Mathematics and Computers in Simulation* **79** (3) (2008) 449–457.

[11] Li, C.D., Liao, X.F. and Zhang, R. Impulsive synchronization of nonlinear coupled chaotic systems. *Physics Letters A* **328** (1) (2004) 47–50.

[12] Li, J.M. and Zhang, C.L. Hybrid Function Projective Synchronization of chaotic systems with uncertain time-varying parameters via Fourier-Series Expansion. *Int. Journal of Automation and Computing* **9** (4) (2012) 388–394.

[13] Liu, W.Q. Antiphase synchronization in coupled chaotic oscillators. *Physical Review E* **73** (5) (2006) Article ID 057203.

[14] Lu, J.H. and Lu, J.A. Controlling uncertain Lu system using linear feedback. *Chaos, Solitons, Fractals* **17** (2003) 127–133.

[15] Lu, J., Wu, X., Han, X. and Lu, J. Adaptive feedback synchronization of a unified chaotic system. *Phys. Lett. A* **329** (2009) 327–333.

[16] Lu, J., Zhou, T. and Zhang, S. Chaos synchronization between linearly coupled chaotic systems. *Chaos, Solitons, Fractals* **14** (2002) 529–541.

[17] Mao, W. and Eke, F.O. A Survey of the Dynamics and Control of Aircraft During Aerial Refueling. *Nonlinear Dynamics and Systems Theory* **8** (4) (2008) 375–388.

[18] Masoller, C. Anticipation in the synchronization of chaotic time-delay systems. *Physica A* **295** (1-2) (2001) 301–304.

[19] Ott, E., Grebogi, C. and Yorke, J.A. Controlling chaos. *Phys. Rev. Lett.* **64** (1990) 1196–1199.

[20] Park, J.H. Adaptive control for modified projective synchronization of a 4-D chaotic system with uncertain parameters. *Journal of Computational & Applied Mathematics* **213** (2008) 288–293.

[21] Park, J.H. Adaptive synchronization of a unified chaotic systems with an uncertain parameter. *Internat. J. Nonlinear Sci. Numer. Simulation* **6** (2) (2005) 201–206.

[22] Park, J.H. Stability criterion for synchronization of linearly coupled unified chaotic systems. *Chaos, Solitons, Fractals* **23** (2005) 1319–1325.

[23] Park, J.H. On synchronization of unified chaotic systems via nonlinear control. *Chaos, Solitons, Fractals* **25** (3) (2005) 699–704.

[24] Park, J.H. and Kwon, O.M. A novel criterion for delayed feedback control of time-delay chaotic systems. *Chaos, Solitons, Fractals* **23** (2005) 495–501.

[25] Pecora, L.M. , Carroll, T.L. Synchronization in chaotic systems. *Phys. Rev. Lett.* **64** (1990) 821–824.

[26] Shahverdiev, E.M., Sivaprakasam, S. and Shore, K.A. Lag synchronization in time-delayed systems. *Physics Letters A* **292** (6) (2002) 320–324.

[27] Vincent U.E. and Guo R. Adaptive Synchronization for Oscillators in ϕ^6 Potentials. *Nonlinear Dynamics and Systems Theory* **13** (1) (2013) 93–106.

[28] Wang, Y., Guan, Z.H. and Wang, H.O. Feedback an adaptive control for the synchronization of Chen system via a single variable. *Phys. Lett. A* **312** (2003) 34–40.

[29] Wu, X. and Lu, J. Parameter identification and backstepping control of uncertain Lu system. *Chaos, Solitons, Fractals* **18** (2003) 721–729.

[30] Xing, J. Adaptive Hybrid Function Projective Synchronization of chaotic systems with time-varying parameters. *Mathematical Problems in Engineering* (2012) Article ID 619708, 18 pages.

[31] Zhou, J.Y., Teo, K.L., Zhou, D. and Zhao, G.H. Optimal Guidance for Lunar Module Soft Landing. *Nonlinear Dynamics and Systems Theory* **10** (2010) 189–201.