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Permanence and Ultimate Boundedness for Discrete-Time Switched Models of Population Dynamics

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Abstract: The problems of permanence and ultimate boundedness for a class of discrete-time Lotka–Volterra type systems with switching of parameter values are studied. Two new approaches for the constructing of a common Lyapunov function for the family of subsystems corresponding to a switched system are suggested. Sufficient conditions in terms of linear inequalities are obtained to guarantee that the solutions of the considered system are ultimately bounded or permanent for an arbitrary switching law. An example is presented to demonstrate the effectiveness of the obtained results.

Keywords: *population dynamics; ultimate boundedness; switched system; discrete-time models; common Lyapunov function; linear inequalities.*

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1 Introduction

The Lotka–Volterra type differential and difference equations systems are extensively used in modeling of population dynamics [6, 7, 9, 12, 14, 15]. A very important ecological problem associated with multispecies population interactions is the following one: whether or not the densities of all species are bounded [5, 7, 9, 15]. Of particular interest is the situation when there exists a bounded region in the phase space of the system, such that every solution enters this region for finite time and remains within it thereafter. Solutions of systems possessing this property are called ultimately bounded [6, 7].

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It is worth mentioning that, in the analysis of population models, it is important not only to check the ultimate boundedness, but also to verify whether or not the considered system is permanent [5, 7, 12, 17]. The permanence property, in addition to the ultimate boundedness of densities of all species, implies that if initially all species are present, even in very small quantities, then after a certain time some sizeable amount of each of them will be present.

Conditions of ultimate boundedness and permanence are well investigated for Lotka–Volterra type models with constant parameters, see, for example, [5–7, 9] and the references cited therein. However, owing to many natural and man-made factors, such as fire, drought, raining season, changing in nutrition, deforestation, radiation, etc., the intrinsic discipline of biological species or ecological environment usually undergoes some discrete changes of relatively short duration at some fixed times. For more adequate modeling of such processes, stochastic, switched or impulsive systems are used [4, 8, 13, 17, 18]. The problem of ultimate boundedness and permanence analysis for these models is much more complicated than that one for differential and difference systems with constant parameters.

In the present paper, a discrete-time switched Lotka–Volterra type system is studied. The system consists of a family of subsystems of difference equations and a switching law determining at each time instant which subsystem is active. We will look for conditions providing the ultimate boundedness or permanence of the considered system for an arbitrary switching law. A general approach to the problem is based on the construction of a common Lyapunov function (CLF) for the family of subsystems corresponding to the switched system. This approach has been effectively used for the analysis of stability and boundedness for many classes of switched systems, see, for instance, [1–3, 10, 11, 16], and the references therein. However, the problem of the existence of a CLF has not got a constructive solution even for the case of family of linear time-invariant systems [11].

In [3], for the investigated switched system, a special form of Lyapunov function has been used. The sufficient condition in terms of linear inequalities was obtained to guarantee the existence of a CLF in the prescribed form, and thereby to ensure that solutions of the switched system are ultimately bounded or permanent for an arbitrary switching signal. In the present paper, two different approaches for the constructing of a CLF are proposed. The usage of these approaches permits to relax the ultimate boundedness and the permanence conditions found in [3].

2 Statement of the Problem

Consider the switched difference system

$$x_i(k+1) = x_i(k)\exp\left(h\left(c_i^{(\sigma)} + \sum_{j=1}^{n} p_{ij}^{(\sigma)} f_j(x_j(k))\right)\right), \quad i = 1,\ldots,n. \tag{1}$$

The system describes interaction of n species in a biological community. Here $x_i(k)$ is the density of population i at the kth generation; functions $f_i(z_i)$ are defined for $z_i \in [0,+\infty)$; $\sigma = \sigma(k)$, $k = 0, 1, \ldots$, with $\sigma(k) \in \{1,\ldots,N\}$ defines a switching law; $c_i^{(s)}$ and $p_{ij}^{(s)}$, $s = 1,\ldots,N$, $i,j = 1,\ldots,n$, are constant coefficients; h is a positive parameter characterizing the transient time between two consecutive generations. Thus, at each time instant, the dynamics of (1) is described by one of the subsystems

$$x_i(k+1) = x_i(k)\exp\left(h\left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(x_j(k))\right)\right), \quad i = 1,\dots,n, \ s = 1,\dots,N. \quad (2)$$

Subsystems of the form (2) are discrete counterparts of the continuous generalized Lotka–Volterra ecosystem models [5–7, 12, 15]. It is known [6, 7, 12] that if the populations have non-overlapping generations, then discrete time models are more appropriate than the continuous ones. Moreover, they provide efficient schemes for the numerical simulation of continuous processes.

In (1), coefficients $c_i^{(s)}$ characterize the intrinsic growth rate of the ith population; the introduction of self-interaction terms $p_{ii}^{(s)} f_i(z_i)$ with $p_{ii}^{(s)} < 0$ is justified by the natural limitation of resources in the environment, the terms $p_{ij}^{(s)} f_j(z_j)$ for $j \neq i$ measure influence of population j on population i. It is supposed that environment fluctuations provoke switching of the system parameters.

According to standard assumptions [6, 7, 15], we assume that functions $f_i(z_i)$, $i = 1,\dots,n$, possess the following properties:

(i) $f_i(z_i)$ are continuous for $z_i \in [0,+\infty)$;

(ii) $f_i(0) = 0$, and for $z_i > 0$ the inequality $f_i(z_i) > 0$ holds, and

(iii) $f_i(z_i) \to +\infty$ as $z_i \to +\infty$.

By R_+^n we denote the non-negative orthant of R^n; $\operatorname{int} R_+^n$ being the interior of R_+^n; $\mathbf{x}(k,\mathbf{x}^{(0)},k_0)$ denotes the solution of (1) starting from $\mathbf{x}^{(0)}$ at $k = k_0$; $\mathbf{P}_s = \left(p_{ij}^{(s)}\right)_{i,j=1}^{n}$, $s = 1,\dots,N$, are given matrices; and $B_Q = \{\mathbf{z} : \ \mathbf{z} \in \operatorname{int} R_+^n, \ \|\mathbf{z}\| \le Q\}$ for a given positive number Q. For biological reasons, we will consider (1) in $\operatorname{int} R_+^n$ which is an invariant set for this system.

Definition 2.1 System (1) is called ultimately bounded in $\operatorname{int} R_+^n$ with the ultimate bound $R > 0$ if, for any $\mathbf{x}^{(0)} \in \operatorname{int} R_+^n$ and $k_0 \ge 0$, there exists $T > 0$, such that $\|\mathbf{x}(k,\mathbf{x}^{(0)},k_0)\| \le R$ for $k \ge k_0 + T$.

Definition 2.2 System (1) is called uniformly ultimately bounded in $\operatorname{int} R_+^n$ with the ultimate bound $R > 0$ if, for any $Q > 0$, there exists $T = T(Q) > 0$, such that $\|\mathbf{x}(k,\mathbf{x}^{(0)},k_0)\| \le R$ for all $k_0 \ge 0$, $\mathbf{x}^{(0)} \in B_Q$, $k \ge k_0 + T$.

Definition 2.3 System (1) is called permanent if there exists a compact set $D \subset \operatorname{int} R_+^n$, such that, for any $\mathbf{x}^{(0)} \in \operatorname{int} R_+^n$ and $k_0 \ge 0$, the solution $\mathbf{x}(k,\mathbf{x}^{(0)},k_0)$ of (1) ultimately remains in D.

Definition 2.4 System (1) is called uniformly permanent if there exist numbers Δ_1 and Δ_2, $0 < \Delta_1 < \Delta_2$, such that, for any δ_1 and δ_2, $0 < \delta_1 < \delta_2$, one can choose $T > 0$ satisfying the following condition: if for the initial values of a solution $\mathbf{x}(k,\mathbf{x}^{(0)},k_0)$ the inequalities $k_0 \ge 0$, $\delta_1 \le x_i^{(0)} \le \delta_2$, $i = 1,\dots,n$, hold, then $\Delta_1 \le x_i(k,\mathbf{x}^{(0)},k_0) \le \Delta_2$, $i = 1,\dots,n$, for $k \ge k_0 + T$.

Conditions of the ultimate boundedness and the permanence are well investigated for individual subsystems from (2) without switching [5–7, 12]. The goal of the present paper is the ultimate boundedness and the permanence analysis for switched system (1).

3 Ultimate Boundedness Conditions

Sufficient conditions of uniform ultimate boundedness for switched system (1) have been obtained in [3]. The case was considered when, for the functions $f_1(z_1), \dots, f_n(z_n)$, in addition to the properties (i)–(iii), the following assumptions are fulfilled.

Assumption 3.1 Let $\int_0^1 \frac{f_i(\tau)}{\tau}\, d\tau < +\infty$, $i = 1, \dots, n$.

Assumption 3.2 The functions $\tilde{f}_i(z_i) = f_i(\exp(z_i))$ satisfy the Lipschitz condition with constant L for all $z_i \in (-\infty, +\infty)$, $i = 1, \dots, n$.

For example, the properties (i)–(iii) and Assumptions 3.1 and 3.2 are fulfilled for functions $f_i(z_i) = \log(z_i + 1)$, $i = 1, \dots, n$.

Let us introduce the auxiliary matrices $\overline{\mathbf{P}}_s = \left(\bar{p}_{ij}^{(s)}\right)_{i,j=1}^{n}$ whose entries are defined by the formulae $\bar{p}_{ii}^{(s)} = p_{ii}^{(s)}$, and $\bar{p}_{ij}^{(s)} = \max\left\{p_{ij}^{(s)};\, 0\right\}$ for $j \neq i$; $i, j = 1, \dots, n$; $s = 1, \dots, N$. Thus, the matrices $\overline{\mathbf{P}}_1, \dots, \overline{\mathbf{P}}_N$ are Metzler ones [9, 10].

Consider the two families of linear inequalities systems

$$\overline{\mathbf{P}}_s \theta < \mathbf{0}, \quad s = 1, \dots, N, \tag{3}$$

$$\overline{\mathbf{P}}_s^{\,T} \mathbf{b} < \mathbf{0}, \quad s = 1, \dots, N, \tag{4}$$

where $\theta = (\theta_1, \dots, \theta_n)^T$, $\mathbf{b} = (b_1, \dots, b_n)^T$. These inequalities in vector form are understood to be component-wise. That is to say, a vector is less than zero if and only if so is each component of the vector. For convenience, one can call a vector to be negative (respectively, positive) if it is less (respectively, greater) than zero.

In [3], a CLF for (2) has been chosen in the form

$$V_1(\mathbf{z}) = \sum_{i=1}^{n} \lambda_i \int_1^{z_i} \frac{f_i(\tau)}{\tau} d\tau, \tag{5}$$

where $\lambda_1, \dots, \lambda_n$ are positive coefficients. By the usage of function (5), the following theorem was proved.

Theorem 3.1 *Let Assumptions 3.1 and 3.2 be fulfilled. If systems* (3) *and* (4) *admit positive solutions, then there exists* $h_0 > 0$ *such that system* (1) *is uniformly ultimately bounded in* $\operatorname{int} R_+^n$ *for any* $h \in (0, h_0)$ *and for arbitrary switching law.*

Remark 3.1 Necessary and sufficient conditions of solvability for inequality systems of the form (3) and (4) with Metzler matrices have been found in [2, 10]. Furthermore, in [2], an effective algorithm based on a modification of Gaussian elimination procedure for the construction of positive solutions of such systems was suggested.

Remark 3.2 It is known [9] that if a matrix $\mathbf{P}$ is Metzler one, then the system of inequalities $\mathbf{P}\theta < \mathbf{0}$ possesses a positive solution if and only if the system of inequalities $\mathbf{P}^T\mathbf{b} < \mathbf{0}$ possesses a positive solution as well. However, it is not true for the families of inequalities (3), (4) [2, 3]. Generally, from the existence of a positive solution for the inequalities (3) with Metzler matrices $\overline{\mathbf{P}}_1, \dots, \overline{\mathbf{P}}_N$, it does not follow that a positive solution also exists for the corresponding inequalities (4).

In the present section, we shall suggest another approach for the constructing of a CLF for family (2). The usage of this approach permits to relax the conditions of Theorem 3.1. In particular, we will prove that in the case when for functions $f_1(z_1), \dots, f_n(z_n)$, instead of Assumptions 3.1 and 3.2, an additional assumption is fulfilled, the existence of a positive solution for (3) is sufficient to ensure that (1) is uniform ultimately bounded for sufficiently small values of h and for any switching law. Thus, another condition of Theorem 3.1, i.e., the condition of the existence of a positive solution for (4), can be dropped.

Assumption 3.3 The functions $\tilde{f}_i(z_i) = f_i(\exp(z_i))$ are continuously differentiable for $z_i \in (-\infty, +\infty)$, and $0 < \tilde{f}_i'(z_i) \le L$, $i = 1, \dots, n$, where L is a positive constant.

Theorem 3.2 *Let Assumption 3.3 be fulfilled. If system* (3) *admits a positive solution, then there exists* $h_0 > 0$ *such that system* (1) *is uniformly ultimately bounded in* $\operatorname{int} R_+^n$ *for any* $h \in (0, h_0)$ *and for arbitrary switching law.*

Proof. Let a positive vector $\theta = (\theta_1, \dots, \theta_n)^T$ satisfy the inequalities (3). Then there exists a number $\gamma > 0$, such that $\sum_{j=1}^n \bar{p}_{ij}^{(s)} \theta_j \le -\gamma$, $i = 1, \dots, n$, $s = 1, \dots, N$.

Construct a CLF for (2) in the form

$$V_2(\mathbf{z}) = \max_{i=1,\dots,n} \frac{f_i(z_i)}{\theta_i}. \tag{6}$$

Function $V_2(\mathbf{z})$ is continuous for $\mathbf{z} \in R_+^n$, and $V_2(\mathbf{z}) \to +\infty$ as $\|\mathbf{z}\| \to \infty$.

For some s in $\{1, \dots, N\}$, consider the difference of the function (6) with respect to the sth subsystem from (2). Let $\hat{\mathbf{x}} \in \operatorname{int} R_+^n$, and $\mathbf{x}(k) = (x_1(k), \dots, x_n(k))^T$ be the solution of the sth subsystem starting from $\hat{\mathbf{x}}$ at $k = 0$. For every $k = 0, 1, \dots$, find

$$B_k = \max_{i=1,\dots,n} \frac{f_i(x_i(k))}{\theta_i}.$$

Denote by A_k a subset of $\{1, \dots, n\}$ such that $f_i(x_i(k))/\theta_i = B_k$ for $i \in A_k$, and $f_i(x_i(k))/\theta_i < B_k$ for $i \notin A_k$.

Choose a nonnegative integer k. Let $r \in A_k$, $i \in A_{k+1}$. We obtain

$$\begin{aligned}
\Delta V_2\big|_{(s)} &= V_2(\mathbf{x}(k+1)) - V_2(\mathbf{x}(k)) = \frac{f_i(x_i(k+1))}{\theta_i} - \frac{f_r(x_r(k))}{\theta_r} \\
&= \left(\frac{f_i(x_i(k+1))}{\theta_i} - \frac{f_i(x_i(k))}{\theta_i} \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \\
&= \left(\frac{\tilde{f}_i(y_i(k+1))}{\theta_i} - \frac{\tilde{f}_i(y_i(k))}{\theta_i} \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)
\end{aligned}$$

$$\le \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} + \sum_{j=1}^n \bar{p}_{ij}^{(s)} f_j(x_j(k)) \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right)$$

$$\le \frac{\tilde{f}_i'(y_i(k) + \xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} + \bar{p}_{ii}^{(s)} f_i(x_i(k)) + \sum_{j=1}^n \bar{p}_{ij}^{(s)} \theta_j \frac{f_r(x_r(k))}{\theta_r} - \bar{p}_{ii}^{(s)} \theta_i \frac{f_r(x_r(k))}{\theta_r} \right)$$

$$
\begin{aligned}
&\quad - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \\
&\leq \frac{\tilde f_i'(y_i(k)+\xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} - \gamma B_k - \bar p_{ii}^{(s)} \theta_i \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \right) \\
&\quad - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \\
&= \frac{\tilde f_i'(y_i(k)+\xi_{ik}\Delta y_i(k))}{\theta_i} h \left(c_i^{(s)} - \gamma B_k \right) - \left(\frac{f_r(x_r(k))}{\theta_r} - \frac{f_i(x_i(k))}{\theta_i} \right) \left(1 + Lh \bar p_{ii}^{(s)} \right).
\end{aligned}
$$

Here $y_i(k) = \log x_i(k)$, $\Delta y_i(k) = y_i(k+1) - y_i(k)$, $\xi_{ik} \in (0,1)$.

Let $D = \max\limits_{s=1,\dots,N} \max\limits_{i=1,\dots,n} |\bar p_{ii}^{(s)}|$,

$$
0 < h_0 < \frac{1}{LD}, \tag{7}
$$

and $h \in (0, h_0)$. Then there exists a positive number H, such that $\Delta V_2\big|_{(s)} < 0$ for $\|\mathbf{x}(k)\| > H$ and for all $s = 1,\dots,N$.

Define the constants M and M_1 by the following formulae:

$$
M = \max_{\mathbf{z} \in R_+^n,\ \|\mathbf{z}\| \leq H} V_2(\mathbf{z}), \quad M_1 > M + hL \max_{s=1,\dots,N} \max_{i=1,\dots,n} \left| \frac{c_i^{(s)}}{\theta_i} \right|.
$$

Consider the region $G = \{\mathbf{z} : \mathbf{z} \in \operatorname{int} R_+^n,\ V_2(\mathbf{z}) \leq M_1\}$. We obtain that $V_2(\mathbf{x}(k+1)) \leq M_1$ if $\|\mathbf{x}(k))\| \leq H$, and $V_2(\mathbf{x}(k+1)) < V_2(\mathbf{x}(k))$ if $\|\mathbf{x}(k))\| > H$. Hence, once a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) enters into G at $k = k_1 \geq k_0$, it remains within the region for $k \geq k_1$.

Choose a positive number Q. We will show that there exists $T = T(Q) \geq 0$ such that $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$ for all $k_0 \geq 0$, $\mathbf{x}^{(0)} \in B_Q$ and $k \geq k_0 + T(Q)$.

Let $U = \max_{\mathbf{z} \in R_+^n,\ \|\mathbf{z}\| \leq Q} V_2(\mathbf{z})$. If $U \leq M_1$, then we can take $T(Q) = 0$.

Now consider the case when $U > M_1$. If $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) > M_1$ for $k = k_0, k_0 + 1, \dots, \tilde k$, then the inequalities

$$
M_1 < V_2(\mathbf{x}(\tilde k, \mathbf{x}^{(0)}, k_0)) \leq V_2(\mathbf{x}^{(0)}) - \rho(\tilde k - k_0) \leq U - \rho(\tilde k - k_0)
$$

hold, where

$$
\rho = - \max_{s=1,\dots,N} \max_{\mathbf{z} \in R_+^n,\ M_1 \leq V_2(\mathbf{z}) \leq U} \Delta V_2\big|_{(s)} > 0.
$$

Hence, $\tilde k < k_0 + (U - M_1)/\rho$. By taking $T(Q) = (U - M_1)/\rho$, one gets $V_2(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \leq M_1$ for $k \geq k_0 + T(Q)$. Thus, system (1) is uniformly ultimately bounded in $\operatorname{int} R_+^n$.

Corollary 3.1 *Let $c_i^{(s)} \leq 0$, $i = 1,\dots,n$; $s = 1,\dots,N$, and Assumption 3.3 be fulfilled. If system* (3) *admits a positive solution, then there exists $h_0 > 0$ such that the zero solution of* (1) *is globally asymptotically stable in* $\operatorname{int} R_+^n$ *for any $h \in (0, h_0)$ and for any switching law.*

Remark 3.3 In the case when all the coefficients $c_i^{(s)}$ are negative, instead of (3), it is sufficient to consider the nonstrict inequalities

$$
\overline{\mathbf{P}}_s \theta \leq \mathbf{0}, \quad s = 1,\dots,N. \tag{8}
$$

Corollary 3.2 *Let $c_i^{(s)} < 0$, $i = 1, \dots, n$; $s = 1, \dots, N$, and Assumption 3.3 be fulfilled. If system* (8) *admits a positive solution, then there exists $h_0 > 0$ such that the zero solution of* (1) *is globally asymptotically stable in* $\operatorname{int} R_+^n$ *for any $h \in (0, h_0)$ and for any switching law.*

4 Permanence Conditions

In this section, we consider the case when, in system (1), parameters $c_i^{(s)}$ and $p_{ij}^{(s)}$ satisfy an additional restriction.

Assumption 4.1 The following inequalities are valid $c_i^{(s)} > 0$, and $p_{ij}^{(s)} \geq 0$ for $j \neq i$; $i, j = 1, \dots, n$; $s = 1, \dots, N$.

Theorem 4.1 *Let Assumptions 3.3 and 4.1 be fulfilled. If system* (3) *admits a positive solution, then there exists $h_0 > 0$ such that system* (1) *is uniformly permanent for any $h \in (0, h_0)$ and for arbitrary switching law.*

Proof. Let for a constant h_0 the condition (7) be valid. Choose a number $h \in (0, h_0)$, and consider the corresponding switched system (1).

According to the proof of Theorem 3.2, there exists $\Delta_2 > 0$, and for given positive numbers δ_1 and δ_2, $0 < \delta_1 < \delta_2$, one can find $\eta > 0$ and $T > 0$, such that if the initial values of a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) satisfy the conditions $k_0 \geq 0$, $\delta_1 \leq x_i^{(0)} \leq \delta_2$, $i = 1, \dots, n$, then $0 < x_i(k, \mathbf{x}^{(0)}, k_0) \leq \eta$, $i = 1, \dots, n$, for $k \geq k_0$, and $0 < x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \dots, n$, for $k \geq k_0 + T$.

The fulfilment of the Assumption 4.1 implies the existence of positive numbers δ and β, such that $c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \geq \beta$ for $0 < z_i \leq \delta$, $i = 1, \dots, n$; $s = 1, \dots, N$. Hence, if $0 < x_i(k, \mathbf{x}^{(0)}, k_0) < \delta$ for some $i \in \{1, \dots, n\}$, then $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq x_i(k, \mathbf{x}^{(0)}, k_0) \exp(h\beta)$.

Let

$$\omega = \min_{s=1,\dots,N} \min_{i=1,\dots,n} \min_{0 \leq z_i \leq \eta} \left(c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \right),$$

$$\tilde{\omega} = \min_{s=1,\dots,N} \min_{i=1,\dots,n} \min_{0 \leq z_i \leq \Delta_2} \left(c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \right).$$

We obtain that $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq \delta \exp(h\omega)$ for $k \geq k_0$, $x_i(k, \mathbf{x}^{(0)}, k_0) \geq \delta$, $i = 1, \dots, n$, and $x_i(k+1, \mathbf{x}^{(0)}, k_0) \geq \delta \exp(h\tilde{\omega})$ for $k \geq k_0 + T$, $x_i(k, \mathbf{x}^{(0)}, k_0) \geq \delta$, $i = 1, \dots, n$.

Therefore, there exists $\widetilde{T} \geq T$, such that $\Delta_1 \leq x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \dots, n$, for $k \geq k_0 + \widetilde{T}$, where $\Delta_1 = \delta \min\{1; \exp(h\tilde{\omega})\}$. This completes the proof.

Consider one more approach for a Lyapunov function constructing which permits to use for the verification of the permanence property system (4) instead of system (3).

Theorem 4.2 *Let Assumptions 3.2 and 4.1 be fulfilled. If system* (4) *admits a positive solution, then there exists $h_0 > 0$ such that system* (1) *is uniformly permanent for any $h \in (0, h_0)$ and for arbitrary switching law.*

Proof. For a constant h_0, let the condition (7) be valid, and $h \in (0, h_0)$. Consider the corresponding switched system (1).

Choose a positive vector $\mathbf{b} = (b_1, \dots, b_n)^T$ satisfying the inequalities (4). There exists a number $\gamma > 0$, such that $\sum_{i=1}^n \bar{p}_{ij}^{(s)} b_i \leq -\gamma$, $j = 1, \dots, n$, $s = 1, \dots, N$.

Construct a CLF for (2) in the form

$$V_3(\mathbf{z}) = \sum_{i=1}^{n} b_i \log z_i. \tag{9}$$

Function $V_3(\mathbf{z})$ is defined and continuous for $\mathbf{z} \in \operatorname{int} R_+^n$.

For some s in $\{1, \ldots, N\}$, consider the difference of the function (9) with respect to the sth subsystem from (2). Let $\hat{\mathbf{x}} \in \operatorname{int} R_+^n$, and $\mathbf{x}(k) = (x_1(k), \ldots, x_n(k))^T$ be the solution of the sth subsystem starting from $\hat{\mathbf{x}}$ at $k = 0$. We obtain

$$\Delta V_3\big|_{(s)} = V_3(\mathbf{x}(k+1)) - V_3(\mathbf{x}(k)) = \sum_{i=1}^{n} b_i \left(\log x_i(k+1) - \log x_i(k)\right)$$

$$= h\sum_{i=1}^{n} b_i \left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(x_j(k)) \right) = h\sum_{i=1}^{n} b_i c_i^{(s)} + h\sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_i p_{ij}^{(s)} \right) f_j(x_j(k))$$

$$\le h\sum_{i=1}^{n} b_i c_i^{(s)} - h\gamma \sum_{j=1}^{n} f_j(x_j(k)).$$

Hence, there exists a positive number H, such that $\Delta V_3\big|_{(s)} < 0$ for $\|\mathbf{x}(k)\| > H$ and for all $s = 1, \ldots, N$.

Let

$$A = H \max_{s=1,\ldots,N} \max_{i=1,\ldots,n} \max_{\|\mathbf{z}\| \le H} \exp\left(h \left(c_i^{(s)} + \sum_{j=1}^{n} p_{ij}^{(s)} f_j(z_j) \right) \right),$$

$$M_1 = \max_{0 < z_i \le A,\ i=1,\ldots,n} V_3(\mathbf{z}) = \log A \sum_{i=1}^{n} b_i.$$

In a similar way as in the proof of Theorem 3.2, it can be shown that for any $Q > 0$ there exists $T = T(Q) \ge 0$, such that if the initial values of a solution $\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)$ of (1) satisfy the conditions $k_0 \ge 0$, $\mathbf{x}^{(0)} \in B_Q$, then $V_3(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \le M_1$ for $k \ge k_0 + T$.

The fulfilment of the Assumption 4.1 implies the existence of positive numbers δ and β, such that $c_i^{(s)} + p_{ii}^{(s)} f_i(z_i) \ge \beta$ for $0 < z_i \le \delta$, $i = 1, \ldots, n$; $s = 1, \ldots, N$. Hence, if $0 < x_i(k) < \delta$ for some $i \in \{1, \ldots, n\}$, then $x_i(k+1) \ge x_i(k) \exp(h\beta)$. Without loss of generality, we assume that $\delta < 1$.

In the case when $x_i(k) \ge \delta$, the following estimates hold

$$x_i(k+1) \ge x_i(k) \exp\left(h \left(p_{ii}^{(s)} f_i(x_i(k)) \right) \right) \ge x_i(k) \exp\left(-hD \tilde{f}_i(y_i(k)) \right)$$

$$\ge x_i(k) \exp\left(-hD(L|y_i(k)| + \tilde{f}_i(0)) \right) \ge \lambda \delta^{1+hLD}.$$

Here $y_i(k) = \log x_i(k)$, $\lambda = \exp(-hD \max_{i=1,\ldots,n} f_i(1))$.

Let positive numbers δ_1 and δ_2, $\delta_1 < \delta_2$, be given. Choose the numbers $T_1 = T_1(\delta_1) > 0$ and $T_2 = T_2(\delta_2) > 0$, such that if $k_0 \ge 0$, $\delta_1 \le x_i^{(0)} \le \delta_2$, $i = 1, \ldots, n$, then $x_i(k, \mathbf{x}^{(0)}, k_0) \ge \lambda \delta^{1+hLD}$, $i = 1, \ldots, n$, for $k \ge k_0 + T_1$, and $V_3(\mathbf{x}(k, \mathbf{x}^{(0)}, k_0)) \le M_1$

for $k \geq k_0 + T_2$. By taking $\hat{T} = \max\{T_1; T_2\}$, we obtain $\Delta_1 \leq x_i(k, \mathbf{x}^{(0)}, k_0) \leq \Delta_2$, $i = 1, \ldots, n$, for $k \geq k_0 + \hat{T}$. Here $\Delta_1 = \lambda\delta^{1+hLD}$, and

$$\Delta_2 = \max_{i=1,\ldots,n} \left(\exp(M_1)/\Delta_1^{\sum\limits_{j\neq i} b_j} \right)^{1/b_i}.$$

This completes the proof.

Remark 4.1 The fulfilment of Assumption 3.2 (Assumption 3.3) with a single constant L for all $z_i \in (-\infty, +\infty)$, $i = 1, \ldots, n$, is quite severe constraint on the admissible functions $f_1(z_1), \ldots, f_n(z_n)$. It is worth mentioning that in a similar way the conditions of permanence can be obtained in the case when, for every $r > 0$, functions $\tilde{f}_i(z_i)$ satisfy Assumption 3.2 (Assumption 3.3) for $z_i \in (-\infty, r)$, $i = 1, \ldots, n$, with the constant $L(r)$, and $L(r) \to +\infty$ as $r \to +\infty$. However, in this case, we can not guarantee the permanence property for all solutions of (1). For any $Q > 0$, there exists a number $h_0 > 0$, such that for any $h \in (0, h_0)$ the conditions of Definition 2.4 are fulfilled only for $\delta_2 < Q$.

5 Example

In (1) let $n = 3$, and the family (2) consists of two subsystems with the matrices

$$\mathbf{P_1} = \begin{pmatrix} -1 & a & 0 \\ 0 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix}, \qquad \mathbf{P_2} = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -1 & 1 \\ d & 0 & -4 \end{pmatrix}.$$

Here a and d are positive parameters. In this case, $\overline{\mathbf{P}}_1 = \mathbf{P_1}$, $\overline{\mathbf{P}}_2 = \mathbf{P_2}$.

On the one hand, it is easy to verify that the system $\overline{\mathbf{P}}_1\theta < \mathbf{0}$, $\overline{\mathbf{P}}_2\theta < \mathbf{0}$ admits a positive solution if and only if

$$a < 3, \quad d < 12, \quad ad < 4. \tag{10}$$

On the other hand, for the existence of a positive solution for the system $\overline{\mathbf{P}}_1^T\mathbf{b} < \mathbf{0}$, $\overline{\mathbf{P}}_2^T\mathbf{b} < \mathbf{0}$ it is necessary and sufficient the fulfilment of the inequalities

$$a < 6, \quad d < 9, \quad ad < 18. \tag{11}$$

The regions (10) and (11) in the parameter space are nonoverlapping. Thus, this example shows that Theorems 4.1 and 4.2 complement each other.

6 Conclusion

In this paper, a discrete-time Lotka–Volterra type system with switching of parameter values is studied. The conditions are determined under which the system is ultimately bounded or permanent for any admissible switching law. Two new approaches for Lyapunov functions constructing are proposed. By the usage of these approaches, the theorems on the ultimate boundedness and permanence conditions are proved. These theorems complement each other and relax the known ultimate boundedness conditions found in [3]. The interesting direction for further research is the extension of the obtained results to switched biological models of more general form.

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