Nonlinear Dynamics and Systems Theory, **14** (1) (2014) 11–29

Approximations of Solutions for a Sobolev Type Fractional Order Differential Equation

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Received: March 11, 2013; Revised: January 21, 2014

Abstract: In this paper, using semigroup theory and Banach fixed point theorem, we establish the existence and uniqueness of approximate solutions of nonlinear Sobolev type fractional order evolution equation in a separable Hilbert space. Also, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results.

Keywords: *analytic semigroup; approximate solution; fractional differential equation; Faedo-Galerkin approximation; Sobolev type evolution equation.*

Mathematics Subject Classification (2010): 34G20, 35R10, 47D06, 47N20.

1 Introduction

In recent few decades, researchers have developed great interest in fractional calculus due to its wide applicability in science and engineering. Tools of fractional calculus have been available and applicable to deal with many physical and real world problems such as anomalous diffusion process, traffic flow, nonlinear oscillation of earthquake, real system characterized by power laws, critical phenomena, scale free process, description of viscoelastic materials and many others. For more details about fractional calculus we refer to [3–5, 7, 10, 12, 13, 16, 18].

In the present paper, we study the convergence of the Faedo-Galerkin approximations of solutions to the nonlinear fractional order Sobolev type evolution equation

$$\begin{aligned} \frac{d^q}{dt^q}[u(t)+g(t,u(t))]+Au(t) &= f(t,u(t)),\ 0<t\leq T\leq\infty,\ 0<q\leq 1,\\ u(0) &= \phi, \end{aligned} \tag{1}$$

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in a separable Hilbert space $(H, \|\cdot\|, (\cdot,\cdot))$, where A is a closed linear operator defined on $D(A)$ which is dense in H. We assume that linear operator $-A$ is the infinitesimal generator of analytic semigroup $\{S(t);\ t \geq 0\}$ in H. The functions f and g are continuous functions and satisfy certain assumptions stated later in Section 2.

The Feado-Galerkin approximations of solutions of the particular case of (1) in which $g = 0$, have been established by Muslim [9]. Author in [9] has discussed the convergence of Feado-Galerkin approximation of the solution to the equation

$$\frac{d^\beta}{dt^\beta}u(t) + Au(t) = f(t, u(t)),\ t \in [0, T],\ \beta \in (0, 1), \tag{2}$$

$$u(0) = \phi. \tag{3}$$

under the assumption that $-A$ generates an analytic semigroup of bounded linear operators defined on a Banach space H and f satisfies certain conditions.

The existence and uniqueness of solution and approximation of solution of functional differential equation

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(t))] &= -Au(t) + f(t, u(t)),\quad t > 0, \\ u(0) &= \phi, \end{aligned} \tag{4}$$

have been discussed by D. Bahuguna and Reeta in [2] with the assumption that $-A$ generates an analytic semigroup and f and g satisfy the conditions such that f and $A^\beta g$ satisfy the Lipschitz condition on $C([0, T] \times D(A^\alpha); H)$.

This paper is organized as follows: we present some basic definitions, lemmas, theorems and assumptions required to establish the convergence result as preliminaries in Section 2. The existence and uniqueness of the approximate solutions are proved using semigroup theory and fixed point theorem in Section 3. In Section 4, we prove the convergence of the solution to each of the approximate integral equations with the limiting function which satisfies the associated integral equation and the convergence of the approximate Feado-Galerkin solutions will be shown in Section 5. In the last section we consider an example as an application.

2 Preliminaries and Assumptions

In this section we provide some basic definitions, results and assumptions on f and g which will be used in the later sections.

Definition 2.1 The fractional derivative of $f : [0, \infty) \to \mathbb{R}$ in the Caputo sense of order q is defined as

$${}^cD_t^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^m(s)ds, \tag{5}$$

for $m - 1 \leq q < m,\ m \in N,\ t > 0$, with the following property:

$${}^cD_t^q f(t) = D_t^q[f(t) - \sum_{k=0}^{m-1} f^k(0)g_{k+1}(t)], \tag{6}$$

where D_t^q denotes the Riemann-Liouville fractional derivative of order q defined as

$$D_t^q f(t) = \frac{d^m}{dt^m}\frac{1}{\Gamma(m-q)}\int_0^t (t-s)^{m-q-1} f(s)ds, \quad t>0, \quad m-1<q<m \tag{7}$$

Definition 2.2 [14]. A function $u \in C([0,T];H)$ is said to be a mild solution of equation (1) if it satisfies

$$\begin{aligned} u(t) &= S_q(t)(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g(s,u(s))ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,u(s))ds, \ t \in [0,T], \\ u(0) &= \phi, \end{aligned} \tag{8}$$

where

$$S_q(t) = \int_0^\infty \zeta_q(\theta) S(t^q\theta)d\theta, \quad T_q(t) = q\int_0^\infty \theta\zeta_q(\theta)S(t^q\theta)d\theta.$$

Here $\zeta_q(\theta)$ is a probability density function defined on the interval $(0,\infty)$, satisfying the following properties

- $\zeta_q(\theta) \geq 0, \ \theta \in (0,\infty)$ and $\int_0^\infty \zeta_q(\theta)d\theta = 1$;
- $\zeta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \times \psi_q(\theta^{-1/q}) \geq 0,$ where

$$\psi_q(\theta) = \frac{1}{\pi}\Sigma_{n=1}^\infty (-1)^{n-1}\theta^{-nq-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \ \theta \in (0,\infty).$$

Now, we consider some assumptions on A, f and g.

Assumptions on A: We assume that linear operator A satisfies the following conditions.

(A1) A is a closed, positive, self-adjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H. We assume that A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots,$$

where $\lambda_m \to \infty$ as $m \to \infty$ and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e. $Au_i = \lambda_i u_i$ and $< u_i, u_j >= \delta_{ij}$, where δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

These assumptions on A imply that $-A$ generates an analytic semigroup, therefore there exist constants $M \geq 1$ and $\delta \geq 0$ such that

$$\|S(t)\| \leq M\ e^{-\delta t}, \quad t \geq 0.$$

So $-A$ is an infinitesimal generator of analytic semigroup. We assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e. $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ denotes the resolvent set of $-A$. If required, for $c > 0$

large enough, we may add cI to A, then $-(A+cI)$ is invertible and generates a bounded analytic semigroup. Also for $t>0$, we have

$$\|AS(t)\| \leq Mt^{-1}, \tag{9}$$
$$\|A^{\alpha}S(t)\| \leq M_{\alpha}t^{-\alpha}. \tag{10}$$

The set of all continuous functions from $[0,T]$ into X, denoted by $C_T = C([0,T];X)$ is a Banach space under the supremum norm given by

$$\|\psi\|_T = \sup_{0\leq t\leq T} \|\psi(t)\|,\ \psi \in C_T.$$

Also, it can be shown easily that $C_T^{\alpha} = X^{\alpha}(T) = C([0,T];D(A^{\alpha}))$ is a Banach space endowed with the supremum norm

$$\|\psi\|_{T,\alpha} = \sup_{0\leq t\leq T} \|\psi(t)\|_{\alpha},\ \psi \in C_T^{\alpha}.$$

It follows that A^{α}, $0\leq\alpha\leq 1$, can be defined as a closed linear invertible operator with domain $D(A^{\alpha})$ which is dense in H. $D(A^{\omega}) \hookrightarrow D(A^{\alpha})$, for $0<\alpha<\omega$ such that embedding is continuous. Also, it can be easily shown that $D(A^{\alpha})$ is a Banach space with norm $\|x\| = \|A^{\alpha}x\|$ and this norm is equivalent to the graph norm of A^{α}. For more details on the fractional powers of closed linear operator, we refer to Pazy [10].

Assumptions on f and g: We list the following assumptions on f and g:

(A2) The nonlinear map $f:[0,T]\times D(A^{\alpha}) \to H$ satisfies a local Lipschitz-like condition

$$\|f(t,x)-f(t,y)\| \leq F_R(t)\|x-y\|_{T,\,\alpha} \tag{11}$$

and

$$\|f(t,x)\| \leq F_R(t), \tag{12}$$

for all $t\in[0,T],\ x,\ y\in B_R(X^{\alpha}(T),\phi)$, where $B_R(X^{\alpha}(T),\phi) := \{u\in X^{\alpha}(T) : \|u-\phi\|_{T,\,\alpha}\leq R\}$, and $F_R(t):\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing function depending on R.

(A3) For $(t,x)\in[0,T]\times D(A^{\alpha})$, there exist positive constants L and β, $0<\alpha<\beta<1$ such that the function $A^{\beta}g$ is a continuous function satisfying the following conditions

$$\|A^{\beta}g(t,x)-A^{\beta}g(s,y)\| \leq L\{\|t-s\|^{\gamma}+\|x-y\|_{T,\,\alpha}\} \tag{13}$$

and

$$L\|A^{\alpha-\beta}\| \leq 1, \tag{14}$$

for all t$\in[0,T]$, $\gamma\in(0,1]$ and $x,y\in B_R(X^{\alpha}(T),\phi)$, L is a constant.

Lemma 2.1 [Zhou and Jiao *[14]*] *For any fixed $t\geq 0$, $S_q(t)$ and $T_q(t)$ are bounded linear operators such that $\|S_q(t)x\|\leq M\|x\|$, $\|T_q(t)x\|\leq \frac{qM}{\Gamma(1+q)}\|x\|$ and $\|A^{\alpha}T_q(t)x\|\leq \frac{qM_{\alpha}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}t^{-q\alpha}$ for all $x\in D(A^{\alpha})$, where M is a constant such that $\|S(t)\|\leq M$, for all $t\in[0,T]$.*

3 Existence and Uniqueness

In this section, we establish the existence and uniqueness of the solution to every approximate integral equations of (1) by using Banach fixed point theorem.

Let H_n denote the finite dimensional subspace of the Hilbert space H which is spanned by $\{u_0, u_1, \cdots, u_n\}$ and let $P^n : H \to H_n$ for $n = 1, 2, \cdots$, be the corresponding projection operators. Let $0 < T_0 \leq T < \infty$ be arbitrary but fixed constant chosen is such a way that

$$B = \max_{\{0 \leq t \leq T_0\}} \|A^\beta g(t, \phi)\|, \tag{15}$$

$$\|(S_q(t) - I)A^\alpha(\phi + g_n(0, \phi))\| \leq \frac{(1-\varsigma)R}{3}, \tag{16}$$

$$\|A^{\alpha-\beta}\| L T_0^\gamma + M_{1+\alpha-\beta} C_1 (L\widetilde{R} + B) \frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} + M_\alpha F_R(T) C_2 \frac{T_0^{q(1-\alpha)}}{(1-\alpha)} < (1-\varsigma)\frac{R}{6}, \tag{17}$$

$$M_{1+\alpha-\beta} L C_1 \frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} + M_\alpha F_R(T) C_2 \frac{T_0^{q(1-\alpha)}}{(1-\alpha)} < 1 - \varsigma, \tag{18}$$

where $L\|A^{\alpha-\beta}\| = \varsigma < 1$, $\widetilde{R} = \sqrt{R^2 + ||\phi_\alpha^2||}$, $C_1 = \frac{\Gamma\{1-(\alpha-\beta)\}}{\Gamma\{1+q(\beta-\alpha)\}}$, $C_2 = \frac{\Gamma(2-\alpha)}{\Gamma\{1+q(1-\alpha)\}}$.

We define

$$g_n : [0, T] \times D(A^\alpha) \to H, \text{ such that } g_n(t, u(t)) = g(t, P^n u(t)) \tag{19}$$

and

$$f_n : [0, T] \times D(A^\alpha) \to H, \text{ such that } f_n(t, u(t)) = f(t, P^n u(t)), \tag{20}$$

for each n.

Now, we consider a map Q_n on $B_R(X^\alpha(T_0), \phi)$ defined by

$$\begin{aligned} Q_n(u)(t) &= S_q(t)(\phi + g_n(0, \phi)) - g_n(t, (t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g_n(s, u(s)) ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) f_n(s, u(s)) ds, \qquad t \in [0, T_0], \end{aligned} \tag{21}$$

for each $n = 0, 1, 2, \cdots$.

Theorem 3.1 *Let the assumptions (A1)-(A3) hold. Then there exists a constant T_0, $0 < T_0 < T$ and a unique fixed point $u_n \in B_R(X^\alpha(T_0), \phi)$ of the operator Q_n for all n i.e. u_n satisfies the approximate integral equations*

$$\begin{aligned} u_n(t) &= S_q(t)(\phi + g_n(0, \phi)) - g_n(t, u_n(t)) + \int_0^t \frac{A T_q(t-s) g_n(s, u_n(s))}{(t-s)^{1-q}} ds \\ &\quad + \int_0^t \frac{T_q(t-s) f_n(s, u_n(s))}{(t-s)^{1-q}} ds, \qquad t \in [0, T_0], \end{aligned} \tag{22}$$

for each $n = 0, 1, 2, \cdots$.

Proof. First we prove the continuity of the map $t \to Q_n u(t)$ from $[0, T_0]$ into $D(A^\alpha)$ with respect to norm $\|\cdot\|_\alpha$. For any $u \in B_R(X^\alpha(T_0), \phi)$ and $t_1,\ t_2 \in [0, T_0]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&A^\alpha[(Q_n u)t_2 - (Q_n u)t_1] \\
&= A^\alpha[(S_q(t_2) - S_q(t_1))(\phi + g(0, \phi))] \\
&\quad -A^{\alpha-\beta}[A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)] \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} T_q(t_2 - s) A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))]ds \\
&\quad + \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] T_q(t_2 - s) A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))]ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{q-1}[T_q(t_2 - s) - T_q(t_1 - s)] A^{1+\alpha-\beta}[A^\beta g_n(s, u(s))]ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{q-1} A^\alpha T_q(t_2 - s) f_n(s, u(s))ds \\
&\quad + \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] A^\alpha T_q(t_2 - s) f_n(s, u(s))ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{q-1} A^\alpha [T_q(t_2 - s) - T_q(t_1 - s)] f_n(s, u(s))ds, \\
&= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8.
\end{aligned}$$

Hence, we have

$$\|(Q_n u)t_2 - (Q_n u)t_1\| \leq \sum_{i=1}^{8} \|K_i\|. \tag{23}$$

We have

$$\begin{aligned}
K_1 &= A^\alpha[(S_q(t_2) - S_q(t_1))(\phi + g(0, \phi))], \\
&= \int_0^\infty \zeta_q(\theta)[\int_{t_1}^{t_2} q\theta t^{q-1} A^\alpha S(t^\beta \theta) A(\phi + g(0, \phi))dt]d\theta,
\end{aligned}$$

taking norm on both the sides, we get (see [7, p. 101] and [8, p. 437])

$$\begin{aligned}
\|K_1\| &\leq \int_0^\infty \zeta_q(\theta)[\int_{t_1}^{t_2} q\theta t^{q-1} \|A^\alpha S(t^\beta \theta)\| \|A(\phi + g(0, \phi))\|dt]d\theta, \\
&\leq M_\alpha \int_0^\infty \theta^{1-\alpha} \zeta_q(\theta) \int_{t_1}^{t_2} t^{q(1-\alpha)-1} \|A(\phi + g(0, \phi))\|dtd\theta, \\
&\leq \frac{C_2 M_\alpha}{(1-\alpha)} \|A(\phi + g(0, \phi))\|(t_2^{q(1-\alpha)} - t_1^{q(1-\alpha)}), \\
&\leq C_2 M_\alpha q \|A(\phi + g(0, \phi))\|(t_1 + \kappa(t_2 - t_1))^{q(1-\alpha)-1}(t_2 - t_1), \\
&\leq C_2 M_\alpha q \|A(\phi + g(0, \phi))\| \kappa^{q(1-\alpha)-1}(t_2 - t_1)^{q(1-\alpha)},
\end{aligned} \tag{24}$$

and

$$\|K_2\| \leq \|A^{\alpha-\beta}\| \|A^\beta g_n(t_2, u) - A^\beta g_n(t_1, u)\| \leq L\|A^{\alpha-\beta}\|(t_1 - t_2)^\gamma. \tag{25}$$

Further, we have

$$\begin{aligned} \|K_3\| &\leq C_1 q M_{1+\alpha-\beta} \int_{t_1}^{t_2} (t_2-s)^{\alpha(\beta-\alpha)-1} \|A^\beta g_n(s,u(s))\| ds, \\ &\leq C_1 M_{1+\alpha-\beta} [(L\widetilde{R}+B)] \frac{(t_2-t_1)^{q(\beta-\alpha)}}{(\beta-\alpha)}, \end{aligned} \tag{26}$$

and

$$K_4 = \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] A^{1+\alpha-\beta} T_q(t_2-s) A^\beta g_n(s,u(s)) ds.$$

Taking norm on both the sides, we get

$$\begin{aligned} \|K_4\| &\leq C_1 q M_{1+\alpha-\beta} \int_0^{t_1} \{(t_1-s)^{-q(1+\alpha-\beta)} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] \\ &\quad \times \|A^\beta g_n(s,u(s))\|\} ds, \\ &\leq C_1 q M_{1+\alpha-\beta} [(L\widetilde{R}+B)] \\ &\quad \times \int_0^{t_1} (t_1-s)^{-q(1+\alpha-\beta)} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] ds, \\ &\leq C_1 q M_{1+\alpha-\beta} [(L\widetilde{R}+B)] \int_0^{t_1} (t_1-s)^{\lambda-1} [(t_2-s)^{-\lambda\mu} - (t_1-s)^{-\lambda\mu}] ds, \end{aligned}$$

where $\lambda = 1 - q(1+\alpha-\beta)$, $\mu = \frac{q-1}{1-q(1+\alpha-\beta)}$ (see Muslim, [8] and El-Borai [9]).

Hence, after some calculations we get

$$\|K_4\| \leq C_1 q M_{1+\alpha-\beta} [(L\widetilde{R}+B)] \mu \delta^{\mu-1} (1-b)^{-\lambda(1-\mu)-1} (t_2-t_1)^{\lambda(1-\mu)}, \tag{27}$$

where $b = (1 - (\frac{\mu}{\lambda})^{\frac{1}{\lambda\mu}})$ and $0 \leq \delta \leq 1$.

Similarly,

$$\|K_5\| \leq C_1 q M_{1+\alpha-\beta} [(L\widetilde{R}+B)] \mu_1 \delta_1^{\mu_1-1} (1-b_1)^{-q(1-\mu_1)-1} (t_2-t_1)^{q(1-\mu_1)}, \tag{28}$$

where $\mu_1 = 1+\alpha-\beta$, $b_1 = (1 - (\frac{\mu_1}{q})^{\frac{1}{q\mu_1}})$ and $0 \leq \delta_1 \leq 1$ (see [8,9]).

$$\begin{aligned} \|K_6\| &\leq \int_{t_1}^{t_2} (t_2-s)^{q-1} \|A^\alpha T_q(t_2-s)\| \|f_n(s,u(s))\| ds, \\ &\leq F_R(T) M_\alpha C_2 \frac{(t_2-t_1)^{q(1-\alpha)}}{(1-\alpha)}. \end{aligned} \tag{29}$$

Also, we notice that

$$\begin{aligned} \|K_7\| &\leq \int_0^{t_1} ((t_2-s)^{q-1} - (t_1-s)^{q-1}) \|A^\alpha T_q(t_2-s)\| \|f_n(s,u(s))\| ds, \\ &\leq M_\alpha C_2 q \int_0^{t_1} ((t_2-s)^{q-1} - (t_1-s)^{q-1}) (t_1-s)^{-q\alpha} \|f_n(s,u(s))\| ds, \\ &\leq M_\alpha C_2 F_R(T) q \int_0^{t_1} ((t_2-s)^{-\mu_2\lambda_1'} - (t_1-s)^{-\mu_2\lambda_1'}) (t_1-s)^{\lambda_1'-1} ds, \\ &\leq M_\alpha C_2 q F_R(T) \mu_2 \delta_2^{\mu_2-1} (1-b_2)^{-\lambda_1'(1-\mu_2)-1} (t_2-t_1)^{\lambda_1'(1-\mu_2)}, \end{aligned} \tag{30}$$

where $\lambda_1^{'} = 1 - q\alpha,\ \mu_2 = \frac{1-q}{1-q\alpha},\ b_2 = (1 - (\frac{\mu_2}{\lambda_1^{'}})^{\frac{1}{\mu_2\lambda_1^{'}}}),\ 0 \le \delta_2 \le 1$, and

$$\begin{aligned}\|K_8\| &\le \int_0^{t_1}(t_1-s)^{q-1}\|A^\alpha[T_q(t_2-s)-T_q(t_1-s)]\|\|f_n(s,u(s))\|ds,\\ &\le C_2qM_\alpha F_R(T)\int_0^{t_1}(t_1-s)^{q-1}[(t_2-s)^{-q\alpha}-(t_1-s)^{-q\alpha}]ds,\\ &\le C_2qM_\alpha F_R(T)\alpha\delta_3^{\alpha-1}(1-b_3)^{-q(1-\alpha)-1}(t_2-t_1)^{q(1-\alpha)}, \qquad (31)\end{aligned}$$

where $b_3 = (1 - (\frac{\alpha}{q})^{\frac{1}{q\alpha}})$ and $0 \le \theta_3 \le 1$. Using (24)-(31) in (23), we get that (Q_nu) is Hölder continuous on $[0, T_0]$. Hence the continuity of the map $t \to (Q_nu)(t)$ is proved. Next we show that $Q_n(B_R(X^\alpha(T_0), \phi)) \subseteq B_R(X^\alpha(T_0), \phi)$. For any element $u \in B_R(X^\alpha(T_0), \phi)$, we have

$$\begin{aligned}\|(Q_nu)(t)-\phi\|_\alpha &\le \|(S_q(t)-I)A^\alpha(\phi+g_n(0,\phi))\|\\ &\quad+\|A^{\alpha-\beta}\|\|A^\beta g_n(0,\phi)-A^\beta g_n(t,u(t))\|\\ &\quad+\int_0^t(t-s)^{q-1}\|A^{1+\alpha-\beta}T_q(t-s)\|\|A^\beta g_n(s,u(s))\|ds\\ &\quad+\int_0^t(t-s)^{q-1}\|T_q(t-s)\|_\alpha\|f_n(s,u(s))\|ds,\\ &\le \|(S_q(t)-I)A^\alpha(\phi+g_n(0,\phi))\|\\ &\quad+\|A^{\alpha-\beta}\|\|A^\beta g_n(0,\phi)-A^\beta g_n(t,u(t))\|\\ &\quad+M_{1+\alpha-\beta}C_1q\int_0^t(t-s)^{q(\beta-\alpha)-1}\|A^\beta g_n(s,u(s))\|ds\\ &\quad+M_\alpha C_2q\int_0^t(t-s)^{q(1-\alpha)-1}\|f_n(s,u(s))\|ds,\\ &\le \|(S_q(t)-I)(\phi+g_n(0,\phi))\|+\|A^{\alpha-\beta}\|L\{T_0^\gamma+\|u(t)-\phi\|\}\\ &\quad+M_{1+\alpha-\beta}C_1\{(L\widetilde{R}+B)\}\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)}\\ &\quad+M_\alpha C_2F_R(T)\frac{T_0^{q(1-\alpha)}}{(1-\alpha)}.\\ &\le R.\end{aligned}$$

Taking supremum over $[0, T_0]$, we get

$$||(Q_nu)-\phi||_{T_0,\,\alpha} \le R. \qquad (32)$$

This implies that $Q_n(B_R(X^\alpha(T_0), \phi)) \subseteq B_R(X^\alpha(T_0), \phi)$.

In the next step, our aim is to show that Q_n is a strict contraction mapping on $B_R(X^\alpha(T_0), \phi)$. Let for all $t \in [0, T_0]$ and $u_1,\ u_2 \in B_R(X^\alpha(T_0), \phi)$, we have

$$\begin{aligned}&\|(Q_nu_1)(t)-(Q_nu_2)(t)\|_\alpha \le \|A^{\alpha-\beta}\|\|A^\beta g_n(t,u_1)-A^\beta g_n(t,u_2)\|\\ &+\int_0^t(t-s)^{q-1}\|A^{1+\alpha-\beta}T_q(t-s)\|[\|A^\beta g_n(s,u_1(s))-A^\beta g_n(s,u_2(s))\|]ds\\ &+\int_0^t(t-s)^{q-1}\|A^\alpha T_q(t-s)\|\|f_n(s,u_1(s))-f_n(s,u_2(s))\|ds, \qquad (33)\end{aligned}$$

From the assumptions $(A2)$ - $(A3)$, we have

$$\|A^{\beta}g_n(t,u_1) - A^{\beta}g_n(t,u_2)\| \leq L\|u_1(t) - u_2(t)\|_{\alpha} \leq L\|u_1 - u_2\|_{T_0,\,\alpha}, \tag{34}$$

$$\|f_n(s,u_1) - f_n(s,u_2)\| \leq F_R(T)\|u_1(s) - u_2(s)\|_{\alpha} \leq F_R(T)\|u_1 - u_2\|_{T_0,\,\alpha}. \tag{35}$$

Using inequalities (34) and (35) in (33), we get

$$\begin{aligned} \|Q_nu_1(t) - Q_nu_2(t)\|_{\alpha} \quad &\leq \quad [\|A^{\alpha-\beta}\|\; L + M_{1+\alpha-\beta}LC_1\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} \\ &\quad + M_{\alpha}F_R(T)C_2\frac{T_0^{q(1-\alpha)}}{(1-\alpha)}]\|u_1(t) - u_2(t)\|_{\alpha}. \end{aligned} \tag{36}$$

Taking supremum over $[0, T_0]$, we get

$$\begin{aligned} \|Q_nu_1 - Q_nu_2\|_{T_0,\,\alpha} \quad &\leq \quad [\|A^{\alpha-\beta}\|\; L + M_{1+\alpha-\beta}LC_1\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)} \\ &+ \quad M_{\alpha}F_R(T)C_2\frac{T_0^{q(1-\alpha)}}{(1-\alpha)}]\|u_1 - u_2\|_{T_0,\,\alpha}. \end{aligned} \tag{37}$$

We use (15)–(18) in the inequality (37) and get that Q_n is a strict contraction on $B_R(X^{\alpha}(T_0), \phi)$. Hence, by the fixed point theorem, there exists a unique $u_n \in B_R(X^{\alpha}(T_0), \phi)$ such that $Q_nu_n = u_n$. which implies that u_n satisfies the integral equation (22) for each $n = 1, 2, \cdots$. This completes the proof of the theorem.

Lemma 3.1 *Suppose that assumptions* $(A1) - -A(3)$ *are satisfied. If* $\phi \in D(A^{\alpha})$, *where* $0 < \alpha < 1$, *then* $u_n(t) \in D(A^{\upsilon})$ *for all* $t \in (0, T_0]$ *with* $0 \leq \upsilon < 1$. *Furthermore, if* $\phi \in D(A)$ *then* $u_n(t) \in D(A^{\upsilon})$ *for all* $t \in [0, T_0]$ *with* $0 \leq \upsilon < 1$.

From Theorem 3.1, we have that there exists a unique $u_n \in B_R(X^{\alpha}(T_0), \phi)$ such that u_n satisfies the equation (22). Theorem 2.6.13 in Pazy [10] implies that $T(t) : H \to D(A^{\upsilon})$ for $t > 0$ and $0 \leq \upsilon < 1$ and for $0 \leq \upsilon \leq \eta < 1$, $D(A^{\eta}) \subseteq D(A^{\upsilon})$. From the assumption $(A3)$ we have that the map $t \mapsto A^{\beta}g(t, u_n(t))$ is Hölder continuous on $[0, T_0]$ with the exponent $\rho = min\{\gamma, \upsilon\}$. It is easy to see that Hölder continuity of u_n can be established using the similar arguments from equation (23), (30)-(31). From Theorem 4.3.2 in Pazy [10], we have

$$\int_0^t (t-s)^{q-1}T_q(t-s)A^{\beta}g_n(s,u_n)ds \in D(A).$$

Also from Theorem 1.2.4 in Pazy [10], we have that $T(t)x \in D(A)$ if $x \in D(A)$. The result follows from these facts and the fact that $D(A) \subseteq D(A^{\upsilon})$ for $0 \leq \upsilon \leq 1$.

Corollary 3.1 *Suppose that* $(A1)$, $(A2)$ *and* $(A3)$ *are satisfied. If* $\phi \in D(A^{\alpha})$ *with* $0 < \alpha < 1$, *then for any* $t_0 \in (0, T_0]$ *there exists a constant* U_{t_0} *such that*

$$\|A^{\upsilon}u_n(t)\| \leq U_{t_0}$$

for all $t_0 \leq t \leq T_0$ *independent of* n, *where* $0 < \alpha < \upsilon < \beta$.

Proof. Let us assume that $\phi \in D(A^\alpha)$. Applying A^υ on both the sides of (22) and using (9)–(10) for $t \in [t_0, T_0]$ and $\alpha < \upsilon < \beta$, we obtain

$$\begin{aligned}
\|u_n(t)\|_\upsilon &\leq \|A^\upsilon S_q(t)(\phi + g_n(0,\phi))\| + \|A^{\upsilon-\beta}\|\|A^\beta g_n(t,u_n)\| \\
&\quad + \int_0^t (t-s)^{q-1}\|A^{1+\upsilon-\beta}T_q(t-s)\|\|A^\beta g_n(s,u_n)\|ds \\
&\quad + \int_0^t (t-s)^{q-1}\|A^\upsilon T_q(t-s)\|\|f_n(s,u_n)\|ds, \\
&\leq M_\upsilon t_0^{-q\upsilon}(\|\phi\| + \|g_n(0,\phi)\|) + \|A^{\upsilon-\beta}\|[(L\widetilde{R} + B)] \\
&\quad + M_{1+\upsilon-\beta}(L\widetilde{R} + B)C_3\frac{T_0^{q(\beta-\upsilon)}}{(\beta-\upsilon)} + M_\upsilon C_4 F_R(T)\frac{T_0^{q(1-\upsilon)}}{(1-\upsilon)} \\
&\leq U_{t_0},
\end{aligned}$$

where $C_3 = \frac{\Gamma(1-\upsilon+\beta)}{\Gamma 1+q(-\upsilon+\beta)}$, $C_4 = \frac{\Gamma(2-\upsilon)}{\Gamma 1+q(1-\upsilon)}$.

Again, for $t \in [0, T_0]$ and $0 < \upsilon \leq \alpha$, $\phi \in D(A^\upsilon)$ and

$$\begin{aligned}
\|u_n(t)\|_\upsilon &\leq M(\|A^\upsilon\phi\| + \|g_n(0,\phi)\|_\upsilon) + \|A^{\upsilon-\beta}\|[LR + B] \\
&\quad + M_{1+\upsilon-\beta}(L\widetilde{R} + B)C_3\frac{T_0^{q(\beta-\upsilon)}}{(\beta-\upsilon)} + M_\upsilon C_4 F_R(T)\frac{T_0^{q(1-\upsilon)}}{(1-\upsilon)} \\
&\leq U_{t_0}.
\end{aligned}$$

Furthermore, we have if $\phi \in D(A^\beta)$ then $\phi \in D(A^\upsilon)$ for $0 < \upsilon \leq \beta$ and required result can be proved easily.

4 Convergence of Solution

In this section we will show the convergence of the solution $u_n \in X^\alpha(T_0)$ of the approximate integral equations (22) to a unique solution $u(\cdot) \in X^\alpha(T_0)$ of the equation (8).

Theorem 4.1 *Let the assumptions (A1)–(A3) hold. If $\phi \in D(A^\alpha)$, then for any $t_0 \in (0, T_0]$,*

$$\lim_{n\to\infty} \sup_{\{n\geq m,\ t_0\leq t\leq T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof. For $n \geq m$, we have

$$\begin{aligned}
A^\alpha[u_n(t) - u_m(t)] &= S_q(t)A^\alpha(g_n(0,\phi) - g_m(0,\phi)) \\
&\quad + \int_0^t (t-s)^{q-1} \times [A^{\alpha+1}T_q(t-s)\{g_n(s,u_n) - g_m(s,u_m)\}]ds \\
&\quad + \int_0^t (t-s)^{q-1}A^\alpha T_q(t-s)[f_n(s,u_n) - f_m(s,u_m)]ds. \qquad (38)
\end{aligned}$$

Now, let $0 < \alpha < \nu < \beta$, then we have

$$\begin{aligned}
\|f_n(t,u_n) - f_m(t,u_m)\| &\leq \|f_n(t,u_n) - f_n(t,u_m)\| + \|f_n(t,u_m) - f_m(t,u_m)\|, \\
&\leq F_R(T)\|u_n(t) - u_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_\alpha.
\end{aligned}$$

Also,

$$\|(P^n - P^m)u_m(t)\|_\alpha \quad \leq \quad \|A^{\alpha-\beta}(P^n - P^m)A^\nu u_m(t)\| \leq \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|.$$

Thus, we have

$$\|f_n(t, u_n) - f_m(t, u_m)\| \leq F_R(T)[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|]. \tag{39}$$

Similarly,

$$\|A^\beta g_n(t, u_n) - A^\beta g_m(t, u_m)\| \quad \leq \quad L[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(t)\|]. \tag{40}$$

From (38), (39) and (40) and for $0 < t_0^{'} < t_0$, we have

$$\|u_n(t) - u_m(t)\|_\alpha \leq \|S_q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\|$$

$$\begin{aligned}
\leq \quad & \|S_q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\| + \|A^{\alpha-\beta}\|\|A^\beta g_n(t, u_n) - A^\beta g_m(t, u_m)\| \\
& +(\int_0^{t_0^{'}} + \int_{t_0^{'}}^t)(t - s)^{q-1}\|A^{1+\alpha-\beta}T_q(t - s)\| \times [\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|]ds \\
& +(\int_0^{t_0^{'}} + \int_{t_0^{'}}^t)(t - s)^{q-1}\|A^\alpha T_q(t - s)\|\|f_n(s, u_n) - f_m(s, u_m)\|ds.
\end{aligned} \tag{41}$$

The first term of (41) is estimated as

$$\begin{aligned}
\|S^q(t)A^\alpha(g_n(0, \phi) - g_m(0, \phi))\| \quad & \leq \quad M\|A^{\alpha-\beta}\|\|A^\beta g(0, P^n\phi) - A^\beta g(0, P^m\phi)\| \\
& \leq \quad M\|A^{\alpha-\beta}\|L\|(P^n - P^m)A^\alpha\phi\|.
\end{aligned} \tag{42}$$

We estimate the first and third integrals as

$$\begin{aligned}
&\int_0^{t_0^{'}}(t - s)^{q-1}\|A^{1+\alpha-\beta}T_q(t - s)\|\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|ds \\
&\qquad \leq 2M_{1+\alpha-\beta}C_1q(LR_1 + B_1) \times (t_0 - t_0^{'})^{q(\beta-\alpha)-1}t_0^{'},
\end{aligned} \tag{43}$$

$$\begin{aligned}
\int_0^{t_0^{'}}(t - s)^{q-1}\|A^\alpha T_q(t - s)\| \quad \times \quad & \|f_n(s, u_n) - f_m(s, u_m)\|ds \\
& \leq 2M_\alpha C_2F_R(T)q(t_0 - t_0^{'})^{q(1-\alpha)-1}t_0^{'}.
\end{aligned} \tag{44}$$

From the second and fourth integrals, we have

$$\begin{aligned}
&\int_{t_0^{'}}^t(t - s)^{q-1}\|A^{1+\alpha-\beta}T_q(t - s)\|\|A^\beta g_n(s, u_n) - A^\beta g_m(s, u_m)\|ds \\
&\leq M_{1+\alpha-\beta}LC_1q\int_{t_0^{'}}^t(t - s)^{q(\beta-\alpha)-1}[\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\nu-\alpha}}\|A^\nu u_m(s)\|]ds, \\
&\leq M_{1+\alpha-\beta}LC_1q(\frac{U_{t_0^{'}}T_0^{q(\beta-\alpha)}}{\lambda_m^{\nu-\alpha}q(\beta - \alpha)} + \int_{t_0^{'}}^t(t - s)^{q(\beta-\alpha)-1}\|u_n(s) - u_m(s)\|_\alpha ds),
\end{aligned} \tag{45}$$

and

$$\begin{aligned}
&\int_{t_0'}^{t}(t-s)^{q-1}\|A^{\alpha}T_q(t-s)\|\|f_n(s,u_n)-f_m(s,u_m)\|ds\\
&\leq M_{\alpha}F_R(T)C_2q\int_{t_0'}^{t}(t-s)^{q(1-\alpha)-1}[\|u_n(s)-u_m(s)\|_{\alpha}+\frac{1}{\lambda_m^{\nu-\alpha}}\|A^{\nu}u_m(s)\|]ds,\\
&\leq M_{\alpha}F_R(T)C_2q(\frac{U_{t_0'}T_0^{q(1-\alpha)}}{\lambda_m^{\nu-\alpha}q(1-\alpha)}+\int_{t_0'}^{t}(t-s)^{q(1-\alpha)-1}\|u_n(s)-u_m(s)\|_{\alpha}ds). \qquad (46)
\end{aligned}$$

Using (42)–(46) in (41), we obtain

$$\begin{aligned}
\|u_n(t)-u_m(t)\|_{\alpha} \leq\ & M\|A^{\alpha-\beta}\|\|(P^n-P^m)A^{\alpha}\phi\|\\
&+\|A^{\alpha-\beta}\|L[\|u_n(t)-u_m(t)\|_{\alpha}+\frac{1}{\lambda_m^{\nu-\alpha}}\|A^{\nu}u_m(t)\|]\\
&+2(\frac{M_{1+\alpha-\beta}C_1q(LR_1+B_1)}{(t_0-t_0')^{q(\alpha-\beta)-1}}+\frac{M_{\alpha}C_2F_R(T)q}{(t_0-t_0')^{q(\alpha-1)-1}})t_0'+M_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}}\\
&+\int_{t_0'}^{t}(\frac{M_{\alpha}qC_2F_R(T)}{(t-s)^{q(\alpha-1)+1}}+\frac{M_{1+\alpha-\beta}qLC_1}{(t-s)^{q(\alpha-\beta)+1}})\\
&\quad\times[\|u_n(s)-u_m(s)\|_{\alpha}]ds, \qquad (47)
\end{aligned}$$

where

$$M_{\alpha,\beta}=M_{\alpha}F_R(T)C_2\frac{T_0^{q(1-\alpha)}}{(1-\alpha)}+M_{1+\alpha-\beta}LC_1\frac{T_0^{q(\beta-\alpha)}}{(\beta-\alpha)}. \qquad (48)$$

Also, we have $\|A^{\alpha-\beta}\|L<1$. Therefore inequality (47) becomes

$$\begin{aligned}
\|u_n(t)-u_m(t)\|_{\alpha} \leq\ & \frac{1}{(1-\|A^{\alpha-\beta}\|L)}\{M\|(P^n-P^m)A^{\alpha}\phi\|+\|A^{\alpha-\beta}\|L\frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}}\\
&+2(\frac{M_{1+\alpha-\beta}C_1q(LR_1+B_1)}{(t_0-t_0')^{q(\alpha-\beta)-1}}+\frac{M_{\alpha}C_2qF_R(T)}{(t_0-t_0')^{q(\alpha-1)-1}})t_0'+M_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}}\\
&+\int_{t_0'}^{t}(\frac{M_{\alpha}qC_2F_R(T)}{(t-s)^{q(\alpha-1)+1}}+\frac{M_{1+\alpha-\beta}LC_1q}{(t-s)^{q(\alpha-\beta)+1}})\\
&\times[\|u_n(s)-u_m(s)\|_{\alpha}]ds\}. \qquad (49)
\end{aligned}$$

Taking supremum over $[t_0,T_0]$, we get

$$\begin{aligned}
&\sup_{t\in[t_0,T_0]}\|u_n(t)-u_m(t)\|_{\alpha}\\
&\quad\leq \frac{1}{(1-\|A^{\alpha-\beta}\|L)}\{M\|(P^n-P^m)A^{\alpha}\phi\|+\|A^{\alpha-\beta}\|L\frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}}\\
&\quad+2(\frac{M_{1+\alpha-\beta}C_1q(LR_1+B_1)}{(t_0-t_0')^{q(\alpha-\beta)-1}}+\frac{M_{\alpha}C_2qF_R(T)}{(t_0-t_0')^{q(\alpha-1)-1}})t_0'+M_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\nu-\alpha}}\\
&\quad+\int_{t_0'}^{t}(\frac{M_{\alpha}qC_2F_R(T)}{(t-s)^{q(\alpha-1)+1}}+\frac{M_{1+\alpha-\beta}LqC_1}{(t-s)^{q(\alpha-\beta)+1}})\|u_n-u_m\|_{T_0,\,\alpha}ds\}. \qquad (50)
\end{aligned}$$

Applying Gronwall's inequality to the above inequality, taking $m \to \infty$, we obtain

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ t_0\leq\ t\ \leq\ T_0\}} \|u_n(t) - u_m(t)\|_\alpha$$
$$\leq \frac{2}{(1 - \|A^{\alpha-\beta}\|L)}[\frac{M_{1+\alpha-\beta}C_1(LR_1 + B_1)}{(t_0 - t_0^{'})^{q(\alpha-\beta)-1}} + \frac{M_\alpha C_2 F_R(T)}{(t_0 - t_0^{'})^{q(\alpha-1)-1}}]t_0^{'} \times C, \quad (51)$$

where C is arbitrary constant. The right hand side of inequality (51) may be made as small as possible by taking $t_0^{'}$ (as $t_0^{'}$ is arbitrary) sufficiently small. This completes the proof of the theorem.

Corollary 4.1 *Let assumptions* $(A1) - (A3)$ *hold. If* $\phi \in D(A)$, *then*

$$\sup_{\{n\geq m,\ 0\leq t\leq T_0\}} \|A^\alpha[u_n(t) - u_m(t)]\| \to 0,$$

as $m \to \infty$.

Proof. In this case, we have

$$\|S_q(t)\phi\|_\alpha \leq M\|\phi\|_\alpha. \quad (52)$$

Then from the inequality (52), Lemma (3.1) and Corollary (3.1) we get that in the proof of Theorem (4.1), we can take $t_0 = 0$ to get the required result.

Theorem 4.2 *Suppose that* $(A1) - (A3)$ *are satisfied and* $\phi \in D(A^\alpha)$. *Then, there exist* T_0, $0 < T_0 \leq T$ *and a unique function* $u \in X^\alpha(T_0)$ *such that* $u_n \to u$ *as* $n \to \infty$ *in* $X^\alpha(T_0)$ *and* $u \in X^\alpha(T_0)$ *satisfies the equation (8) on* $[0, T_0]$.

Proof. Let $\phi \in D(A^\alpha)$. Since $A^\alpha u_n(t) \to A^\alpha u(t)$ as $n \to \infty$, for $0 < t \leq T_0$ and $u_n(0) = u(0) = \phi$ for all n. Since $u_n \in B_R(X^\alpha(T_0), \phi)$, it follows that $u \in B_R(X^\alpha(T_0), \phi)$. Further, for any $0 < t_0 \leq T_0$, we have

$$\sup_{\{t_0\leq t\leq T_0\}} \|u_n(t) - u(t)\|_\alpha = 0.$$

Also,

$$\begin{aligned} \|f_n(t, u_n) - f(t, u)\| &= \|f(t, P^n u_n) - f(t, u)\|, \\ &\leq F_R(T)[\|u_n - u\|_\alpha + \|(P^n - I)u\|_\alpha], \end{aligned} \quad (53)$$

and

$$\begin{aligned} \|A^\beta g_n(t, u_n) - A^\beta g(t, u)\| &= \|A^\beta g(t, P^n u_n) - A^\beta g(t, u)\|, \\ &\leq L[\|u_n - u\|_\alpha + \|(P^n - I)u\|]. \end{aligned} \quad (54)$$

Taking supremum on $[t_0, T_0]$, we get

$$\begin{aligned} \sup_{\{t_0\leq t\leq T_0\}} \|f_n(t, u_n) - f(t, u)\| &\leq F_R(T)[\|u_n - u\|_{T_0,\ \alpha} + \|(P^n - I)u\|_{T_0,\ \alpha}], \\ &\to 0, \end{aligned}$$

as $n \to \infty$ and

$$\begin{aligned} \sup_{\{t_0\leq t\leq T_0\}} \|A^\beta g_n(t, u_n) - A^\beta g(t, u)\| &\leq L[\|u_n - u\|_{T_0,\ \alpha} + \|(P^n - I)u\|_{T_0,\alpha}], \\ &\to 0, \end{aligned}$$

as $n \to \infty$. Now, for $0 < t_0 < t$, we may rewrite (22) as

$$\begin{aligned} u_n(t) &= S_q(\phi + g_n(0,\phi)) - g_n(t,u_n) + (\int_0^{t_0} + \int_{t_0}^{t})(t-s)^{q-1}AT_q(t-s)g_n(s,u_n)ds \\ &\quad +(\int_0^{t_0} + \int_{t_0}^{t})(t-s)^{q-1}T_q(t-s)f_n(s,u_n)ds. \end{aligned} \tag{55}$$

We have

$$\begin{aligned} \|\int_0^{t_0}(t-s)^{q-1}AT_q(t-s)g_n(s,u_n)ds\| &\leq \int_0^{t_0}(t-s)^{q-1}\|A^{1-\beta}T_q(t-s)\| \\ &\quad \times[\|A^{\beta}g_n(s,u_n)\|]ds, \\ &\leq M_{1-\beta}C_1^{'}\{(L\widetilde{R}+B)\}T_0^{q\beta-1}t_0, \end{aligned} \tag{56}$$

and

$$\begin{aligned} \|\int_0^{t_0}(t-s)^{q-1}T_q(t-s)f_n(s,u_n)ds\| &\leq \int_0^{t_0}(t-s)^{q-1}\|T_q(t-s)\|\|f_n(s,u_n)\|ds, \\ &\leq MC_1^{'}\{(L\widetilde{R}+B)\}T_0^{q\beta-1}t_0, \end{aligned} \tag{57}$$

where $C_1^{'} = \frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}$ and $C_2^{'} = \frac{q}{\Gamma(1+q)}$. Thus, we have

$$\begin{aligned} &\|u_n(t) - S_q(t)(\phi + g_n(0,\phi)) + g_n(t,u_n) - \int_{t_0}^{t}(t-s)^{q-1}AT_q(t-s)g_n(s,u_n)ds \\ &-\int_{t_0}^{t}(t-s)^{q-1}T_q(t-s)f_n(s,u_n)ds\| \\ &\leq M_{1-\beta}C_1^{'}\{(L\widetilde{R}+B)\}T_0^{q\beta-1}t_0 + MC_2^{'}F_R(T)T_0^{q-1}t_0. \end{aligned}$$

Let $n \to \infty$, in the above inequality, we get

$$\begin{aligned} &\|u(t) - S_q(t)(\phi + g(0,\phi)) + g(t,u(t)) - \textstyle\int_{t_0}^{t}(t-s)^{q-1}AT_q(t-s)g(s,u(s))ds \\ &\qquad - \int_{t_0}^{t}(t-s)^{q-1}T_q(t-s)f(s,u(s))ds\| \\ &\qquad \leq M_{1-\beta}C_1^{'}\{(L\widetilde{R}+B)\}T_0^{q\beta-1}t_0 + MC_2^{'}F_R(T)T_0^{q-1}t_0. \end{aligned} \tag{58}$$

Since $0 < t_0 \leq T_0$ is arbitrary, we get that u satisfies the integral equation (8).

Now, let $\phi \in D(A)$. Corollary 4.1 implies that there exists $u \in X^{\alpha}(T_0)$ such that $u_n \to u$ in $X^{\alpha}(T_0)$. Since $u_n \in B_R(X^{\alpha}(T_0),\phi)$ for each n, u is also in $B_R(X^{\alpha}(T_0),\phi)$. Further, we have

$$\begin{aligned} \sup_{\{0\leq t\leq T_0\}}\|f_n(t,u_n) - f(t,u)\| &\leq F_R(T)[\|u_n - u\|_{T_0,\,\alpha} + \|(P^n - I)u\|_{T_0,\,\alpha}], \\ &\to 0, \ \text{ as } \ n \to \infty, \end{aligned} \tag{59}$$

and

$$\begin{aligned} \sup_{\{0\leq t\leq T_0\}}\|A^{\beta}g_n(t,u_n) - A^{\beta}g(t,u)\| &\leq L[\|u_n - u\|_{T_0,\,\alpha} + \|(P^n - I)u\|_{T_0,\,\alpha}], \\ &\to 0, \ \text{ as } \ n \to \infty. \end{aligned} \tag{60}$$

Using (59), (60) and (22), we obtain

$$u(t) = S_q(t)(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g(s,u(s)) ds + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,u(s)) ds. \qquad (61)$$

Hence, this completes the proof of the theorem.

Now, we shall show the uniqueness of the solution to equation (61). Let u_1 and u_2 be the two solutions of (61). We have

$$\begin{aligned} u_1(t) - u_2(t) &= -\{g(t,u_1(t)) - g(t,u_2(t))\} \\ &\quad + \int_0^t (t-s)^{q-1} A T_q(t-s)[g(s,u_1) - g(s,u_2)] ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s)[f(s,u_1) - f(s,u_2)] ds, \end{aligned}$$

and thus

$$\begin{aligned} \|A^\alpha[u_1(t) - u_2(t)]\| &\leq \|A^{\alpha-\beta}\| \|A^\beta g(t,u_1(t)) - A^\beta g(t,u_2(t))\| \\ &\quad + \int_0^t (t-s)^{q-1} \|A^{1+\alpha-\beta} T_q(t-s)\| \|A^\beta g(s,u_1) - A^\beta g(s,u_2)\| ds \\ &\quad + \int_0^t (t-s)^{q-1} \|A^\alpha T_q(t-s)\| \|f(s,u_1) - f(s,u_2)\| ds, \\ &\leq \|A^{\alpha-\beta}\| L \|u_1(t) - u_2(t)\|_\alpha \\ &\quad + M_{1+\alpha-\beta} C_1 L q \int_0^t (t-s)^{q(\beta-\alpha)-1} \|u_1(t) - u_2(t)\|_\alpha ds \\ &\quad + M_\alpha F_R(T) C_2 q \int_0^t (t-s)^{q(1-\alpha)-1} \|u_1(t) - u_2(t)\|_\alpha ds. \end{aligned}$$

Since $\|A^{\alpha-\beta}\| L < 1$, therefore we obtain
$\|u_1(t) - u_2(t)\|_\alpha$

$$\leq \frac{1}{(1 - L\|A^{\alpha-\beta}\|)} [\int_0^t \{ \frac{M_{1+\alpha-\beta} C_1 q L}{(t-s)^{1-q(\beta-\alpha)}} + \frac{M_\alpha F_R(T) C_2 q}{(t-s)^{1-q(1-\alpha)}} \} \|u_1(t) - u_2(t)\|_\alpha ds].$$

Applying Gronwall's inequality, we obtain

$$\|u_1(t) - u_2(t)\|_\alpha = 0$$

for all $0 \leq t < T_0$. From the fact

$$\|u_1(t) - u_2(t)\| \leq \frac{1}{\lambda_0^\alpha} \|u_1(t) - u_2(t)\|_\alpha,$$

therefore, $u_1 = u_2$ on $[0, T_0]$. The proof of the theorem is complete.

5 Faedo-Galerkin Approximation

In this section, we will discuss the Faedo-Galerkin approximations of solutions and prove some convergence result for such approximations.

We know that for any $0 < T_0 < T$, we have a unique $u \in X^\alpha(T_0)$ satisfying the integral equation

$$\begin{aligned} u(t) &= S_q(t)(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g(s,u(s)) ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) f(s,u(s)) ds, \end{aligned} \tag{62}$$

Also, there is a unique solutions $u_n \in X^\alpha(T_0)$ of the approximate integral equations

$$\begin{aligned} u_n(t) &= S_q(t)(\phi + g_n(0,\phi)) - g_n(t,u_n(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g_n(s,u_n(s)) ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) f_n(s,u_n(s)) ds. \end{aligned} \tag{63}$$

We apply the projection on the above equation, then Faedo-Galerkin approximation is given by $v_n(t) = P^n u_n(t)$ satisfying

$$\begin{aligned} P^n u_n(t) &= v_n(t) = S_q(t)(P^n\phi + P^n g(0,P^n\phi)) - P^n g(t,v_n(t)) \\ &\quad + \int_0^t (t-s)^{q-1} A T_q(t-s) P^n g(s,v_n(s)) ds \\ &\quad + \int_0^t (t-s)^{q-1} T_q(t-s) P^n f(s,v_n(s)) ds. \end{aligned} \tag{64}$$

Let the solution u of (62) and v_n of (64) have the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) u_i, \quad \alpha_i(t) = (u(t),u_i) \;\; i = 0,1,2,\cdots, \tag{65}$$

$$v_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) u_i, \quad \alpha_i^n(t) = (v_n(t),u_i) \;\; i = 0,1,2,\cdots, \tag{66}$$

Using (66) in (64), we obtain a system of fractional order integro-differential equation of the form

$$\frac{d^q}{dt^q}(\alpha_i^n(t) + g_i^n(t,\alpha_0^n(t),\alpha_1^n(t)...,\alpha_n^n)) + \lambda_i \alpha_i^n(t) = f_i^n(\alpha_0^n(t),\alpha_1^n(t)...,\alpha_n^n), \tag{67}$$

$$\alpha_i^n(0) = \phi_i, \tag{68}$$

where

$$g_i^n(t,\alpha_0^n(t),\alpha_1^n(t)...,\alpha_n^n)) = (g(t,\sum_{i=0}^n \alpha_i^n(t)u_i),u_i),$$

$$f_i^n(\alpha_0^n(t),\alpha_1^n(t)...,\alpha_n^n) = (f(t,\sum_{i=0}^n \alpha_i^n(t)u_i),u_i),$$

and $\phi_i = (\phi,u_i)$, for $i = 1,2,\cdots,n$. The system (67)–(68) determines the $\alpha_i^n(t)$'s.

As a consequence of Theorems 3.1 and 4.1, we have the following convergence result.

Theorem 5.1 *Let* $(A1)-(A3)$ *hold and* $\phi \in D(A^{\alpha})$. *Then there exist functions* $v_n \in C([0,T_0],D(A^{\alpha}))$,

$$\begin{aligned} v_n(t) = S_q(t)(P^n\phi + P^n g(0,P^n\phi)) - P^n g(t,v_n(t)) \\ + \quad & \int_0^t (t-s)^{q-1} A T_q(t-s) P^n g(s,v_n(s))ds \\ + \quad & \int_0^t (t-s)^{q-1} T_q(t-s) P^n f(s,v_n(s))ds, \;\; t\in[0,T_0] \end{aligned}$$

and $u \in C([0,T_0],D(A^{\alpha}))$,

$$u(t) = S_q(t)(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t (t-s)^{q-1} A T_q(t-s) g(s,u(s))ds$$

$$+\int_0^t (t-s)^{q-1} T_q(t-s) f(s,u(s))ds, \;\; t\in[0,T_0]$$

such that $v_n \to u$ *in* $C([0,T_0],D(A^{\alpha}))$ *as* $n\to\infty$.

Now, we show the convergence of $\alpha_i^n(t) \to \alpha_i(t)$. Consider the following

$$A^{\alpha}[u(t)-v_n(t)] = A^{\alpha}[\sum_{i=0}^{\infty}(\alpha_i(t)-\alpha_i^n(t))u_i] = \sum_{i=0}^{\infty}\lambda_i^{\alpha}(\alpha_i(t)-\alpha_i^n(t))u_i.$$

Therefore, we have

$$\|A^{\alpha}[u(t)-v_n(t)]\|^2 \geq \sum_{i=0}^{n}\lambda_i^{2\alpha}(\alpha_i(t)-\alpha_i^n(t))^2.$$

We have the following convergence theorem.

Theorem 5.2 *We have the following result:*
(a) If $\phi \in D(A^{\alpha})$ *for all* $t_0 \in (0,T_0]$, *then*

$$\lim_{n\to\infty}\sup_{t_0\leq t\leq T_0}[\sum_{i=0}^{n}\lambda_i^{2\alpha}\{\alpha_i(t)-\alpha_i^n(t)\}^2] = 0.$$

(b) If $\phi \in D(A)$ *for all* $t\in[0,T_0]$, *then*

$$\lim_{n\to\infty}\sup_{0\leq t\leq T_0}[\sum_{i=0}^{n}\lambda_i^{2\alpha}\{\alpha_i(t)-\alpha_i^n(t)\}^2] = 0.$$

The assertion of this theorem follows from the facts mentioned above and the following result.

Proposition 5.1 *Let* $(H1)-(H3)$ *hold and let* T_0 *be any number such that* $0<T_0<T$, *then we have the following.*

(a) If $\phi \in D(A^{\alpha})$ *for all* $t_0\in(0,T_0]$ *then*

$$\lim_{n\to\infty}\sup_{n\geq m,\; t_0\leq t\leq T}\| A^{\alpha}[v_n(t)-v_m(t)]\| = 0. \tag{69}$$

(b) If $\phi \in D(A)$ for all $t_0 \in [0, T_0]$ then

$$\lim_{n\to\infty} \sup_{n\geq m,\ 0\leq t\leq T} \| A^\alpha[v_n(t) - v_m(t)]\| = 0. \tag{70}$$

Proof. For $n \geq m$, we have

$$\begin{aligned} \| A^\alpha[v_n(t) - v_m(t)]\| &= \| A^\alpha[P^n u_n(t) - P^m u_m(t)]\|, \\ &\leq \| P^n[u_n(t) - u_m(t)]\|_\alpha + \| (P^n - P^m)u_m(t)\|_\alpha, \\ &\leq \| [u_n(t) - u_m(t)]\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}} \| A^\vartheta u_m\|. \end{aligned} \tag{71}$$

If $\phi \in D(A^\alpha)$, then the result in (a) follows from Theorem 4.1. If $\phi \in D(A)$, (b) follows from Corollary 4.1.

6 Application

Consider the following partial differential equation of fractional order of the form

$$\frac{d^q}{dt^q}[u(t,x) - \Delta u(x,t)] + \Delta^2 u(x,t) = F(x,t,u(t,x)), 0 < q \leq 1, \tag{72}$$

$$u(x,0) = u_0, \ \ x \in \Omega, \tag{73}$$

with the homogenous boundary conditions. Were Ω is a bounded domain in the $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and Δ is N-dimensional Laplacian and function h is sufficiently smooth in all arguments. We take $X = L^2(\Omega)$ and let A be the operator defined as $-Au = \Delta u$ with the domain

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{74}$$

Then equation (72) can be written as

$$\frac{d^q}{dt^q}[v(t) + Av(t)] + A^2 v(t) = F(t, v(t)), \tag{75}$$

$$v(0) = u_0. \tag{76}$$

It is well known that A is not invertible but $(A + cI)$ is invertible and $||(A + cI)^{-1}|| \leq C$ for large enough $c > 0$. Therefore equation (75) can be written of the form (1) with $g(t,\ v) = (1 - c)(A + cI)^{-1}v$ and $f(t,\ v) = cA(A + cI)^{-1}v + F(t,\ (A + cI)^{-1}v)$. It is easy to see that operator A satisfies $(A1)$ and f and g satisfy $(A2)$ and $(A3)$ respectively. By applying the results of the earlier sections, we have the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (72)-(73).

Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions. The work of the first author is supported by the UGC (University Grants Commission, India) under Grant No $(6405 - 11 - 061)$.nt.

References

[1] Bahuguna, D. and Muslim, M., Approximation of solutions to a class of second order history-valued delay differential equations. *Nonlinear Dynamics and Systems Theory* **8** (3) (2008) 237–254.

[2] Bahuguna, D., Shukla, Reeta. Approximations of solutions to nonlinear Sobolev type evolution equations. *Electron. J. Differential Equations* **31** (2003) 1–16.

[3] Li, Fang and N'Guérékata, Gaston M. Existence and uniqueness of mild solution for fractional Integrodifferential equations. *Advances in Diff. Eqn.* (2010) 1–10.

[4] Kilbas, A. A., Srivastava, H. M. and Trujillo J.J. *Theory and Applications of Fractional Differential Equations. Elsevier*, Amsterdam, 2006.

[5] Miller, K. S. and Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations. *John Wiley and Sons, Inc.*, New York, 1993.

[6] Miletta, P. D. Approximation of solutions to evolution equations. *Math. Method in the Appl.Sci.* **17** (1994) 753–763.

[7] El-Borai, M. M. Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Soliton Fractals* **14** (2002) 433–440.

[8] Muslim, M. and Nandakumaran, A. K. Existence and Approximations of solutions to some fractional order functional integral equation. *J. Integral Equations Appl.* **22** (2010) 95–114.

[9] Muslim, M. Existence and approximation of solutions to fractional differential equations. *Math. Comput. Modelling* **49** (2009) 1164–1172.

[10] Pazy A. *Semigroup of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag*, 1983.

[11] Pandey, Dwijendra N., Ujlayan, Amit and Bahuguna, D. On nonlinear abstract neutral differential equations with deviated argument, *Nonlinear Dynamics and Systems Theory* **10** (3) (2010) 283–294.

[12] Balachandran, K., Kiruthika, S. and Trujillo, J. J. Existence results for fractional impulsive integrodifferential equations in Banach space. *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 1970–1977.

[13] Chang, Yong-Kui, Kavitha, V. and Mallika Arjunan, M. Existence and uniqueness of mild solutions to a semilinear integrodifferential equation of fractional order. *Nonlinear Anal.* **71** (2009) 5551–5559.

[14] Zhou, Y. and Jiao, Feng. Existence of mild solutions for fractional neutral evolution equations. *Comput. Math. Appl.* **59** (2010) 1063–1077.

[15] Dubey, Reeta Shukla. Approximation of solutions to abstract neutral functional differential equation. *Numer. Funct. Anal. Optim.* **32** (2011) 286–308.

[16] Podlubny, I. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Academic Press, Inc., San Diego, 1999.

[17] Samko, S.G., Kilbas, A.A. and Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach Sci. Publishers, Yverdon, 1993.

[18] Liang, J., Yan, Sheng-Hua, Li, Fang and Huang, Ting-Wen. On the existence of mild solution to the Cauchy problem for a class of fractional evolution equations. *Adv. Difference Equ.* (2012) 1–16.