Nonlinear Dynamics and Systems Theory, **13** (4) (2013) 425–436

Synchronization Between a Fractional Order Chaotic System and an Integer Order Chaotic System

Ayub Khan [1] and Priyamvada Tripathi [2*]

[1] *Department of Mathematics, Zakir Husain Delhi College, University of Delhi-110002, India*
[2] *Department of Mathematics, University of Delhi-110007, India*

Received: November 7, 2012; Revised: October 25, 2013

Abstract: This paper deals with synchronization between a fractional order Coullet chaotic system and an integer order Rabinovich-Fabrikant chaotic system by using tracking control and stability theory of fractional order system. An effective controller is designed to synchronize these two systems. Numerical simulations have been done by using Mathematica and Matlab both. Numerical solutions via Grünwald-Letnikov method have been used in Matlab. Numerical results show that method is effective and feasible.

Keywords: *synchronization; fractional order derivatives; fractional order coullet system, integer order Rabinovich-Fabrikant chaotic system, tracking control method, Grünwald-Letnikov method.*

Mathematics Subject Classification (2010): 37B25, 37D45, 37N30, 37N35, 70K99.

1 Introduction

Synchronization is the dynamical process by which two or more oscillators adjust their rhythms due to a weak interaction [38]. This problem has received the great attention in the literature due to its importance in engineering and physical sciences, as well as in the challenging biological and social entities [38,39,44]. Chaotic synchronization did not attract much attention until Pecora and Carroll [34] introduced a method to synchronize two identical chaotic systems with different initial conditions in 1990 and they demonstrated that chaotic synchronization could be achieved by driving or replacing one of the variables of a chaotic system with a variable of another similar chaotic device. From then

* Corresponding author: mailto:priyam_du@yahoo.in

on, enormous studies have been done by researchers on the synchronization of dynamical systems. In the last two decades considerable research has been done in non-linear dynamical systems and their various properties. One of the most important properties is synchronization. Synchronization techniques have been improved in recent years and many different methods are applied theoretically as well as experimentally to synchronize the chaotic-systems including adaptive control [7,10,27], backstepping design [46–48], active control [5,23,52], nonlinear control [6,33] and observer based control method [11,50]. Using these methods, numerous synchronization problems of well-known chaotic systems such as Lorenz, Chen, Lü and Rössler system have been worked on by many researchers. In sequel to the study of chaotic systems, chaotic dynamics of fractional order systems has also been studied popularly. Since many real objects are generally fractional, so fractional calculus opens wide ways to describe a real object more accurately than the classical integer methods. So the fractional order methods become global and allow greater degree of flexibility in the study of dynamical models. Due to advantage over integer methods they have a lot of important applications in the various fields such as control theory [35, 42], viscoelastic [3], diffusion [9, 25], bioengineering [29], dielectric polarization [45], electrode-electrolyte polarization [24], electromagnetic waves [22], medicine [19] etc. Chaotic dynamics of fractional order systems is becoming an important field of investigation in nonlinear dynamics. Although the fractional calculus is more than three century old subject, yet in past few years it has increased rapidly. Analysis of fractional order dynamical systems has been studied by authors in [31,32,40]. Geometric and physical interpretation of fractional integration and fractional differentiation has also been studied by Podlubny [41]. In the continuation of study of chaos in fractional order dynamical systems, one of the important property synchronization of dynamical systems of fractional order has also got much attention. We can see many works on chaos in fractional order systems: in Chen's system [26], Volta system [37], Rössler system [12], Chua system [21], Duffing oscillators [18], cellular network [1], Lorenz system [20] etc. And synchronization between identical as well as non-identical fractional order systems has been presented in [4, 13, 14, 16, 17, 28, 49, 51, 53, 54].

The aim of this study is to investigate the synchronization behavior between an integer order system and a fractional order chaotic system. Synchronization between different orders has its own importance since it plays an important role in security of communication as well as it can also generate hybrid chaotic transient signals before final states. So it is necessary to synchronize two different order systems. Here we have used tracking control method to synchronize Rabinovich-Fabrikant integer order system and Coullet chaotic system of fractional order. Numerical simulations have been done by using both Matlab and Mathematica. For fractional order system we have used Grünwald-Letnikov method [40].

2 Preliminaries

In this section we mention some fundamental properties and definitions of fractional order derivatives.

2.1 Fractional derivatives and its approximations

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^{\alpha}$, where a and t are the limits of the operation and α is the fractional order which can be a complex number, $R(\alpha)$ denotes the real part of α. The continuous integro-differential operator is defined as:

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \int_a^t d\tau^{-\alpha}, & R(\alpha) < 0. \end{cases} $$

The three definitions used for general fractional differintegral are Grünwald-Letnikov(GL) definition, the Riemann-Liouville (RL) and Caputo's definition [40]. The GL definition is given as:

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh), $$

where $[\cdot]$ denotes the integer part. And the RL definition is given by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n, $$

where $\Gamma(\cdot)$ is the gamma function. The Caputo fractional derivative is

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha < n. $$

For numerical calculations of fractional-order derivatives we have used Grünwald-Letnikov method which is derived from Grünwald-Letnikov definition. It is also called Power Series Expansion method [8, 15, 36].

2.2 Methodology for synchronization between fractional order and integer order chaotic system

In this section we put a glimpse of methodology for synchronization between fractional order and integer order chaotic system via tracking control. Consider the following n-dimensional fractional order chaotic system as drive (master) system

$$ \frac{d^{q_\alpha} x}{dt^{q_\alpha}} = f(x), \tag{2.1} $$

where $x \in \mathbb{R}^n$, fractional order $q_\alpha = (q_{\alpha_1}, q_{\alpha_2}, ...q_{\alpha_n})^T$; $(0 < q_{d_i} < 1)$ may be unequal. $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function. Now, consider the following n-dimensional chaotic system of integer order as :

$$ \frac{dy}{dt} = g(y), $$

where $y \in \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function and construct the following integer order response system:

$$ \frac{dy}{dt} = g(y) + u(y, x), \tag{2.2} $$

where $u(y, x)$ is the controller to be designed via tracking control method.

Our goal in this paper is to design controller $u(y, x)$ such that

$$ \lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y - x\| = 0, $$

where $\| \cdot \|$ is the Euclidean norm (here x in response system (2.2) belongs to chaotic system (2.1)), then the systems (2.1) and (2.2) will be synchronized.

2.3 Stability of Fractional order systems

An autonomous system $D^q X = AX, X(0) = 0$, with $0 < q \leq 1$, $X \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is asymptotically stable iff $|\arg \lambda| > q\pi/2$ is satisfied for all eigenvalues (λ)of matrix A. Also this system is stable iff $|\arg \lambda| \geq q\pi/2$ is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy $|\arg \lambda| > q\pi/2$ have geometric multiplicity one [30].

3 System Description

3.1 Rabinovich-Fabrikant chaotic system of integer order

The Rabinovich-Fabrikant chaotic system is a set of three coupled ordinary differential equations exhibiting chaotic behavior for certain values of parameters. They are named after Mikhail Rabinovich and Anatoly Fabrikant, who described them in 1979 [43]. The equations of system are:

$$\left.\begin{aligned} \dot{y_1} &= y_2(y_3 - 1 + y_1^2) + \gamma y_1, \\ \dot{y_2} &= y_1(3y_3 + 1 - y_1^2) + \gamma y_2, \\ \dot{y_3} &= -2y_3(y_1 y_2 + \alpha), \end{aligned}\right\} \tag{3.1}$$

where α and γ are constant parameters that control the evolution of the system. For some values of α and γ, the system is chaotic but for other it tends to a stable periodic orbit. Figures given below show the chaotic behavior of Rabinovich-Fabrikant system with different values of parameters.

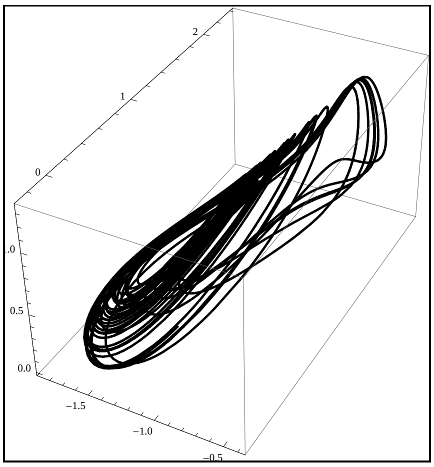

Figure 1: Chaotic behavior of the system with $\alpha = 0.87$ and $\gamma = 1.1$.

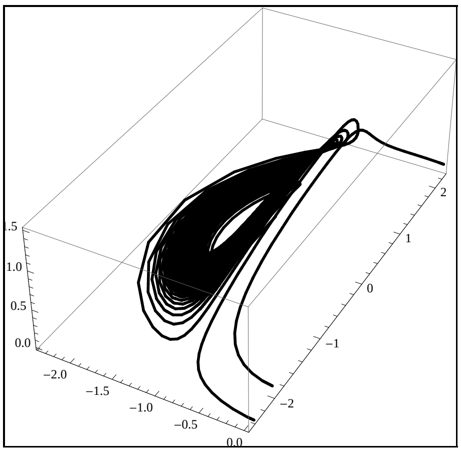

Figure 2: Chaotic behavior having tendency of stable periodic orbit with $\alpha = 0.14$ and $\gamma = 0.1$.

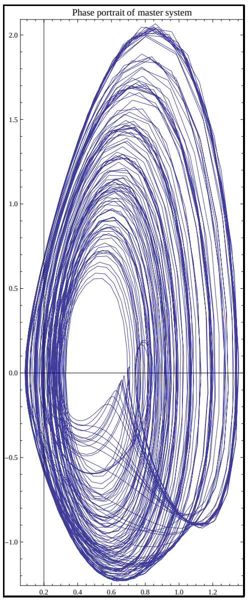

Figure 3: Phase portrait shows chaotic behavior of the system.

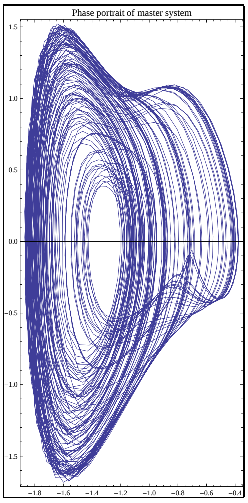

Figure 4: Phase portrait shows chaotic behavior of the system.

3.2 Coullet chaotic system of fractional order

The Coullet chaotic system consists of three fractional order differential equations with orders q_1, q_2, and q_3, respectively [2]

$$\left.\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= x_2, \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= x_3, \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= ax_1 + bx_2 + cx_3 + dx_1^3, \end{aligned}\right\} \tag{3.2}$$

where $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. We can vary values of q_1, q_2, and q_3 accordingly. Figures given below show chaotic behavior of the system with different values of q_1, q_2, and q_3, respectively.

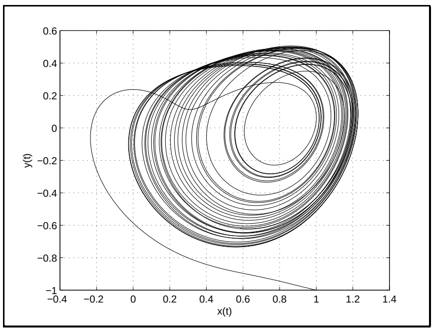

Figure 5: Chaotic attractor of the Coullet system in xy-plane with $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$.

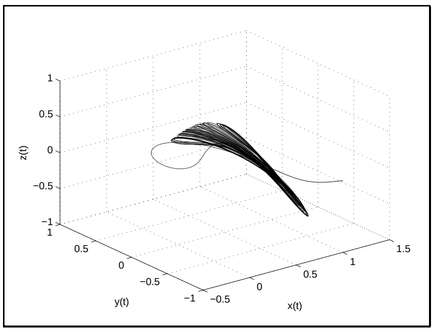

Figure 6: 3D chaotic attractor of the Coullet system with $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$.

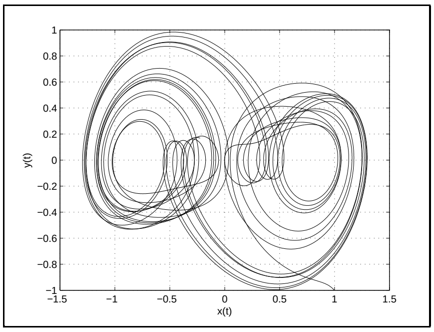

Figure 7: Chaotic attractor of the Coullet system in xy-plane with $q_1 = q_2 = q_3 = 0.98$.

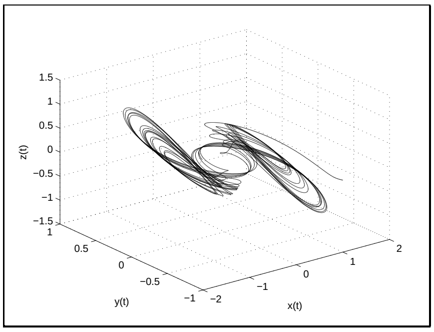

Figure 8: 3D chaotic attractor of the Coullet system with $q_1 = q_2 = q_3 = 0.98$.

3.3 Synchronization between Coullet chaotic system of fractional order and Rabinovich-Fabrikant chaotic system of integer order via Tracking control

In this section, we synchronize a fractional order derivative and an integer order derivative via tracking control. Consider fractional order Coullet system as a drive (master) system:

$$\left.\begin{aligned} \frac{d^{q_1}x_1}{dt^{q_1}} &= x_2, \\ \frac{d^{q_2}x_2}{dt^{q_2}} &= x_3, \\ \frac{d^{q_3}x_3}{dt^{q_3}} &= ax_1 + bx_2 + cx_3 + dx_1^3, \end{aligned}\right\} \tag{3.3}$$

where $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. Here we have taken $q_1 = q_2 = q_3 = 0.98$ and $q_1 = 0.90$, $q_2 = 0.97$, and $q_3 = 0.95$. One can take any other values of q_1, q_2 and q_3 $(0 < q \leq 1)$.

Now integer order Rabinovich-Fabrikant chaotic system is:

$$\left.\begin{aligned} \dot{y_1} &= y_2(y_3 - 1 + y_1^2) + \gamma y_1, \\ \dot{y_2} &= y_1(3y_3 + 1 - y_1^2) + \gamma y_2, \\ \dot{y_3} &= -2y_3(y_1y_2 + \alpha), \end{aligned}\right\} \tag{3.4}$$

where α and γ are constant parameters. For $\alpha = 0.87$ and $\gamma = 1.1$ system is chaotic but for $\alpha = 0.14$ and $\gamma = 0.1$ (see Figures 1 and 2) it tends to a stable periodic orbit. Now construct Rabinovich-Fabrikant chaotic system as response system. The response system is:

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1y_2 + \alpha) \end{pmatrix} + \theta(x) + \tau(y, x), \tag{3.5}$$

where $\theta(x)$ is compensation controller and $\tau(y, x)$ is feedback controller.

According to methodology, we can obtain compensation controller for response system (3.5) as follows:

$$\theta(x) = \frac{dx}{dt} - g(x) = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} - \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix}. \tag{3.6}$$

So from equation (3.5) and (3.6),

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1y_2 + \alpha) \end{pmatrix} + \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix}$$

$$- \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix} + \tau(y, x). \tag{3.7}$$

Let error $e_i = y_i - x_i$; $i = 1, 2, 3$. Then error system can be obtained from (3.7) described by

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} - \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} y_2(y_3 - 1 + y_1^2) + \gamma y_1 \\ y_1(3y_3 + 1 - y_1^2) + \gamma y_2 \\ -2y_3(y_1y_2 + \alpha) \end{pmatrix}$$

$$- \begin{pmatrix} x_2(x_3 - 1 + x_1^2) + \gamma x_1 \\ x_1(3x_3 + 1 - x_1^2) + \gamma x_2 \\ -2x_3(x_1x_2 + \alpha) \end{pmatrix} + \tau(y, x).$$

This implies

$$\begin{pmatrix} \frac{de_1}{dt} \\ \frac{de_2}{dt} \\ \frac{de_3}{dt} \end{pmatrix} = G_1(x, e) + G_2(x, e), \tag{3.8}$$

where $G_1(x, e) = g(x_i + e_i) - g(x_i)$ and $G_2(x, e) = \tau(x_i + e_i, x_i)$; $i = 1, 2, 3$. Now the vector function $G_1(x, e)$ is

$$\begin{pmatrix} 2x_1x_2e_1 + 2\gamma x_1e_1 + 2x_1e_1e_2 + x_2e_1^2 + e_2x_1^2 + e_2x_3 + e_3x_2 + e_2e_3 - e_2 + e_2e_1^2 + \gamma e_1^2 \\ 3x_1e_3 + 3x_3e_1 - 3e_1x_1^2 - 3x_1e_1^2 + 3e_1e_3 + e_1 - e_1^3 + 2\gamma x_2e_2 + \gamma e_1^2 \\ -2\alpha e_3 - 2x_1x_2e_3 - 2x_1x_3e_2 - 2x_1e_2e_3 - 2e_1x_2x_3 - 2x_2e_1e_3 - 2x_3e_1e_2 - 2e_1e_2e_3 \end{pmatrix}.$$

Hence, we can choose that

$$\overline{e_1} = e_1, \overline{e_2} = (e_2, e_3)^T, A_1 = (0), A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha \end{pmatrix},$$

$F_1(x,\overline{e_1},\overline{e_2})$

$$= \left(\; 2x_1x_2e_1 + 2\gamma x_1e_1 + 2x_1e_1e_2 + x_2e_1^2 + e_2x_1^2 + e_2x_3 + e_3x_2 + e_2e_3 - e_2 + e_2e_1^2 + \gamma e_1^2 \; \right),$$

$$F_{21}(x,\overline{e_1},\overline{e_2}) = \begin{pmatrix} 3x_3e_1 - 3e_1x_1^2 - 3x_1e_1^2 + 3e_1e_3 + e_1 - e_1^3 \\ -2e_1x_2x_3 - 2x_2e_1e_3 - 2x_3e_1e_2 - 2e_1e_2e_3 \end{pmatrix},$$

$$F_{22}(x,\overline{e_1},\overline{e_2}) = \begin{pmatrix} 3x_1e_3 - 2\gamma e_2x_2 + \gamma e_2^2 \\ -2e_3x_1x_2 \end{pmatrix}.$$

So, the vector function

$$G_1(x,e) = \begin{pmatrix} A_1\overline{e_1} + F_1(x,\overline{e_1},\overline{e_2}) \\ A_2\overline{e_2} + F_{21}(x,\overline{e_1},\overline{e_2}) + F_{22}(x,\overline{e_1},\overline{e_2}) \end{pmatrix}. \tag{3.9}$$

Obviously, $\lim_{e_1\to 0} F_{21}(x,\overline{e_1},\overline{e_2}) = 0$. According to tracking control method we can define feedback controller as

$$G_2(x,e) = \begin{pmatrix} \Omega_1(x,\overline{e_1},\overline{e_2}) \\ \Omega_2(x,\overline{e_1},\overline{e_2}) \end{pmatrix} = \begin{pmatrix} \Lambda_1\overline{e_1} - F_1(x,\overline{e_1},\overline{e_2}) \\ \Lambda_2\overline{e_2} - F_{22}(x,\overline{e_1},\overline{e_2}) \end{pmatrix}. \tag{3.10}$$

So, from equations (3.9) and (3.10) response system (3.8) can be rewritten as

$$\begin{cases} \frac{d\overline{e_1}}{dt} = (A_1+\Lambda_1)\overline{e_1}, \\ \frac{d\overline{e_2}}{dt} = (A_2+\Lambda_2)\overline{e_2} + F_{21}(x,\overline{e_1},\overline{e_2}), \end{cases} \tag{3.11}$$

so, we choose now suitable $A_1+\Lambda_1 \in \mathbb{R}^1$ and $A_2+\Lambda_2 \in \mathbb{R}^{2\times 2}$, which satisfy $|\arg\lambda| > \pi/2$ (here $q=1$). As equation (3.11) is asymptotically stable with equilibrium point $e_1 = 0$ and $\overline{e_2} = 0$. Obviously, $\lim_{t\to\infty}\|e_1\| = 0$ and $\lim_{e_1\to 0} F_{21}(x,\overline{e_1},\overline{e_2}) = 0$, then the synchronization between response system and master system can be achieved.

4 Numerical Simulations

Parameters of the integer order Rabinovich-Fabrikant chaotic system are $\alpha = 0.87$ and $\gamma = 1.1$ and for fractional order Coullet system $a = 0.8$, $b = -1.1$, $c = -0.45$, and $d = -1$. The fractional order is taken to be $q = q_1 = q_2 = q_3 = 0.98$ and $q_1 = 0.97, q_2 = 0.95$ and $q_3 = 0.90$ for which the systems are chaotic. In equation (3.11) we have chosen $\Lambda_1 = (-1)$ and $\Lambda_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, which leads to stability conditions as eigenvalue of $A_1+\Lambda_1$ is $\lambda_1 = -1$ and eigenvalues of $A_2+\Lambda_2$ are $\lambda_2 = -1, \lambda_3 = -1.74$ when $\alpha = 0.87$ and $\lambda_2 = -1, \lambda_3 = -0.28$ when $\alpha = 0.14$. The initial conditions for master and slave systems $[x_1(0), x_2(0), x_3(0)] = [0.1, 0.4, 0.3]$ and $[y_1(0), y_2(0), y_3(0)] = [-1, 0, 0.5]$ respectively and for $[e_1(0), e_2(0), e_3(0)] = [-1.1, -0.4, 0.2]$ diagrams of convergence of errors given below are the witness of achieving synchronization between master and slave system.

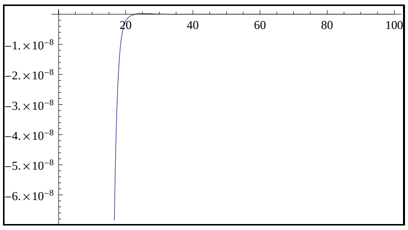

Figure 9: Convergence error of e_1, $t = [0, 100]$ with $\alpha = 0.87$.

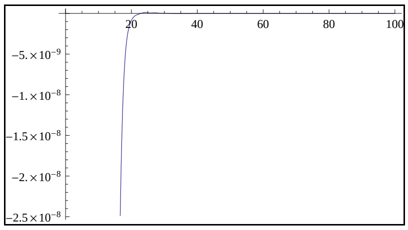

Figure 10: Convergence error of e_2, $t = [0, 100]$ with $\alpha = 0.87$.

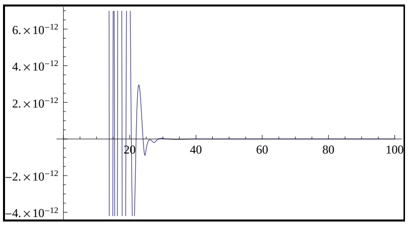

Figure 11: Convergence error of e_3, $t = [0, 100]$ with $\alpha = 0.87$.

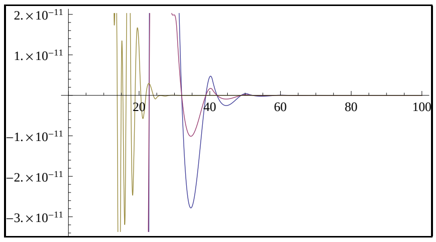

Figure 12: Combined Convergence error of e_1, e_2, and e_3, $t = [0, 100]$ with $\alpha = 0.87$.

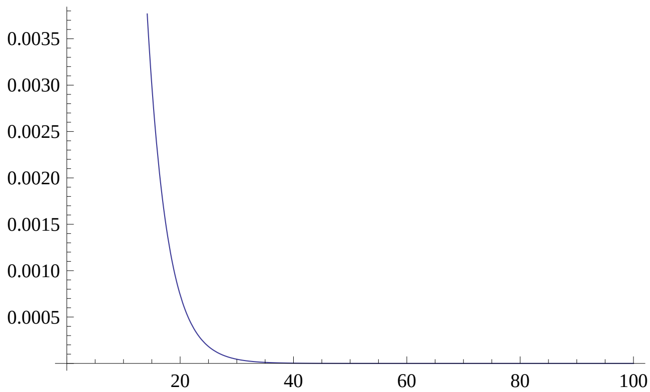

Figure 13: Graph between $e = \sqrt{e_1^2 + e_2^2 + e_3^2}$ and $t = [0, 100]$ with $\alpha = 0.87$ shows synchronization between drive and response system.

5 Conclusion

In this paper, we have investigated synchronization behavior between an integer order Rabinovich-Fabrikant chaotic system and fractional order Coullet chaotic system via tracking control method and stability of fractional order systems. The results are validated by numerical simulations using Mathematica and Matlab both. Synchronization between two different orders has more advantage over synchronization between same order systems. Synchronization between two different order chaotic systems is more beneficial to enhance security of communication.

References

[1] Arena, P., Fortuna, L. and Porto, D. Chaotic behavior in noninteger-order cellular neural networks. *Phys. Rev. E* **61** (2000) 776–781.

[2] Arneodo, A., Coullet, P. and Tresser, C. Possible new strange attractors with spiral structure. *Communication in Mathematical Physics* **79(4)** (1981) 573-579.

[3] Bagley, R. L. and Calico, R. A. Fractional order state equations for the control of viscoelastically damped structures. *J Guid Control Dynam* **14** (1991) 304-311.

[4] Bhalekar, S. and Daftardar-Gejji, V. Synchronization of different fractional order chaotic systems using active control. *Commun. Nonlinear Sci. Numer. Simul.* **15** (2010) 3536-3546.

[5] Chen, H. K. Synchronization of two different chaotic system: A new system and each of the dynamical systems Lorenz, Chen and Lu. *Chaos Solitons & Fractals* **25** (2005) 1049–1056.

[6] Chen, H. K. Global chaos synchronization of new chaoic systems via nonlinear control. *Chaos Solitons & Fractals* **23** (2005) 1245–1251.

[7] Chen, S. H. and Lu, J. Synchronization of an uncertain unified system via adaptive control. *Chaos Solitons & Fractals* **14** (2002) 643–647.

[8] Chen, Y. Q. and Moore, K. L. Discretization Schemes for Fractional-Order Differentiators and Integrators. *IEEE Trans. Circuits and Systems - I: Fundamental Theory and Applications* **49(3)** (2002) 363-367.

[9] Daftardar-Gejji, V. and Bhalekar, S. Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method. *Appl. Math. Comput.* **202** (2008) 113-120.

[10] Ge, Z. M. and Chen, Y. S. Adaptive synchronization of unidirectional and mutual coupled chaotic systems. *Chaos Solitons & Fractals* **26** (2005) 881–888.

[11] Ge, Z. M., Yu, T. C. and Chen, Y.S. Chaos synchronization of a horizontal platform system. *J. Sound Vibrat.* **268** (2003) 731–749.

[12] Chunguang, L. and Guanrong, C. Chaos and hyperchaos in the fractional-order Rössler equations. *Physica A* **341** (2004) 55–61.

[13] Das, S. and Gupta, P. K. A mathematical model on fractional Lotka V olterra equations. *J. Theor. Biol.* **277** (2011) 1–6.

[14] Deng, W. H. and Li, C. P. Chaos synchronization of the fractional Lü system. *Physica A* **353** (2005) 61–72.

[15] Dorčàk, Ľ. *Numerical models for simulation of the fractional-order control systems.* UEF-04-94, The Academy of Sciences, Inst. of Experimental Physic, Košice, Slovakia, 1994.

[16] Erjaee, G.H. and Taghvafard, H. Stability analysis of phase synchronization in coupled chaotic system presented by fractional differential equations. *Nonlinear Dyn. Syst. Theory* **11**(2) (2011) 147–154.

[17] Filali, R. L., Hammami, S., Benrejeb, M. and Borne, P. On synchronization, anti-synchronization and Hybrid synchronization of 3D discrete generaized Henon map. *Nonlinear Dyn. Syst. Theory* **12**(1) (2012) 81–95.

[18] Gao, X. and Yu, J. Chaos in the fractional-order periodically forced complex Duffing s oscillators. *Chaos Solitons & Fractals* **24** (2005) 1097-1104.

[19] Glockle, W. G., Mattfeld, T., Nonnenmacher, T. F. and Weibel, E. R. *Fractals in biology and medicine.* Basel, Birkhauser, 1998.

[20] Grigorenko, I. and Grigorenko, E. Chaotic dynamics of the fractional Lorenz system. *Phys. Rev. Lett.* **91** (2003) 034101–4.

[21] Hartley, T. T., Lorenzo, C. F. and Qammer, H. K. Chaos on a fractional Chua s system. *IEEE Trans. Circ. Syst. Theory Appl.* **42(8)** (1995) 485-490.

[22] Heaviside, O. *Electromagnetic Theory.* New York, Chelsea, 1971.

[23] Ho, M. C. and Hung, Y. C. Synchronization two different systems by using generalized active control. *Phys. Lett. A* **301** (2002) 424–428.

[24] Ichise, M., Nagayanagi, Y. and Kojima, T. An analog simulation of noninteger order transfer function for analysis of electrode process. *J. Electroanal Chen.* **33** (1971) 253–265.

[25] Jesus, I. S. and Machado, J. Fractional control of heat diffusion systems. *Nonlinear Dyn.* **54(3)** (2008) 263–282.

[26] Li, C. P. and Peng, G. J. Chaos in Chens system with a fractional order. *Chaos Solitons & Fractals* **22** (2004) 443 -450.

[27] Liao, T. L. and Lin, S. H. Adaptive control and synchronization of Lorenz systems. *J. Franklin Inst.* **336** (1999) 925– 937.

[28] Lu, J. G. Chaotic dynamics and synchronization of fractional-order Arneodos systems. *Chaos Solitons & Fractals* **26** (2005) 1125-1133.

[29] Magin, R. L. *Fractional Calculus in Bioengineering.* USA, Begll House Publishers, 2006.

[30] Matignon, D. Stability results of fractional differential equations with applications to control processing. In: *Proc. of the IMACS-IEEE Multiconference on Computational Engineering in Systems Applications (CESA '96)*, Lille, France, July, 1996, 963–968.

[31] Miller, K. and Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* San Fransisco, A Wiley-Interscience Publication, 1993.

[32] Oldham, K. B. and Spanier, J. *The Fractional Calculus.* Academic Press, A subsidiary of Harcourt Brace Jovanovich, Publishers, New York-London, 1974.

[33] Park, J. H. Chaos synchronization between two different chaotic dynamical systems. *Chaos Solitons & Fractals* **27** (2006) 549–554.

[34] Pecora, L. M. and Carroll, T. L. Synchronization in chaotic systems. *Phys. Rev. Lett.* **64** (1990) 821–824.

[35] Petráš, I. and Dorčàk, Ľ. Fractional order control systems: modelling and simulation. *Fract. Calculus Appl. Anal.* **6(2)** (2003) 205–232.

[36] Petráš, I. Digital fractional-order differentiator/integrator fir type. MathWorks, Inc. Matlab Central File Exchange, URL: www.mathworks.com/matlabcentral /fileexchange/3673, May 26 (2008).

[37] Petráš, I. Chaos in the fractional-order Volta s system: modeling and simulation. *Nonlinear Dyn.* **57** (2009) 157-170.

[38] Pikovsky, A., Rosenblum, M. and Kurths, J. *Synchronization: A Universal Concept in Nonlinear Science.* Cambridge University Press, 2002.

[39] Glass, L. Synchronization and rhythmic processes in physiology. *Nature* (2001) 277–477.

[40] Podlubny, I. *Fractional Differential Equations.* New York, Academic Press, 1999.

[41] Podlubny, I. Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calculus Appl. Anal.* **5(4)** (2002) 367–386.

[42] Sabatier, J., Poullain, S., Latteux, P., Thomas, J. and Oustaloup, A. Robust speed control of a low damped elecromechanical system based on CRONE control: application to a four mass experimental test bench. *Nonlinear Dyn.* **38** (2004) 383–400.

[43] Rabinovich, I. M. and Fabrikant, A. L. Stochastic Self- Modulation of Waves in Nonequilibrium Media. *Sov. Phys. JETP* **50** (1979) P. 311.

[44] Strogatz, S. *Sync*. Hyperion, 2003.

[45] Sun, H. H., Abdelwahad, A. A. and Onaral, B. Linear approximation of transfer function with a pole of fractional order. *IEEE Automat. Control* **29(5)** (1984) 441–444.

[46] Tan, X., Zhang, J. and Yang, Y. Synchronizing chaotic systems using backstepping design. *Chaos Solitons & Fractals* **16** (2003) 37–45.

[47] Wang, C. and Ge, S. S. Synchronization of two uncertain chaotic systems via adaptive backstepping. *Int. J. Bifurcat. Chaos* **11** (2001) 1743-1751.

[48] Wang, C. and Ge, S. S., Adaptive synchonizaion of uncertain chaotic systems via adaptive backstepping design. *Chaos Solitons & Fractals* **12** (2001) 1199–1206.

[49] Wang, X. Y. and Wang, M. J. Dynamic analysis of the fractional-order Liu system and its synchronization. *Chaos,* Article ID 033106, **17(3)** (2007) 1–6.

[50] Wen, G. L. and Xu, D. Observer-based control for full-state projective synchronization of a general class of chaotic maps in any dimension. *Phys. Lett. A* **330** (2004) 420–425.

[51] Yan, J. and Li, C. On chaos synchronization of fractional differential equations. *Chaos Solitons & Fractals* **32(2)** (2007) 725–735.

[52] Yassen, M. T. Chaos synchronization between two different chaotic systems using active control. *Chaos Solitons & Fractals* **23** (2005) 131–140.

[53] Zhou, P. Chaotic synchronization for a class of fractional-order chaotic systems. *Chinese Physics* **16(5)** (2007) 1263–1266.

[54] Zhoua, T. and Lib, C. Synchronization in fractional-order differential systems. *Physica D* **212** (2005) 111-125.