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Infinitely Many Solutions for a Discrete Fourth Order Boundary Value Problem

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Abstract: By using variational methods and critical point theory, the authors obtain criteria for the existence of infinitely many solutions to the fourth order discrete boundary value problem

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = \lambda f(t, u(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) - \alpha\Delta u(T) = \mu g(u(T+1)), \end{cases}$$

where $T \geq 2$ is an integer, $[1, T]_{\mathbb{Z}} = \{1, 2, \dots, T\}$, $\alpha, \beta, \lambda, \mu \in \mathbb{R}$ are parameters, $f \in C([1, T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, and $g \in C(\mathbb{R}, \mathbb{R})$. Several consequences of their main theorems are also presented. One example is included to show the applicability of the results.

Keywords: *discrete boundary value problem; infinitely many solutions; fourth order; variational methods.*

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1 Introduction

Throughout this paper, for any integers a and b with $a \leq b$, let $[a, b]_{\mathbb{Z}}$ denote the discrete interval $\{a, a+1, \ldots, b\}$. Here, we are concerned with the existence of solutions of the four-parameter fourth order discrete boundary value problem (BVP)

$$\begin{cases} \Delta^4 u(t-2) - \alpha \Delta^2 u(t-1) + \beta u(t) = \lambda f(t, u(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) - \alpha \Delta u(T) = \mu g(u(T+1)), \end{cases} \tag{1.1}$$

where $T \geq 2$ is an integer, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^k u(t) = \Delta^{k-1}(\Delta u(t))$ for $k = 2, 3, 4$, $\alpha, \beta, \lambda, \mu$ are four parameters with $\alpha, \beta \in \mathbb{R}$, $\lambda \in (0, \infty)$, $\mu \in [0, \infty)$, $f \in C([1, T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$, and $g \in C(\mathbb{R}, \mathbb{R})$. By a *solution* of (1.1), we mean a function $u \in C([-1, T+2]_{\mathbb{Z}}, \mathbb{R})$ satisfying (1.1). We assume throughout, and without further mention, that the following condition holds:

(H1) α and β satisfy

$$1 + \alpha_-(T+1)^2 + \beta_- T^2 (T+1)^2 > 0,$$

where $\alpha_- = \min\{\alpha, 0\}$ and $\beta_- = \min\{\beta, 0\}$.

Difference equations appear in numerous settings and forms, both in mathematics and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields ([1,19]). In recent years, many researchers have paid a lot of attention to fourth order BVPs for difference equations with various boundary conditions. The reader may refer to [2,6,7,11,13,14,16–18,20,22,26,28] and the included references for some recent work.

We point out, depending on the values of the parameters α, β, λ, and μ, that BVP (1.1) covers many problems as special cases. For instance, if $\alpha = \beta = 0$ and $\mu = 1$, BVP (1.1) becomes

$$\begin{cases} \Delta^4 u(t-2) = \lambda f(t, u(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0, \ \Delta^3 u(T-1) = g(u(T+1)). \end{cases} \tag{1.2}$$

The continuous version of BVP (1.2), i.e., the problem

$$\begin{cases} u^{(4)}(t) = \lambda f(t, u), \quad t \in (0, 1), \\ u(0) = u'(0) = u''(1) = 0, \ u'''(1) = g(u(1)), \end{cases}$$

has recently been investigated in [24] where results for the existence of three solutions are obtained. Notice that BVPs for fourth order differential equations have been extensively studied in the literature. For a small sample of recent work, see [9,12,14,15,23–25].

The existence of three solutions of BVP (1.1) has been studied in [11]. In this paper, we continue our study on BVP (1.1). We apply variational methods and critical point theorem to establish some criteria for the existence of infinitely many solutions of BVP (1.1). We also present several consequences of our main theorems. Our analysis is mainly based on a recent theorem on critical points that appeared in [3,21]; see Lemma 4.1 below. This lemma and its variations have been frequently used to obtain multiplicity results for nonlinear problems of a variational nature; see, for example, [3–5,8,10,21] and the references therein. Our proofs are partly motivated by these papers.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Section 3 contains the main results of this paper and one illustrative example, and the proofs of the main results are presented in Section 4.

2 Preliminary Lemmas

We define a real vector space

$$X = \left\{u : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R} \ : \ u(-1) = u(0) = 0,\ \Delta^2 u(T) = 0\right\}. \tag{2.1}$$

For any $u \in X$, we let

$$||u||_X = \left(\sum_{t=1}^{T+1} \left(|\Delta^2 u(t-2)|^2 + \alpha|\Delta u(t-1)|^2\right) + \beta \sum_{t=1}^{T} |u(t)|^2\right)^{1/2}.$$

Let

$$\rho = (T+1)^{3/2}\left(1 + \alpha_-(T+1)^2 + \beta_- T^2(T+1)^2\right)^{-1/2}. \tag{2.2}$$

Clearly, $\rho > 0$ by condition (H1).

The following result is taken from [11, Lemma 2.1].

Lemma 2.1 *For any $u \in X$, we have*

$$\sum_{t=1}^{T+1} \left(|\Delta^2 u(t-2)|^2 + \alpha|\Delta u(t-1)|^2\right) + \beta \sum_{t=1}^{T} |u(t)|^2 \geq 0$$

and

$$|u(t)| \leq \rho ||u||_X \quad \text{for } t \in [1, T+1]_{\mathbb{Z}}. \tag{2.3}$$

Hence, $||\cdot||_X$ is a norm on X with which X becomes a $T+1$ dimensional separable and reflexive Banach space.

For any $u \in X$, let the functionals Φ and Ψ be defined by

$$\Phi(u) = \frac{1}{2}||u||_X^2 \tag{2.4}$$

and

$$\Psi(u) = \sum_{t=1}^{T} F(t, u(t)) - \frac{\mu}{\lambda} G(u(T+1)), \tag{2.5}$$

where

$$F(t, x) = \int_0^x f(t, s)ds, \quad (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}, \tag{2.6}$$

and

$$G(x) = \int_0^x g(s)ds, \quad x \in \mathbb{R}. \tag{2.7}$$

Then, Φ and Ψ are well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at $u \in X$ are the functionals $\Phi'(u)$ and $\Psi'(u)$ given by

$$\Phi'(u)(v) = \sum_{t=1}^{T+1} \left(\Delta^2 u(t-2)\Delta^2 v(t-2) + \alpha \Delta u(t-1)\Delta v(t-1)\right) + \beta \sum_{t=1}^{T} u(t)v(t)$$

and

$$\Psi'(u)(v) = \sum_{t=1}^{T} f(t, u(t))v(t) - \frac{\mu}{\lambda} g(u(T+1))v(T+1)$$

for any $v \in X$.

Lemma 2.2 below follows from [11, Lemma 2.3].

Lemma 2.2 *The function $u \in X$ is a critical point of the functional $\Phi - \lambda\Psi$ if and only if u is a solution of BVP* (1.1).

3 Main Results

In this section, we present our main results. In what follows, let X, ρ, F, and G be defined by (2.1), (2.2), (2.6), and (2.7), respectively. For convenience, we use the following notation:

$$A = \liminf_{\xi\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi} F(t,x)}{\xi^2}, \quad B = \limsup_{\xi\to\infty} \frac{\sum_{t=1}^{T} F(t,\xi)}{\xi^2}, \tag{3.1}$$

$$C = \liminf_{\xi\to 0^+} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi} F(t,x)}{\xi^2}, \quad D = \limsup_{\xi\to 0^+} \frac{\sum_{t=1}^{T} F(t,\xi)}{\xi^2}, \tag{3.2}$$

$$\lambda_1 = \frac{2+\alpha+\beta T}{2B}, \quad \lambda_2 = \frac{1}{2\rho^2 A}, \tag{3.3}$$

$$\lambda_3 = \frac{2+\alpha+\beta T}{2D}, \quad \lambda_4 = \frac{1}{2\rho^2 C}.$$

In the following, we assume that

(H2) $A,\ B,\ C,\ D \ge 0$.

We also use the convention that $1/a = \infty$ when $a = 0$.

We now state our main results in the paper.

Theorem 3.1 *Assume that*

$$A < \frac{B}{\rho^2(2+\alpha+\beta T)}. \tag{3.4}$$

Then, for each $\lambda \in (\lambda_1, \lambda_2)$, for each function $g \in C(\mathbb{R},\mathbb{R})$ with

$$g(x) \le 0 \text{ on } \mathbb{R} \quad \text{and} \quad G_\infty = \liminf_{\xi\to\infty} \frac{G(\xi)}{\xi^2} > -\infty, \tag{3.5}$$

and for each $\mu \in [0, \overline{\mu}_1)$ with

$$\overline{\mu}_1 = \frac{1-2\rho^2\lambda A}{-2\rho^2 G_\infty}, \tag{3.6}$$

BVP (1.1) *has a sequence of solutions that is unbounded in X.*

Theorem 3.2 *Assume that*

$$C < \frac{D}{\rho^2(2+\alpha+\beta T)}. \tag{3.7}$$

Then, for each $\lambda \in (\lambda_3, \lambda_4)$, for each function $g \in C(\mathbb{R},\mathbb{R})$ satisfying (3.4)*, and for each $\mu \in [0, \overline{\mu}_2)$ with*

$$\overline{\mu}_2 = \frac{1-2\rho^2\lambda C}{-2\rho^2 G_\infty},$$

BVP (1.1) *has a sequence of solutions converging uniformly to zero in X.*

Remark 3.1 For Theorems 3.1 and 3.2, we make the following comments.

(a) It is easy to verify that condition (H) implies $2+\alpha+\beta T>0$. Thus, $\lambda_1 \geq 0$ and $\lambda_3 \geq 0$.

(b) By the assumptions (3.4) and (3.7), we see that $\lambda_1 < \lambda_2$ and $\lambda_3 < \lambda_4$. This assures that the intervals (λ_1, λ_2) and (λ_3, λ_4) are nonempty.

(c) The interval $[0, \overline{\mu}_1)$ is well defined since $\overline{\mu}_1 > 0$ under the condition that $\lambda < \lambda_2$.

(d) The interval $[0, \overline{\mu}_2)$ is well defined since $\overline{\mu}_2 > 0$ under the condition that $\lambda < \lambda_4$.

The following results are direct consequences of Theorems 3.1 and 3.2.

Corollary 3.1 *Assume that* (3.4) *holds. Then, for each* $\lambda \in (\lambda_1, \lambda_2)$*, the BVP*

$$\begin{cases} \Delta^4 u(t-2) - \alpha \Delta^2 u(t-1) + \beta u(t) = \lambda f(t, u(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0,\ \Delta^3 u(T-1) - \alpha \Delta u(T) = 0, \end{cases} \tag{3.8}$$

has a sequence of solutions which is unbounded in X*.*

Corollary 3.2 *Assume that* (3.7) *holds. Then, for each* $\lambda \in (\lambda_3, \lambda_4)$*, BVP* (3.8) *has a sequence of solutions converging uniformly to zero in* X*.*

Corollary 3.3 *Assume that* $A = 0$ *and* $B = \infty$*. Then, for each* $\lambda \in (0, \infty)$*, for each function* $g \in C(\mathbb{R}, \mathbb{R})$ *with*

$$g(x) \leq 0 \quad \text{on } \mathbb{R} \quad \text{and} \quad G_\infty = \liminf_{\xi \to \infty} \frac{G(\xi)}{\xi^2} = 0, \tag{3.9}$$

and for each $\mu \in [0, \infty)$*, BVP* (1.1) *has a sequence of solutions which is unbounded in* X*.*

Corollary 3.4 *Assume that* $C = 0$ *and* $D = \infty$*. Then, for each* $\lambda \in (0, \infty)$*, for each function* $g \in C(\mathbb{R}, \mathbb{R})$ *satisfying* (3.9)*, and for each* $\mu \in [0, \infty)$*, BVP* (1.1) *has a sequence of solutions converging uniformly to zero in* X*.*

Corollary 3.5 *Assume that* $A < \frac{B}{2(T+1)^3}$*. Then, for each* $\lambda \in \left(\frac{1}{B}, \frac{1}{2A(T+1)^3}\right)$ *and each function* $g \in C(\mathbb{R}, \mathbb{R})$ *satisfying* (3.9)*, BVP* (1.2) *has a sequence of solutions which is unbounded in* X*.*

Corollary 3.6 *Assume that* $C < \frac{D}{2(T+1)^3}$*. Then, for each* $\lambda \in \left(\frac{1}{D}, \frac{1}{2C(T+1)^3}\right)$ *and each function* $g \in C(\mathbb{R}, \mathbb{R})$ *satisfying* (3.9)*, BVP* (1.2) *has a sequence of solutions converging uniformly to zero in* X*.*

We conclude this section with the following example where the construction of the nonlinear function $f(t, x)$ is partly motivated by [10, Example 3.1].

Example 3.1 Let $T \geq 2$ be an integer, $\{a_n\}$ and $\{b_n\}$ be sequences defined by $b_1 = 2$, $b_{n+1} = b_n^6$, and $a_n = b_n^4$ for $n \in \mathbb{N}$. Let $f : [0,T]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ be a positive continuous function defined by

$$f(t,x) = t^2 \begin{cases} b_1^3\sqrt{1-(1-x)^2}+1, & x \in [0,b_1], \\ (a_n - b_n^3)\sqrt{1-(a_n-1-x)^2}+1, & x \in \cup_{n=1}^{\infty}[a_n-2,a_n], \\ (b_{n+1}^3 - a_n)\sqrt{1-(b_{n+1}-1-x)^2}+1, & x \in \cup_{n=1}^{\infty}[b_{n+1}-2,b_{n+1}], \\ 1, & \text{otherwise}. \end{cases}$$

Let α, $\beta \in \mathbb{R}$ satisfy (H). We claim that for each $\lambda \in (0,\infty)$ and $\mu \in [0,\infty)$, the BVP

$$\begin{cases} \Delta^4 u(t-2) - \alpha\Delta^2 u(t-1) + \beta u(t) = \lambda f(t,u(t)), \quad t \in [1,T]_{\mathbb{Z}}, \\ u(0) = \Delta u(-1) = \Delta^2 u(T) = 0,\ \Delta^3 u(T-1) - \alpha\Delta u(T) = -\mu(u(T+1))^{2/3}, \end{cases} \tag{3.10}$$

has a sequence of solutions which is unbounded in X.

In fact, with $g(x) = -x^{2/3}$, it is clear that BVP (3.10) is a special case of BVP (1.1) and that (3.9) holds. Let $F(t,x)$ be defined by (2.6). Then, for $t \in [1,T]_{\mathbb{Z}}$, simple computations yield

$$\begin{aligned} F(t,a_n) &= t^2\Bigg(\int_0^{a_n} 1ds + b_1^3\int_0^2 \sqrt{1-(1-s)^2}\, ds \\ &\quad + \sum_{i=1}^{n}\int_{a_i-2}^{a_i}(a_i-b_i^3)\sqrt{1-(a_i-1-s)^2}\, ds \\ &\quad + \sum_{i=1}^{n-1}\int_{b_{i+1}-2}^{b_{i+1}}(b_i^3-a_i)\sqrt{1-(b_{i+1}-1-s)^2}\, ds\Bigg) \\ &= t^2\left(\frac{\pi}{2}a_n + a_n\right) \end{aligned}$$

and

$$\begin{aligned} F(t,b_n) &= t^2\Bigg(\int_0^{b_n} 1ds + b_1^3\int_0^2 \sqrt{1-(1-s)^2}\, ds \\ &\quad + \sum_{i=1}^{n-1}\int_{a_i-2}^{a_i}(a_i-b_i^3)\sqrt{1-(a_i-1-s)^2}\, ds \\ &\quad + \sum_{i=1}^{n-1}\int_{b_{i+1}-2}^{b_{i+1}}(b_i^3-a_i)\sqrt{1-(b_{i+1}-1-s)^2}\, ds\Bigg) \\ &= t^2\left(\frac{\pi}{2}b_n^3 + b_n\right). \end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{F(t,a_n)}{a_n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{F(t,b_n)}{b_n^2} = \infty \quad \text{for } t \in [1,T]_{\mathbb{Z}}.$$

Then, for A and B defined in (3.1), it is easy to see that

$$A = \liminf_{\xi\to\infty}\frac{F(t,\xi)\sum_{t=1}^{T} t^2}{\xi^2} = 0 \quad \text{and} \quad B = \limsup_{\xi\to\infty}\frac{F(t,\xi)\sum_{t=1}^{T} t^2}{\xi^2} = \infty. \tag{3.11}$$

Thus, all the conditions of Corollary 3.3 are satisfied. The claim then follows directly from Corollary 3.3.

4 Proofs of the Main Results

The proofs of our theorems are based on the following lemma obtained in [3, Theorem 2.1]. This result is a supplement of the variational principle of Ricceri [21, Theorem 2.5].

Lemma 4.1 *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}, \tag{4.1}$$

and

$$\gamma := \liminf_{r\to\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum that is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to\infty} \Phi(u_n) = \infty.$$

(c) If $\delta < \infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either

(c_1) there is a global minimum of Φ which is a local minimum of I_λ, or

(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which converges weakly to a global minimum of Φ.

The proof of Theorem 3.1 relies on Lemma 4.1 (b).

Proof of Theorem 3.1. Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by (2.4) and (2.5), respectively. Then, it is clear that Φ and Ψ satisfy all the regularity assumptions given in Lemma 4.1.

By the definition of A in (3.1), there exists a sequence $\{\xi_n\}$ of positive numbers such that $\lim_{n\to\infty} \xi_n = \infty$ and

$$A = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t, x)}{\xi_n^2}. \tag{4.2}$$

Let $r_n = \frac{\xi_n^2}{2\rho^2}$. Then, for any $u \in X$ with $\Phi(u) < r_n$, from (2.3), we have

$$\max_{t\in[1,T+1]_{\mathbb{Z}}} |u(t)| \le \rho||u||_X < \rho(2r_n)^{1/2} = \xi_n. \tag{4.3}$$

Note that $0 \in \Phi^{-1}(-\infty, r_n)$ and $\Psi(0) = 0$. Then, by (4.1) and (3.5),

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\right) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n} \\ &\le \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x) - \frac{\mu}{\lambda} \min_{|s|\le\xi_n} G(s)}{r_n} \\ &= 2\rho^2 \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x) - \frac{\mu}{\lambda} G(\xi_n)}{\xi_n^2}.\end{aligned}$$

Thus, from (3.5) and (4.2), we see that, for γ defined in Lemma 4.1,

$$\gamma \le \liminf_{n\to\infty} \varphi(r_n) \le 2\rho^2 \left(A - \frac{\mu}{\lambda} G_\infty\right) < \infty. \tag{4.4}$$

We claim that

$$\text{if } \lambda \in (\lambda_1, \lambda_2) \text{ and } \mu \in [0, \overline{\mu}_1), \text{ then } \lambda \in (0, 1/\gamma). \tag{4.5}$$

In fact, it is clear that $\lambda > 0$. Now, when $\lambda \in (\lambda_1, \lambda_2)$ and $\mu \in [0, \overline{\mu}_1)$, from (3.6) and (4.4), we have

$$\gamma \le 2\rho^2 \left(A - \frac{\overline{\mu}_1}{\lambda} G_\infty\right) = 2\rho^2 \left(A + \frac{1 - 2\rho^2\lambda A}{2\rho^2\lambda}\right) = \frac{1}{\lambda},$$

and so, $\lambda < 1/\gamma$. Thus, (4.5) holds.

Let $\lambda \in (\lambda_1, \lambda_2)$ and $\mu \in [0, \overline{\mu}_1)$ be fixed. Then, in view of (4.4) and (4.5), by Lemma 4.1 (b), it follows that one of the following alternatives holds

(b$_1$) either $I_\lambda := \Phi - \lambda\Psi$ has a global minimum, or

(b$_2$) there exists a sequence $\{u_n\}$ of critical points of I_λ such that $\lim_{n\to\infty} ||u_n||_X = \infty$.

In what follows, we show that alternative (b$_1$) does not hold. By the definition of B in (3.1), there exists a sequence $\{\eta_n\}$ of positive numbers such that $\lim_{n\to\infty} \eta_n = \infty$ and

$$B = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} F(t, \eta_n)}{\eta_n^2}. \tag{4.6}$$

For each $n \in \mathbb{N}$, define a function $w_n : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R}$ by

$$w_n(t) = \begin{cases} 0, & t = -1, 0, \\ \eta_n, & t \in [1, T+2]_{\mathbb{Z}}. \end{cases} \tag{4.7}$$

Then, $w_n \subseteq X$. Moreover, from (2.4) and (2.5), it is easy to see that

$$\Phi(w_n) = \frac{1}{2}(2+\alpha+\beta T)\eta_n^2$$

and

$$\Psi(w_n) = \sum_{t=1}^{T} F(t,\eta_n) - \frac{\mu}{\lambda}G(\eta_n).$$

Note that $G(\eta_n) \le 0$ by (3.5). Then, we have

$$\begin{aligned} I_\lambda(w_n) &= \Phi(w_n) - \lambda\Psi(w_n) \\ &= \frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda\sum_{t=1}^{T} F(t,\eta_n) + \mu G(\eta_n) \\ &\le \frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda\sum_{t=1}^{T} F(t,\eta_n). \end{aligned} \tag{4.8}$$

Now, we consider two cases.

Case 1: $B < \infty$. From the fact that $\lambda > \lambda_1$ and the definition of λ_1 in (3.3), we have $B - \frac{2+\alpha+\beta T}{2\lambda} > 0$. Let

$$\epsilon \in \left(0, B - \frac{2+\alpha+\beta T}{2\lambda}\right). \tag{4.9}$$

From (4.6), there exists $N_1 \in \mathbb{N}$ such that

$$\sum_{t=1}^{T} F(t,\eta_n) > (B-\epsilon)\eta_n^2 \quad \text{for } n \ge N_1.$$

This, together with (4.8), implies that

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda(B-\epsilon)\right)\eta_n^2.$$

Thus, from (4.9) and the fact that $\lim_{n\to\infty}\eta_n = \infty$, we have $\lim_{n\to\infty} I_\lambda(w_n) = -\infty$.

Case 2: $B = \infty$. Choose

$$M > \frac{2+\alpha+\beta T}{2\lambda}. \tag{4.10}$$

Then, (4.6) implies that there exists $N_2 \in \mathbb{N}$ such that

$$\sum_{t=1}^{T} F(t,\eta_n) > M\eta_n^2 \quad \text{for } n \ge N_2.$$

Thus, from (4.8),

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda M\right)\eta_n^2.$$

Then, from (4.10) and the fact that $\lim_{n\to\infty}\eta_n = \infty$, we have $\lim_{n\to\infty} I_\lambda(w_n) = -\infty$.

Combining the above two cases, we see that the functional I_λ is always unbounded from below. Hence, the alternative (b_1) does not hold. Therefore, there exists a sequence

$\{u_n\}$ of critical points of I_λ such that $\lim_{n\to\infty} ||u_n||_X = \infty$. Applying Lemma 2.2 completes the proof of the theorem. □

Using Lemma 4.1 (c) and arguing as in the proof of Theorem 3.1, we can prove Theorem 3.2. For the completeness, we give the proof below.

Proof of Theorem 3.2. Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by (2.4) and (2.5), respectively. Then, as before, Φ and Ψ satisfy all the regularity assumptions given in Lemma 4.1.

By the definition of C in (3.2), there exists a sequence $\{\xi_n\}$ of positive numbers such that $\lim_{n\to\infty} \xi_n = 0$ and

$$C = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} \max_{|x|\le\xi_n} F(t,x)}{\xi_n^2}.$$

By the fact that $\inf_X \Phi = 0$ and the definition δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$. Then, as in showing (4.4) and (4.5) in the proof of Theorem 3.1, we can prove that $\delta < \infty$ and that if $\lambda \in (\lambda_3, \lambda_4)$ and $\mu \in [0, \overline{\mu}_2)$, then $\lambda \in (0, 1/\delta)$. Let $\lambda \in (\lambda_3, \lambda_4)$ and $\mu \in [0, \overline{\mu}_2)$ be fixed. Then, by Lemma 4.1 (c), we see that one of the following alternatives holds

(c_1) either there is a global minimum of Φ which is a local minimum of $I_\lambda = \Phi - \lambda\Psi$, or

(c_2) there exists a sequence $\{u_n\}$ of critical points of I_λ which converges weakly to a global minimum of Φ.

In the following, we show that alternative (c_1) does not hold. By the definition of C in (3.2), there exists a sequence $\{\eta_n\}$ of positive numbers such that $\lim_{n\to\infty} \eta_n = 0$ and

$$C = \lim_{n\to\infty} \frac{\sum_{t=1}^{T} F(t,\eta_n)}{\eta_n^2}. \tag{4.11}$$

For each $n \in \mathbb{N}$, let $w_n : [-1, T+2]_{\mathbb{Z}} \to \mathbb{R}$ be defined by (4.7) with the above η_n. Then, as in the cases 1 and 2 of the proof of Theorem 3.1, we can obtain that, for n large enough, if $C < \infty$, then

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda(C-\epsilon)\right)\eta_n^2,$$

where

$$\epsilon \in \left(0, C - \frac{2+\alpha+\beta T}{2\lambda}\right),$$

and if $C = \infty$, then

$$I_\lambda(w_n) \le \left(\frac{1}{2}(2+\alpha+\beta T)\eta_n^2 - \lambda M\right)\eta_n^2,$$

where M satisfies (4.10). Therefore, we always have $I_\lambda(w_n) < 0$ for large n. Then, since $\lim_{n\to\infty} I_\lambda(w_n) = I_\lambda(0) = 0$, we see that 0 is not a local minimum of I_λ. This, together with the fact that 0 is the only global minimum of Φ, shows that alternative (c_1) does not hold. Therefore, there exists a sequence $\{u_n\}$ of critical points of I_λ which converges weakly (and thus also strongly) to 0. An application of Lemma 2.2 completes the proof of the theorem. □

Finally, we point out that Corollaries 3.1, 3.3, and 3.5 follow directly from Theorem 3.1, and Corollaries 3.2, 3.4, and 3.6 are obviously consequences of Theorem 3.2.

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