Nonlinear Dynamics and Systems Theory, **14** (1) (2014) 30–43

# Stability of Dynamic Graph on Time Scales

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Received: October 15, 2012; Revised: January 23, 2014

**Abstract:** This paper is aimed at establishing stability conditions for a dynamic graph on a time scale in terms of the matrix Lyapunov function and the principle of comparison. Dynamic graphs on time scales are defined in a linear spaces as a one-parameter mapping of the space of graphs with $N$ nodes into itself. In the analysis of the dynamic graph this mapping is referred to as a motion of the corresponding dynamic graph. A notion of motion stability of a dynamic graph is introduced together with a notion of stability of an equilibrium adjacent matrix of dynamic graph. The dynamic graph on a time scale is considered for the first time and the necessity of introducing these objects is caused by the presence of a series of unsolved problems on stability of complex systems, whose subsystem interconnections are changing in time continuous-discrete mode. A method of matrix-valued function is proposed to solve the motion stability problem for the dynamic graph on a time scale. The essence of this method is that the problem on stability of an equilibrium graph of the given dynamic graph is replaced by a simpler problem on stability of the equilibrium state of a matrix equation. The application of the theory of dynamic graphs to the modeling of time-varying interconnections between subsystems of complex system of Lotka-Volterra type is proposed for the first time. A mathematical model is constructed in the form of a dynamic graph for the equilibrium adjacency matrix of which the existence conditions are established as well as the sufficient stability conditions.

**Keywords:** *dynamic graphs on time scales; matrix Lyapunov functions; comparison principle; Lotka-Volterra systems; stability.*

**Mathematics Subject Classification (2010):** 34A34, 34A40, 34D20, 39A13, 39A11.

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## 1 Introduction

The notion of a dynamic graph (not on a time scale) was introduced by D.D.Siljak (see [1] and bibliography therein). This notion was justified by the fact that it makes possible to present the effect of interconnections between subsystems of a complex system on its whole dynamics in a more precise way (see [3]). In a series of works that followed paper [1] (see [2] and bibliography therein) the idea of a dynamic graph for continuous complex system was extended for controlled and other systems.

This paper is aimed at establishing stability conditions for a dynamic graph on a time scale (see Bohner and Peterson [4] and bibliography therein) in terms of the matrix Lyapunov function and the principle of comparison (see [5] and bibliography therein). The paper is arranged as follows.

Section 1 presents a notion of dynamic graph as a one-parameter mapping of the space of graphs with N nodes into itself. In the analysis of the dynamic graph this mapping is referred to as a motion of the corresponding dynamic graph.

In Section 2 a notion of motion stability of a dynamic graph is introduced together with a notion of stability of an equilibrium adjacent matrix of dynamic graph. The latter is considered in the case when the properties of the dynamic graph are studied in terms of the adjacent matrix.

Section 3 deals with a partial case of the dynamic graph, i.e. the dynamic graph on a time scale. This type of dynamic graphs is considered for the first time and the necessity of introducing these objects is caused by the presence of a series of unsolved problems on stability of complex systems, whose subsystem interconnections are changing in time continuous-discrete mode.

In Section 4 a method of matrix-valued function is proposed to solve the motion stability problem for the dynamic graph on a time scale. The essence of this method is that the problem on stability of an equilibrium graph of the given dynamic graph is replaced by a simpler problem on stability of the equilibrium state of a matrix equation. The answer to the question when the solution of the second problem guarantees the solution of the first one is given in Section 5. Also, in this section the procedure of constructing an auxiliary equation is specified.

In Section 6 the application of the theory of dynamic graphs to the modeling of time-varying interconnections between subsystems of complex system of Lotka-Volterra type is proposed for the first time. A mathematical model is constructed in the form of a dynamic graph for the equilibrium adjacency matrix of which the existence conditions are established as well as the sufficient stability conditions.

## 2 The Description of a Dynamic Graph

Consider a weighted directed graph (later referred to as a graph) $D = (V, E)$ which is an ordered pair where $V$ is a nonempty finite set of $N$ nodes and $E$ is a set of the ribs of the graph. The nodes $(v_1, v_2, \dots, v_N)$ tie the ribs of the graph $(v_j, v_i)$ so that each rib is oriented from $v_j$ to $v_i$ at all $(i, j) \in \mathcal{N} = \{1, 2, \dots, N\}$. Each rib $(v_j, v_i)$ is put in correspondence with the weight $e_{ij}$, if the rib $(v_j, v_i) \in D$ while $e_{ij} = 0$ if $(v_j, v_i) \neq D$. Put the concept of isomorphism $N \times N$ of the matrix $E = (e_{ij})$ in correspondence with the digraph $D$. Later we will use this concept of isomorphism and the permutation of the symbols $D$ and $E$ as applied to the concerned situation.

Now define the space of graphs $D$ with the fixed number $N$ of nodes, as a linear space above the field $\mathcal{F}$ of real numbers. For any $D_1, D_2 \in \mathcal{D}$ there exists a single graph

$$D_1 + D_2 \in \mathcal{D}, \tag{1}$$

which is called a sum of graphs $D_1$ and $D_2$, and for any $D \in \mathcal{D}$ and an arbitrary number $\alpha \in \mathcal{F}$ there exists a single graph

$$\alpha D \in \mathcal{D}. \tag{2}$$

If in the formula (2) we assume $\alpha = 0$, then $\alpha D = 0$, which corresponds to the zero graph $D = 0 \in \mathcal{D}$. This graph consists of $N$ disconnected nodes, and therefore the matrix $E$ is empty.

The above operations defining $\mathcal{D}$ as a linear space can be interpreted in the context of a linear space $\mathcal{C}$ of adjacent matrices. For the two $N \times N$ matrices $E_1 = (e^1_{ij})$ and $E_2 = (e^2_{ij})$ the sum is

$$(e^1_{ij}) + (e^2_{ij}) = (e^1_{ij} + e^2_{ij}) \in \mathcal{C} \tag{3}$$

and for any $N \times N$ matrix $E = (e_{ij}) \in \mathcal{C}$ and a scalar quantity $\alpha \in \mathcal{F}$ obtain

$$\alpha e_{ij} = (\alpha e_{ij}) \in \mathcal{C}. \tag{4}$$

Note that the zero element of the space $\mathcal{C}$ is an $N \times N$ matrix $E = 0 \in \mathcal{C}$.

Now, in order to introduce the notion of the motion of the graph and its stability in the space $\mathcal{D}$, introduce the norm of the graph $\nu(D)$ with the following properties:

$$\begin{aligned} &\text{(a)}\ \ \nu(D) > 0 \ \text{ at all } \ D \in \mathcal{D}\,(D \neq 0);\\ &\text{(b)}\ \ \nu(\alpha D) = |\alpha|\nu(D) \ \text{ at all } \ D \in \mathcal{D} \ \text{ and } \ \alpha \in \mathcal{F};\\ &\text{(c)}\ \ \nu(D_1 + D_2) \le \nu(D_1) + \nu(D_2) \ \text{ at all } \ (D_1, D_2) \in \mathcal{D}. \end{aligned} \tag{5}$$

For the space of adjacent matrices $\mathcal{C}$ isomorphic to the space $\mathcal{D}$, consider the matrix norm $\nu\colon \mathbb{R}^{N\times N} \to \mathbb{R}_+$ in the space $\mathbb{R}^{N\times N}$ with the properties:

$$\begin{aligned} &\text{(a)}\ \ \nu(E) > 0 \ \text{ at all } \ E \in \mathbb{R}^{N\times N}\,(E \neq 0);\\ &\text{(b)}\ \ \nu(\alpha E) = |\alpha|\nu(E) \ \text{ at all } \ E \in \mathbb{R}^{N\times N} \ \text{ and at all } \ \alpha \in \mathcal{F};\\ &\text{(c)}\ \ \nu(E_1 + E_2) \le \nu(E_1) + \nu(E_2) \ \text{ at all } \ (E_1, E_2) \in \mathbb{R}^{N\times N}. \end{aligned} \tag{6}$$

Using these norms, introduce the metric in the space $\mathcal{D}$ by the formula

$$\rho(D_1, D_2) = \nu(D_1 - D_2) \ \text{ at all } \ (D_1, D_2) \in \mathcal{D}. \tag{7}$$

and in the matrix space $\mathcal{D}$ by the formula

$$\rho(E_1, E_2) = \nu(E_1 - E_2) \ \text{ at all } \ (E_1, E_2) \in \mathcal{C}. \tag{8}$$

Taking into account some of the results of the monograph [6], consider the axiomatic specification of a dynamic graph as a mapping of the abstract space $\mathcal{D}$ into itself.

Let the family of mappings $\Phi(t, \mathcal{D})$ in the space $\mathcal{D}$ for any $D \in \mathcal{D}$ and an arbitrary $t \in \mathbb{R}$ be put into correspondence with some graph $\Phi \in \mathcal{D}$.

**Definition 2.1** A dynamic graph $D$ is a one-parameter mapping $\Phi\colon \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ of the space $\mathcal{D}$ into itself, which satisfies the following axioms:

(a) $\Phi(t_0, D_0) = D_0$ at all $t_0 \in \mathbb{R}$ and at all $D_0 \in \mathcal{D}$;

(b) $\Phi(t, D)$ is continuous at all $t \in \mathbb{R}$ and at all $D \in \mathcal{D}$;

(c) $\Phi(t_2, \Phi(t_1, D)) = \Phi(t_1 + t_2, D)$ at all $(t_1, t_2) \in \mathbb{R}$ and at all $D \in \mathcal{D}$.

The axiom (a) establishes the fact of the existence of an initial graph $D(t_0) = D_0$. The axiom (b) specifies the continuity of the mapping $\Phi(t, D)$ with respect to all $t$ and all $D$, including $t_0$ and $D_0$. The axiom (b) determines that the dynamic graph is a one-parameter group of transformations of the space $\mathcal{D}$ into itself.

In applications of the theory of dynamic graphs the notion of an adjacent matrix plays a key role, therefore the introduction of such a notion is justified.

**Definition 2.2** A dynamic adjacent matrix $E$ is a one-parameter mapping $\Psi\colon \mathbb{R} \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ of the space $\mathbb{R}^{N\times N}$ into itself, satisfying the following axioms:

(a) $\Psi(t_0, E_0) = E_0$ at all $t_0 \in \mathbb{R}$ and at all $E_0 \in \mathbb{R}^{N\times N}$;

(b) the mapping $\Psi(t, E)$ is continuous at all $t \in \mathbb{R}$ and at all $E \in \mathbb{R}^{N\times N}$;

(c) $\Psi(t_2, \Psi(t_1, E)) = \Psi(t_1 + t_2, E)$ at all $(t_1, t_2) \in \mathbb{R}$ and at all $E \in \mathbb{R}^{N\times N}$.

In the process of the analysis of the dynamic graph $\Phi(t, D)$ the mapping is called the motion of the dynamic graph $D$, while $\Psi(t, E)$ is called the motion of the adjacent matrix $E$. The graph of stationary motion determined by the formula

$$\Phi(t, D^e) = D^e \quad \text{at all} \quad t \in \mathbb{R}. \tag{9}$$

is of interest. The graph $D^e$ will also be called the equilibrium graph.

Analogously, the adjusent equilibrium matrix is determined by the formula

$$\Psi(t, E^e) = E^e \quad \text{at all} \quad t \in \mathbb{R}. \tag{10}$$

Now consider the notion of stability (instability) of a dynamic graph, if a graph of stationary motion (equilibrium) is specified.

## 3 Setting of a Problem of Stability of a Dynamic Graph

The analysis of the form and the character of motions of a graph in the neighbourhood of an equilibrium graph or an equilibrium adjacent matrix is of interest, since this analysis allows to identify the conditions for the conservation in time of a certain structure of a complex system described by the specified graph. Introduce some definitions, taking into account the notion of stability in the Lyapunov sense and the two metrics: $\rho_0(\cdot, D^e)$ and $\rho(\cdot, D^e)$ for the characteristic of the initial and the current state of the dynamic graph.

**Definition 3.1** The equilibrium graph $D^e$ is called

(a) $(\rho_0, \rho)$-stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\Delta = \Delta(t_0, \epsilon) > 0$ such that the inequality

$$\rho_0(D_0, D^e) < \Delta \tag{11}$$

implies the estimate

$$\rho(D(t, D_0), D^e) < \epsilon \tag{12}$$

at all $t \geq t_0$;

(b) uniformly $(\rho_0, \rho)$-stable, if in the conditions of Definition 3.1 (a) the quantity $\Delta$ does not depend on $t_0 \in \mathbb{R}$;

(c) asymptotically $(\rho_0, \rho)$-stable, if it is $(\rho_0, \rho)$-stable and for any $t_0 \in \mathbb{R}$ there exists $\eta > 0$ such that at

$$\rho_0(D_0, D^e) < \eta \tag{13}$$

the following relation holds:

$$\lim_{t\to\infty} \rho(D(t, D_0), D^e) = 0; \tag{14}$$

(d) globally asymptotically $(\rho_0, \rho)$-stable if the conditions of Definition 3.1 (c) are satisfied at an arbitrary large $\eta$ and at all $D \in \mathcal{D}$;

(e) $(\rho_0, \rho)$-unstable if the conditions of Definition 3.1 (a) are not satisfied.

In the case when the properties of a dynamic graph are studied on the basis of an adjacent equilibrium matrix it makes sense to consider the following definition.

**Definition 3.2** An equilibrium adjacent matrix $E^e \in \ \in \mathbb{R}^{N\times N}$ is said to be:

(a) $(\rho_0, \rho)$-stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\Delta = \Delta(t_0, \epsilon) > 0$ such that the inequality

$$\rho_0(E_0, E^e) < \Delta \tag{15}$$

implies the estimate

$$\rho(E(t, E_0), E^e) < \epsilon \tag{16}$$

at all $t \geq t_0$;

(b) uniformly $(\rho_0, \rho)$-stable if all the conditions of Definition 3.2 (a) are satisfied with $\Delta$ not depending on $t_0 \in \mathbb{R}$;

(c) asymptotically $(\rho_0, \rho)$-stable if it is $(\rho_0, \rho)$-stable and for any $t_0 \in \mathbb{R}$ there exists $\zeta > 0$ such that at

$$\rho_0(E_0, E^e) < \zeta \tag{17}$$

the following relation holds:

$$\lim_{t\to\infty} \rho(E(t, E_0), E^e) = 0;$$

(d) globally asymptotically $(\rho_0, \rho)$-stable if the conditions of Definition 3.2 (c) are satisfied at an arbitrary fixed $\zeta$ and at any matrix $E_0 \in \mathbb{R}^{N\times N}$.

**Remark 3.1** Since for the selection of two measures some variants are admissible, Definitions 3.1 and 3.2 can have different interpretations. Let us dwell on some of them:

(1) let $D^e = 0$ and $\rho_0(t, \cdot) = \rho(t, \cdot) = \|D\|$, where $\|\cdot\|$ is an Euclidean norm. Then Definition 3.1 characterises the stability of a dynamic graph with respect to the zero graph;

(2) let $E^e = 0$ and $\rho_0(t, \cdot) = \rho(t, \cdot) = \|E\|$. Then Definition 3.2 characterises the stability of the dynamic adjacent matrix with respect to the zero adjacent matrix $E = 0 \in \mathcal{C}$.

## 4 The Evolution of a Dynamic Graph on a Time Scale

Let a time scale $\mathbb{T}$ with a graininess function $\mu(t) = \sigma(t) - t$, where $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ be specified. The function $\sigma(t)$ determines the operator of a jump forward $\sigma\colon \mathbb{T} \to \mathbb{T}$. Determine $\mathbb{T}^k$ by the formula $\mathbb{T}/\{M\}$, if $\mathbb{T}$ has a right scattered maximum $M$, and in the rest cases $\mathbb{T}^k = \mathbb{T}$ (see [5] and the bibliography therein).

**Definition 4.1** Fix $t \in \mathbb{T}^k$ and let $D\colon \mathbb{T} \to \mathcal{D}$. Determine some matrix $D^{\Delta}(t)$ (provided that it exists) with the following properties: for any $\epsilon > 0$ there exists a neighbourhood $W$ of a point $t$ for which

$$\|[D(\sigma(t)) - D(s)] - D^{\Delta}(t)[\sigma(t) - s]\| \le |\sigma(t) - s|$$

at all $s \in W$.

In this case we will say that $D^{\Delta}(t)$ is a delta derivative of the graph $D(t)$ in a point $t$.

The evolution of the dynamic graph on a time scale $\mathbb{T}$ will be described by the matrix equation

$$D^{\Delta}(t) = G(t, D), \quad D(t_0) = D_0 \in \mathcal{D}, \tag{18}$$

where $G\colon \mathbb{T}\times\mathcal{D} \to \mathcal{D}$. In terms of the dynamic matrix of adjacency $E(t)$ the equation (18) takes the form

$$E^{\Delta}(t) = F(t, E), \quad E(t_0) = E_0 \in \mathbb{R}^{N\times N}, \tag{19}$$

where $F\colon \mathbb{T} \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and $E^{\Delta} = \frac{dE}{dt}$ and the initial problem (19) becomes the initial problem for the matrix ordinary differential equation

$$\frac{dE}{dt} = F(t, E), \quad E(t_0) = E_0 \in \mathbb{R}^{N\times N}. \tag{20}$$

If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ and $E^{\Delta} = \Delta E(t) = E(t+1) - E(t)$ and the initial problem (19) becomes the initial problem for the matrix difference equation

$$E(t+1) - E(t) = F(t, E(t)), \quad E(t_0) = E_0 \in \mathbb{R}^{N\times N}. \tag{21}$$

The objective of the qualitative analysis of a dynamic graph is the study of the solutions of the matrix system of dynamic equations (19).

## 5 The Application of Matrix-Valued Functions Method in the Study of Stability

Now, connect with the system (19) the matrix-valued function $V(t, E)\colon \mathbb{T} \times \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ and its full dynamic derivative along the solutions of the system (19)

$$\begin{aligned} V^{\Delta}(t, E) &= V_t^{\Delta}(t, E(\sigma(t))) + \\ &\quad + \int_0^1 \dot{V}_E\big(t, E(t) + H\mu(t)E^{\Delta}(t)\big)dH\, E^{\Delta}(t) = \\ &= V_t^{\Delta}(t, E(\sigma(t))) + \\ &\quad + \int_0^1 \dot{V}_E\big(t, E(t) + H\mu(t)F(t, E(t))\big)dH\, F(t, E(t)), \end{aligned} \tag{22}$$

where $V_t^\Delta$ is calculated as a $\Delta$-derivative of the matrix- valued function $V(t,E)$ with respect to $t$ in compliance with Definition 5.4.5, and $\dot V_E$ is a partial derivative of the matrix- valued function $V(t,E)$ with respect to the matrix argument $E \in \mathbb{R}^{N\times N}$.

Assume that for the expression (22) there exists a matrix- valued function $G(t, V(t,E))$ such that

$$V^\Delta(t,E)|_{(19)} \le G(t, V(t,E)). \tag{23}$$

Along with the matrix inequality (23) consider the matrix equation

$$M^\Delta(t) = G(t, M(t)), \quad M(t_0) = M_0 \in \mathbb{R}^{N\times N}, \tag{24}$$

where $M(t) = U(t, E(t))$, $E(t) = E(t; t_0, E_0)$ at all $t \in \mathbb{T}$.

Now introduce some notions and definitions for the dynamic equations (19) and (24).

Assume that for the system (19) a time scale $\mathbb{T}$ with the graininess function $\mu(t)$ is chosen. Let $X_1 = \mathbb{R}^{N\times N}$ and $A_1 \subset X_1$ be the space of initial data $E_0$, such that $E(t_0; t_0, E_0) = E_0 \in A_1$. Denote $S_E$ which is a family of motions of the dynamic graph on the time scale $\mathbb{T}$.

Then the sequence of sets and spaces $\{\mathbb{T}, X_1, A_1, I, S_E\}$ determines the evolution of the dynamic graph on a time scale.

Analogously, for the system (24) keep the time scale $\mathbb{T}$ with the same graininess function $\mu(t)$ and denote $X_2 = \mathbb{R}^{N\times N}$, $A_2 \subset X_2$ is a space of initial values $M_0$ such that $M(t_0; t_0, M_0) = M_0 \in A_2$. Let $S_M$ be a family of motions of the matrix system (24).

Then the sequence $\{\mathbb{T}, X_2, A_2, I, S_M\}$ determines the evolution of the matrix dynamic equation (24) on a time scale.

Let the sets $N_1 \subset X_1$ and $N_2 \subset X_2$ be invariant with respect to the families of motions $S_E$ and $S_M$ respectively.

By the matrix mapping $U\colon \mathbb{T}\times X_1 \to X_2$ connect the sets $N_2$ and $N_1$ by the relation

$$\begin{aligned} N_2 = U(\mathbb{T}\times N_1) = \{M \in X_2\colon M = U(t_*, E_1) \\ \text{for some } \ E_1 \subset N_1 \ \text{ and } \ t^* \in \mathbb{T}\}. \end{aligned} \tag{25}$$

The family of motions $S_M$ of the system (24) and the family of motions $S_E$ of the dynamic graph (19) will be connected by the relation

$$S_m = \mathcal{M}(S_E), \tag{26}$$

where $\mathcal{M}(S_E) = \{M(\cdot; t_0, B) : M(t; t_0, B) = U(t, E(t; t_0, A))$ for any $E(t; t_0, A) \in S_E$, $B = U(t_0, A)$, $A \in A_1$ and $t_0 \in \mathbb{T}\}$.

It seems interesting to obtain conditions under which the dynamic properties of the pairs $(S_M, N_2)$ and $(S_E, N_1)$ would be equivalent.

Note that the systems (19) and (24) are determined in the same space of variables $\mathbb{R}^{N\times N}$, but the system (24), in view of its construction according to the inequality (23), can prove to be more traceable compared with the initial system (19).

## 6 The Comparison Principle

Before we start obtaining the conditions for the stability of the system of dynamic equations (24), formulate a lemma determining the connection between the dynamic properties of the pairs $(S_M, N_2)$ and $(S_E, N_1)$. Let $\nu_1(E, N_1)$ be a metric in a space $X_1$ and $\nu_2(U(t,E), N_2)$ be a metric in a space $X_2$.

The function $\psi\colon [0, r_1] \to \mathbb{R}_+$ (respectively $\psi\colon [0, \infty] \to \mathbb{R}_+$) belongs to the Hahn class if $\psi(0) = 0$ and $\psi(r)$ is strictly increasing over $[0, r_1]$ (on $\mathbb{R}_+$). Functions of this class play the part of comparison functions in the theory of stability of motion.

**Lemma 6.1** *Assume that evolutions of the systems* (19) *and* (24) *are determined and there exists a matrix-valued function* $U\colon \mathbb{T} \times X_1 \to X_2$*, such that:*

(a) *the sets of motions* $S_M$ *and* $S_E$ *are connected by the relation* (26)*;*

(b) *the sets* $V_1$ *and* $N_2$ *are closed and connected by the relation* (25)*;*

(c) *there exist comparison functions* $\psi_1, \psi_2 \in K$*-class, such that*

$$\psi_1(\nu_1(E, N_1)) \le \nu_2(U(t, E), N_2) \le \psi_2(\nu_1(E, N_1))$$

*at all* $t \in \mathbb{T}$ *and* $E \in \mathbb{R}^{N\times N}$*.*

*Then the following statements hold:*

(a) *the invariance of the pair* $(S_E, N_1)$ *implies the invariance of the pair* $(S_M, N_2)$*;*

(b) *the stability of a certain type of the pair* $(S_M, N_2)$ *implies the stability of the same type of the pair* $(S_E, N_1)$*;*

(c) *the exponential stability of the pair* $(S_M, N_2)$ *implies the exponential stability of the pair* $(S_E, N_1)$ *if the comparison functions have the form* $\psi_i(r) = a_i r^{b_0}$*,* $a_i > 0$*,* $b_0 > 0$*,* $i = 1, 2$*.*

***Proof.*** Consider the statement (b) and assume that the pair $(S_M, N_2)$ is stable. Here for any $\epsilon_2 > 0$ and any $t_0 \in \mathbb{T}$ one can find $\Delta_2 = \Delta_2(\epsilon_2, t_0) > 0$ such that $\nu_2(M(t; t_0, B), N_2) < \epsilon_2$ at all $M(\cdot; t_0, B) \in S_M$ and at all $t \in T(B, t_0) \subset \mathbb{T}$ as soon as $\nu_2(B, N_2) < \Delta_2$.

To prove the stability of the pair $(S_E, N_1)$ for an arbitrary $\epsilon > 0$ and $t_0 \in \mathbb{T}$ choose $\epsilon_2 = \psi_1(\epsilon)$ and $\Delta = \psi_2^{-1}(\Delta_2)$. If $\nu_1(A, N_1) < \Delta$, then, according to the condition (c) of Lemma 6.1 obtain $\nu_2(B, N_2) \le \psi_2(\nu_1(A, N_1)) < \psi_2(\Delta) = \Delta_2$. It means that for any solution $M(t, t_0, B) \in S_M$ the estimate $\nu_2(M(t; t_0, B)) < \epsilon_2$ is true at all $t \in T(B, t_0)$. From the conditions (a), (b) of Lemma 6.1 obtain that $E(\cdot; t_0, A) \in N_1$ at all $t \in T(A, t_0) = T(B, t_0)$, where $B = U(t_0, A)$. From the condition (c) of Lemma 6.1 it follows that

$$\begin{aligned}\nu_1(E(t; t_0, A), N_1) &\le \psi^{-1}(U(t, E(t; t_0, A)), N_2) = \\ &= \psi^{-1}(\nu_2(M(t; t_0, B), N_2) \le \psi^{-1}(\epsilon_2) = \epsilon,\end{aligned}$$

at all $t \in T(A, t_0) = T(B, t_0)$ as soon as $\nu_1(A, N_1) < \Delta$. The statement (b) is proved.

The proof of the other statements of the comparison principle is performed in a similar way. □

To obtain the sufficient conditions for the stability of a dynamic graph on the basis of the analysis of the system (24) define concretely the choice of the matrix-valued function $U(t, E)$ and the matrix of the function $G(t, U)$ in the inequality (23).

Let

$$U(t, E) = EE^T \text{ and } G(t, U) = AU, \tag{27}$$

where $A$ is an $N\times N$-constant matrix, and $E\in\mathbb{R}^{N\times N}$.

Taking into account the relation

$$E(\sigma(t)) = E(t) + \mu(t)E^{\Delta}(t)$$

on a time scale $\mathbb{T}$ with the graininess $\mu(t)$, obtain

$$U^{\Delta}(E(t)) = EF^T(t,E) + F(t,E)E^T + \mu(t)F(t,E)F^T(t,E). \tag{28}$$

Taking into account (28), the inequality (23) takes the form

$$U^{\Delta}(E(t))|_{(19)} \le AU(E(t)) \tag{29}$$

at all $t\in\mathbb{T}$, and the matrix comparison equation (24)

$$M^{\Delta}(t) = AM(t), \quad M(t_0) = M_0 \in \mathbb{R}^{N\times N} \tag{30}$$

is linear.

## 7 Applications

From the analysis of the literature on complex systems [1, 3], mathematical biology [7] etc., it becomes clear, that complex systems with the time-varying interaction between subsystems have not been researched. Indeed, in the literature complex systems are described by the system of differential equations:

$$\frac{dx_i}{dt} = g_i(t,x_i) + h_i(t, e_{i1}x_1, e_{i2}x_2, ..., e_{iN}x_N), \quad i = 1,2,...,N. \tag{31}$$

where equations

$$\frac{dx_i}{dt} = g_i(t,x_i), \quad i = 1,2,...,N,$$

describe motion of the disconnected subsystems. Functions $h_i$ describe action of all subsystems of the complex system on the $i$-th subsystem. Parameter $e_{ik}$ replies for the action of the $k$-th subsystem on the $i$-th one and $e_{ik}$ is *constant*. So, *the actual problem is to construct the mathematical model and research the complex systems with time-varying interconnection between their subsystems*.

Since interconnection matrix $E = [e_{ij}]_{i,j=1}^N$ in the complex system (31) may be considered as an adjacent matrix of some graph $G = (V,\mathcal{E})$, where $V = \{V_1, V_2, ..., V_N\}$ is a nonempty finite set of $N$ nodes and $\mathcal{E} = \{(V_i,V_j)|V_i,V_j\in V, i,j=\overline{1,N}\}$ is a set of ribs, then the earlier mentioned problem is to construct the example of complex systems, in which interconnections between subsystems would assign some *time-varying* or, perhaps, *dynamic graph* [6].

Following the setting problem, consider the generalization of the well-known in mathematical biology and ecology Volterra model of the community of $n$ species. The generalized system is described by the system of dynamic equations on some time scale $\mathbb{T}$:

$$N_i^{\Delta}(t) = N_i\Big(\varepsilon_i - \sum_{j=1}^{n}\gamma_{ij}N_j\Big), \quad i = 1,2,...,n, \tag{32}$$

where $N_i(t)$ is a number of individuals of the $i$-th species at the moment $t\in\mathbb{T}$, $N_i^{\Delta}(t)$ is a delta derivative of the function $N_i(t)$ in a point $t\in\mathbb{T}$. In the case when $\mathbb{T}=\mathbb{R}$

(when the number of the species changes quickly enough, such scales are considered; communities of bacteria are an example), $N_i^\Delta(t) = \frac{dN_i}{dt}$. Such a case is considered in [7]. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$ (when the number of the species changes over long periods of time such scales are considered; communities of higher animals are an example), then $N_i^\Delta(t) = \Delta N_i(t) = N_i(t+h) - N_i(t)$. When the number of species changes with the different intensity on the different time intervals, the scale with inconstant graininess function $\mu(t)$ ($\mu(t) \equiv 0$, when $\mathbb{T} = \mathbb{R}$, and $\mu(t) \equiv h$, when $\mathbb{T} = h\mathbb{Z}$) can be applied to such species dynamics modelling. The intensity can be affected, for example, by habitat conditions (climate, geography, forage base, etc.)

In addition, in (32) $\varepsilon_i$ denotes a rate of natural growth or mortality of the $i$-th species in the absence of other species. The sign and the absolute value of $\gamma_{ij}$ $(i \neq j)$ represent the nature and intensity of influence of the $j$-th species to $i$-th; $\gamma_{ii}$ is an indicator of infraspecific competition.

We assume now, that $n$ species whose dynamics are described by the system (32), are the preys and identify interconnections in a community of $m$ species, where the individuals are predators, feeding on individuals of preys.

Denote by $S_k$ $(k = 1, 2, ..., m)$ the set of those $n$ species of the preys community, which form the forage base of the $k$-th species of the predator community. Also define $N(S_k)$ by the formula:

$$N(S_k) = \sum_{i \in S_k} N_i,$$

that is, $N(S_k)$ is equal to the volume of the $k$-th predator's forage base. Predator's community dynamics can be described by the system (32):

$$M_i^\Delta(t) = M_i\Big(\alpha_i - \sum_{j=1}^{m} \beta_{ij} M_j\Big), \quad i = 1, 2, ..., m, \tag{33}$$

where $M_i(t)$ is a number of individuals of the $i$-th species at the moment $t \in \mathbb{T}$, $M_i^\Delta(t)$ is a delta derivative of the function $M_i(t)$. Also in (33) $\alpha_i$ denotes a rate of natural growth or mortality of the $i$-th species in the absence of other species, and $\beta_{ij}$ represent the nature and intensity of influence of the $j$-th species to the $i$-th. In this case, it seem natural to assume that the effect of the $j$-th to the $i$-th is dependent on percentage of the species, forming the mutual forage base, in the $j$-th species forage base. That is:

$$\beta_{ij} = \beta_{ij}\Big(\frac{N(S_i \cap S_j)}{N(S_j)}\Big).$$

The more large the ratio $\frac{N(S_i \cap S_j)}{N(S_j)}$ is (the interval $[0, 1]$ is the range of the ratio), the larger the $j$-th species makes bids for the mutual with the $i$-th species forage base, thereby affecting on the $i$-th species of community of the predators.

So, we have constructed an example of the complex system, that is described by the system of equations

$$M_i^\Delta(t) = M_i\Big(\alpha_i - \sum_{j=1}^{m} \beta_{ij}\Big(\frac{N(S_i \cap S_j)}{N(S_j)}\Big) M_j\Big), \quad i = 1, 2, ..., m, \tag{34}$$

and the interconnections between the subsystems are described by the system of equations (32).

So, adjacent matrix $E(t) = [e_{ij}]_{i,j=1}^m$ of some dynamic graph $\mathfrak{G}$ is constructed. The matrix satisfies the following system of equations:

$$\begin{aligned} E(t) &= B\Big(\frac{N(S_i \cap S_j)}{N(S_j)}\Big), \\ N_i^{\Delta}(t) &= N_i\Big(\varepsilon_i - \sum_{j=1}^{n} \gamma_{ij} N_j\Big), \quad i = 1, 2, ..., n, \end{aligned} \tag{35}$$

Let us consider the particular case when the functions $\beta_{ij}$ are linear:

$$\beta_{ij}\Big(\frac{N(S_i \cap S_j)}{N(S_j)}\Big) = Q_{ij}\frac{N(S_i \cap S_j)}{N(S_j)}.$$

Let the community of the preys consist of the 3 species $z_1, z_2, z_3$, and the community of the predators consist of 2 species. Suppose that the forage base $S_1$ of the first species of the predators is $\{z_1, z_2\}$, and the forage base $S_2$ of the second species of the predators is $\{z_2, z_2\}$. Then the interconnections parameters $\beta_{ij}$ satisfy the following relations:

$$\begin{aligned} \beta_{11} = Q_{11}, \quad \beta_{12} &= Q_{12}\frac{N_2}{N_2 + N_3}, \\ \beta_{21} = Q_{21}\frac{N_2}{N_1 + N_2}, \quad \beta_{22} &= Q_{22}, \\ N_i^{\Delta}(t) = N_i\Big(\varepsilon_i - \sum_{j=1}^{3} \gamma_{ij} N_j\Big), &\quad i = 1, 2, 3. \end{aligned} \tag{36}$$

The equations (36) describe the evolution of a dynamic graph, consisting of two preys. The value $\beta_{ij}(t)$, as it was mentioned, denotes the weight of the edge $(V_i, V_j)$.

For the dynamic graph $\mathfrak{G}$, which is represented by equations (36), consider the problem of existence of the adjacent equilibrium matrix and of its stability in terms of Definition 3.2.

As we see from the formula (36), the value of the adjacent equilibrium matrix $E^e$ is assigned by the equilibrium state of the system of dynamic equations on the time scale (32). That is, adjacent equilibrium matrix $E^e$ equals

$$E^e = \begin{pmatrix} Q_{11} & \beta_{12}^e \\ \beta_{21}^e & Q_{22} \end{pmatrix}$$

if and only if components $N_i^e$ $(i = 1, 2, 3)$ of the equilibrium vector of the system (32) satisfies the system of equations:

$$\begin{cases} N_i\Big(\varepsilon_i - \sum\limits_{j=1}^{3} \gamma_{ij} N_j\Big) = 0, \quad i = 1, 2, 3, \\ \frac{Q_{12} N_2}{N_2 + N_3} = \beta_{12}^e, \\ \frac{Q_{21} N_2}{N_1 + N_2} = \beta_{21}^e. \end{cases} \tag{37}$$

Suppose now, that the adjacent matrix equals to $E^e = E^*$ and let $N^* = (N_1^*, N_2^*, N_3^*)^T$ be a corresponding state vector of the system (32) (that is, the solution of the system (37)). Establish the stability conditions of the state $N^*$. It is easy to see, that stability conditions of the state $N^*$ of the system (32) are also stability conditions of the equilibrium matrix $E^e = E^*$. In the system (32) replace the value $N_i$ to $x_i$ by the formula:

$$x_i = N_i - N_i^*, \quad i = 1, 2, 3, \tag{38}$$

to obtain stability conditions. We obtain the system of dynamic equations

$$\begin{aligned} x_i^\Delta = N_i^\Delta = (x_i + N_i^*)\Big(\varepsilon_i - \sum_{j=1}^{3}\gamma_{ij}(x_j + N_j^*)\Big) = \\ = \Big(x_i\Big(\varepsilon_i - \sum_{j=1}^{3}\gamma_{ij}N_j^*\Big) - \sum_{j=1}^{3}N_i^*\gamma_{ij}x_j\Big) - \sum_{j=1}^{3}\gamma_{ij}x_ix_j, \quad i = 1, 2, 3, \end{aligned} \tag{39}$$

and

$$\begin{cases} x_1^\Delta = \Big(\varepsilon_1 - \sum\limits_{j=1}^{3}\gamma_{1j}N_j^* - N_1^*\gamma_{11}\Big)x_1 - N_1^*\gamma_{12}x_2 - N_1^*\gamma_{13}x_3 - \sum\limits_{j=1}^{3}\gamma_{1j}x_1x_j, \\ x_2^\Delta = -N_2^*\gamma_{21}x_1 + \Big(\varepsilon_2 - \sum\limits_{j=1}^{3}\gamma_{2j}N_j^* - N_2^*\gamma_{22}\Big)x_2 - N_2^*\gamma_{23}x_3 - \sum\limits_{j=1}^{3}\gamma_{2j}x_2x_j, \\ x_3^\Delta = -N_3^*\gamma_{31}x_1 - N_3^*\gamma_{32}x_2 + \Big(\varepsilon_3 - \sum\limits_{j=1}^{3}\gamma_{3j}N_j^* - N_3^*\gamma_{33}\Big)x_3 - \sum\limits_{j=1}^{3}\gamma_{3j}x_3x_j. \end{cases} \tag{40}$$

Denoting

$$x = (x_1, x_2, x_3)^T,$$

$$A = \begin{pmatrix} \varepsilon_1 - \sum\limits_{j=1}^{3}\gamma_{1j}N_j^* - N_1^*\gamma_{11} & -N_1^*\gamma_{12} & -N_1^*\gamma_{13} \\ -N_2^*\gamma_{21} & \varepsilon_2 - \sum\limits_{j=1}^{3}\gamma_{2j}N_j^* - N_2^*\gamma_{22} & -N_2^*\gamma_{23} \\ -N_3^*\gamma_{31} & -N_3^*\gamma_{32} & \varepsilon_3 - \sum\limits_{j=1}^{3}\gamma_{3j}N_j^* - N_3^*\gamma_{33} \end{pmatrix},$$

$$F(x) = (F_1(x), F_2(x), F_3(x))^T, \quad F_i(x) = -\sum_{j=1}^{3}\gamma_{ij}x_ix_j,$$

we obtain the vector form of the system (40):

$$x^\Delta = Ax + F(x), \tag{41}$$

with the conditions

$$\lim_{\|x\|\to 0}\|F(x)\| = 0. \tag{42}$$

Now the stability conditions of the equilibrium state $N^*$ of the system (32) are the stability conditions of the trivial equilibrium of the system (41), which can be obtained by the generalized Lyapunov's direct method [5]. According to the method, consider the positive definite function:

$$v(x) = x^Tx = x_1^2 + x_2^2 + x_3^2,$$

and compute the total $\Delta$-derivative of $v(x)$ with respect to the solutions of the system (41). Using the product rule (see [5]), we find:

$$\begin{aligned} v^{\Delta}\Big|_{(41)} &= \left(x^{\Delta}\right)^T x^{\sigma} + x^T x^{\Delta}\Big|_{(41)} = \left(x^{\Delta}\right)^T \left(x + \mu(t)x^{\Delta}\right) + x^T x^{\Delta}\Big|_{(41)} = \\ &= (Ax + F(x))^T (x + \mu(t)(Ax + F(x))) + x^T (Ax + F(x))\Big|_{(41)} = \\ &= x^T (A^T + A + \mu(t)A^T A)x + \Psi(\mu(t), x) = x^T (A^T \oplus A)x + \Psi(\mu(t), x), \end{aligned} \tag{43}$$

where

$$\Psi(\mu(t), x) = F^T(x)x + x^T F(x) + \mu(t)(x^T A^T F(x) + F^T(x)Ax + F^T(x)F(x)).$$

Here we have used a symbol of regressive sum: $A^T \oplus A = A^T + A + \mu(t)A^T A$.

Now if there exists the negative definite matrix $B \in \mathbb{R}^{3\times3}$ such that inequality:

$$x^T (A^T \oplus A)x \le x^T Bx, \quad \forall t \in \mathbb{T}, \quad \forall x \in D \subseteq \mathbb{R}^3, \tag{44}$$

holds, then the equilibrium state $x = 0$ is stable by Theorem 3.3.2 from [5]. Indeed, conditions (1), (2) and (2b) for the function $v(x)$ hold. From (43) and (44) we obtain:

$$v^{\Delta}\Big|_{(41)} \le x^T Bx + \Psi(\mu(t), x),$$

where the function $\Psi(\mu(t), x)$ satisfies the inequality:

$$\|\Psi(\mu(t), x)\| \le 2\|F(x)\|\|x\|(1 + \mu(t)\|A\|).$$

Using the equality (42), we compute

$$\lim_{\|x\|\to0} \frac{\|\Psi(\mu(t), x)\|}{\|x\|} \le \lim_{\|x\|\to0} 2\|F(x)\|(1 + \mu(t)\|A\|) = 0.$$

That is, conditions (2b) and (2c) of Theorem 3.3.2 hold, therefore by Theorem 3.3.2 the equilibrium state $x = 0$ of the system (41) is asymptotically stable which implies the asymptotical stability of the state $N = N^*$ of the system (32).

So, in the case when the system (37) can be solved with respect to $N_1$, $N_2$ and $N_3$, there exists the equilibrium matrix

$$E^e = \begin{pmatrix} Q_{11} & \beta^e_{12} \\ \beta^e_{21} & Q_{22} \end{pmatrix},$$

which is asymptotically stable, when (44) holds.

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