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Existence and Uniqueness of a Nontrivial Solution for Second Order Nonlinear m-Point Eigenvalue Problems on Time Scales

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Abstract: In this paper, by introducing a new operator, improving and generating a p-Laplace operator for some $p \geq 2$, we study the existence and uniqueness of a nontrivial solution for nonlinear m-point eigenvalue problems on time scales. We obtain several sufficient conditions of the existence and uniqueness of nontrivial solution of the eigenvalue problems when λ is in some interval. Our approach is based on the Leray - Schauder nonlinear alternative.

Keywords: *nontrivial solutions; eigenvalue problems; fixed point theorems; time scales.*

Mathematics Subject Classification (2010): 34B15, 39A10.

1 Introduction

In this paper, we are concerned with the existence and uniqueness of a nontrivial solution for the following second order m-point eigenvalue problems on time scales:

$$(\varphi(h(t)u^{\triangle}(t)))^{\nabla} + \lambda f(t, u(t), u^{\triangle}(t)) = 0, \qquad t \in [0, T], \tag{1}$$

$$\alpha u(\rho(0)) - \beta u^{\triangle}(\rho(0)) = C_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \qquad u^{\triangle}(T) = 0, \tag{2}$$

where $\varphi : R \to R$ is an increasing homeomorphism and homomorphism such that $\varphi(0) = 0$, $\lambda > 0$ is a parameter, $\xi_i \in [0, T]$ with $0 < \xi_1 < ... < \xi_{m-2} < T$, $\alpha > 0$ and $\beta \geq 0$.

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A projection $\varphi : R \to R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:
1) If $x \leq y$, then $\varphi(x) \leq \varphi(y)$, for all $x, y \in R$.
2) φ is a continuous bijection and its inverse mapping is also continuous.
3) $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x, y \in R$.

Moreover, throughout the paper the following conditions hold for α_i, f, h, C_0 and φ^{-1}:
(A_1) $\alpha_i \in [0, \infty), i = 1, 2, ..., m-2$ and $f \in C_{ld}([0, T] \times R \times R)$.
(A_2) $h \in C([\rho(0), T], (0, \infty))$ and h is increasing on $[\rho(0), T]$.
(A_3) $C_0(v)$ is a continuous function on R and satisfies the condition that there exists $A > 0$ such that $|C_0(v)| \leq A|v|$, for all $v \in R$.
(A_4) For all $x, y \in R$, $|C_0(x) - C_0(y)| \leq C_0(|x - y|)$.
(A_5) For all $x, y \geq 0$, $\varphi^{-1}(x + y) \leq \varphi^{-1}(x) + \varphi^{-1}(y)$.

A time scale T is a nonempty closed subset of R. We make the assumption that $0 \in \mathrm{T}_k$ and $T \in \mathrm{T}^k$. By an interval $[0, T]$, we always mean the intersection of the real interval $[0, T]$ with T_k^k; that is $[0, T] \cap \mathrm{T}_k^k$. Some basic definitions and theorems on time scales can be found in the books [4, 5].

Recently, for $\phi_p(u) = |u|^{p-2}u, p > 1$, p-Laplacian problems with two-point, three-point and multi-point boundary value conditions for ordinary differential equations and finite difference equations have been studied extensively, see [8, 11, 13, 15]. For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results; for details, see [2, 7, 9, 10, 12, 14, 16] and the references therein. However, for the increasing homeomorphism and homomorphism operator, the research has proceeded very slowly. Especially for the existence of countably many positive solutions for dynamic equations on time scales still remain unknown.

In this paper we define a new operator φ which is an increasing homeomorphism and homomorphism with $\varphi(0) = 0$. For existence result we need that the assumption (A_5) is provided by this operator. Since the condition (A_5) is not satisfied for $\phi_p(u), 1 < p < 2$, our paper generalizes p-Laplacian operator ϕ_p for $p \geq 2$.

In [9], He considered the existence of positive solutions of the p-Laplacian dynamic equations on time scales:

$$\begin{aligned} &(\phi_p(u^{\triangle}(t)))^{\nabla} + a(t)f(u(t)) = 0, \qquad t \in (0, T), \\ &\alpha u(0) - B_0(u^{\triangle}(\eta) = 0, \qquad u^{\triangle}(T) = 0, \end{aligned}$$

or

$$\alpha u^{\triangle}(0) = 0, \qquad u(T) - B_1(u^{\triangle}(\eta)) = 0,$$

where $\eta \in (0, \rho(T))$. He obtained the existence of at least double and triple positive solutions of this problem by using a new double fixed-point theorem and triple fixed-point theorem, respectively.

In [15], Yao studied the existence of positive solutions for the following semipositone second-order boundary value problem:

$$\begin{aligned} &u''(t) = \lambda q(t) f(t, u(t), u'(t)), \qquad t \in (0, 1), \\ &\alpha u(0) - \beta u'(0) = d, \qquad u(1) = 0, \end{aligned}$$

where $d > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta > 0$ and $q(t)f(t, u(t), u'(t)) \geq 0$ on a suitable subset of $[0, 1] \times [0, \infty) \times (-\infty, \infty)$. His proofs are based on the Leray-Schauder fixed-point theorem and the localization method.

In [12], Lianga and Zhanga show the sufficient conditions for the existence of countably many positive solutions by using the fixed-point index theory and a new fixed-point theorem in cones for the following boundary value problem on time scales:

$$(\varphi(u^{\triangle}(t)))^{\nabla} + a(t)f(u(t)) = 0, \qquad t \in [0,T]_{\mathrm{T}},$$
$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \qquad u^{\triangle}(T) = 0,$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$, $\xi_i \in [0,T]_{\mathrm{T}}$ with $0 < \xi_1 < ... < \xi_{m-2} < T$ and $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $a(t) : [0,T]_{\mathrm{T}} \to [0,\infty)$ and has countably many singularities in $[0,T]_{\mathrm{T}}$.

This paper is organized as follows. In Section 2, we present some lemmas that will be used to prove our main results and we will establish two new theorems of existence and uniqueness of nontrivial solutions of (1.1)–(1.2). In Section 3, we will give some examples to illustrate the main results in this paper.

2 Main Results

To prove the main results in this paper, we will employ some several lemmas. The following lemma is based on the linear BVP

$$(\varphi(h(t)u^{\triangle}(t)))^{\nabla} + \lambda y(t) = 0, \qquad t \in [0,T], \tag{3}$$
$$\alpha u(\rho(0)) - \beta u^{\triangle}(\rho(0)) = C_0(\sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \qquad u^{\triangle}(T) = 0. \tag{4}$$

Lemma 2.1 *If $y \in C_{ld}([0,T], R)$, then the problem $(2.3)-(2.4)$ has a unique solution*

$$u(t) = \int_{\rho(0)}^{t} \frac{1}{h(s)}\varphi^{-1}(\int_{s}^{T} \lambda y(r)\nabla r)\Delta s + \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T} \lambda y(r)\nabla r)$$
$$+\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T} \lambda y(r)\nabla r)).$$

Let Y denote the Banach space $C^1_{ld}[0,T]$ with the norm
$\|u\|_1 = \|u\| + \|u^{\triangle}\| = max_{t\in[0,T]}|u(t)| + max_{t\in[0,T]}|u^{\triangle}(t)|$.

Lemma 2.2 [6] *Let X be a real Banach space and Ω be a bounded open subset of X, $0 \in \Omega$, $F : \overline{\Omega} \to X$ be a completely continuous operator.Then either there exist $x \in \partial\Omega$, $\mu > 1$ such that $F(x) = \mu x$ or there exists a fixed point $x^* \in \overline{\Omega}$.*

The main results of this paper are the following.

Theorem 2.1 *Suppose that $(A_1), (A_2), (A_3), (A_5)$ hold, $f(t,0,0) \not\equiv 0$, $t \in [0,T]$ and there exist nonnegative functions $p, q, a \in L^1[0,T]$ such that*

$$|f(t,u,v)| \le p(t)\varphi(|u|) + q(t)\varphi(|v|) + a(t),\ forall(t,u,v) \in [0,T] \times R^2,$$

and there exists $t_0 \in [0,T]$ such that $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Then there exists a constant $\lambda^* > 0$ such that for any $0 < \lambda \le \lambda^*$, the problem $(1.1) - (1.2)$ has at least one nontrivial solution $u^* \in Y$.

Proof. By Lemma 2.1, the problem (1.1) – (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = Fu(t) = \int_{\rho(0)}^{t} \frac{1}{h(s)}\varphi^{-1}(\int_{s}^{T} \lambda y(r)\nabla r)\Delta s + \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T} \lambda y(r)\nabla r)$$
$$+\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T} \lambda y(r)\nabla r))$$

in Y. So we only need to seek a fixed point of F in Y. It follows that this operator $F : Y \to Y$ is a completely continuous operator from the references $[1, 3, 14]$.

From the condition (A_3), there exists $A > 0$ such that

$$|C_0(\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T} \lambda f(r, u, u^{\triangle})\nabla r))| \leq A|\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T} \lambda f(r, u, u^{\triangle})\nabla r)|.$$

Let $M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M$ and $N^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)N$. where $M = \varphi^{-1}(\int_{\rho(0)}^{T} (p(r) + q(r))\nabla r), N = \varphi^{-1}(\int_{\rho(0)}^{T} a(r)\nabla r)$ and $K = A\sum_{i=1}^{m-2} \alpha_i$.

Since $|f(t, 0, 0)| \leq a(t)$ for all $t \in [0, T]$, we know that $N > 0$, from $p(t_0) \neq 0$ or $q(t_0) \neq 0$, we readily obtain $M > 0$. Moreover, $M^*, N^* > 0$ since $\alpha, M, N > 0$.

Let $r = \frac{N^*}{M^*}$ and $\Omega = \{u \in C^1_{ld}([0, T]) : \|u\|_1 < r\}$. Suppose $u \in \partial\Omega$, $\mu > 1$ such that $Fu = \mu u$. Then

$$\mu r = \mu\|u\|_1 = \|Fu\|_1 = \|Fu\| + \|(Fu)^{\triangle}\|.$$

For all $t \in [0, T]$, we have

$$|Fu(t)| \leq \int_{\rho(0)}^{T} \frac{1}{h(s)}\varphi^{-1}(\int_{s}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)\Delta s + \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)$$
$$+\frac{A}{\alpha}\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T} \lambda|f(r, u, u^{\triangle})|\nabla r)$$
$$\leq \int_{\rho(0)}^{T} \frac{1}{h(\rho(0))}\varphi^{-1}(\int_{s}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)\Delta s$$
$$+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)$$
$$+\frac{A}{\alpha}\sum_{i=1}^{m-2} \alpha_i \frac{1}{h(\rho(0)))}\varphi^{-1}(\int_{\xi_i}^{T} \lambda[p(r)\varphi(|u|) + q(r)\varphi(|u^{\triangle}|) + a(r)]\nabla r)$$
$$\leq \int_{\rho(0)}^{T} \frac{1}{h(\rho(0))}\varphi^{-1}(\lambda[\int_{\rho(0)}^{T} \varphi(\|u\|_1)(p(r) + q(r))\nabla r + \int_{\rho(0)}^{T} a(r)\nabla r])\Delta s$$
$$+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda[\int_{\rho(0)}^{T} \varphi(\|u\|_1)(p(r) + q(r))\nabla r + \int_{\rho(0)}^{T} a(r)\nabla r])$$

$$
\begin{aligned}
&+\frac{A}{\alpha}\frac{1}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\lambda[\int_{\rho(0)}^{T}\varphi(\|u\|_1)(p(r)+q(r))\nabla r+\int_{\rho(0)}^{T}a(r)\nabla r])\\
\le\ & \frac{1}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\lambda)[\|u\|_1M+N]\Delta s\\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)[\|u\|_1M+N]+\frac{1}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0)}[\|u\|_1M+N]A\sum_{i=1}^{m-2}\alpha_i.
\end{aligned}
$$

Then

$$
\|Fu\|\le\frac{\varphi^{-1}(\lambda}{h(\rho(0))}\|u\|_1\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)N.
$$

For all $t\in[0,T]$, we have

$$
\begin{aligned}
|(Fu)^{\triangle}(t)|&\le\frac{1}{h(t)}\varphi^{-1}(\int_t^T\lambda|f(r,u,u^{\triangle})|\nabla r)\\
&\le\frac{1}{h(\rho(0))}\varphi^{-1}(\int_t^T\lambda[p(r)\varphi(|u|)+q(r)\varphi(|u^{\triangle}|)+a(r)]\nabla r)\\
&\le\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda[\varphi(\|u\|_1)\int_{\rho(0)}^{T}(p(r)+q(r))\nabla r+\int_{\rho(0)}^{T}a(r)\nabla r])\\
&\le\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1\varphi^{-1}(\int_{\rho(0)}^{T}(p(r)+q(r))\nabla r)+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}a(r)\nabla r)\\
&=\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N.
\end{aligned}
$$

Then $\|(Fu)^{\triangle}\|\le\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1M+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N$. Thus, we get

$$
\|Fu\|_1\le\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u\|_1M^*+\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}N^*.
$$

Choose $\lambda^*=\varphi(\frac{h(\rho(0))}{2M^*})$. Then when $0<\lambda\le\lambda^*$, we have

$$
\mu r=\mu\|u\|_1=\|Fu\|_1\le\frac{1}{2M^*h(\rho(0))}M^*h(\rho(0))\|u\|_1+\frac{N^*}{2M^*}.
$$

Consequently, $\mu r\le\frac{1}{2}r+\frac{1}{2}r=r$.

This contradicts $\mu>1$, by Lemma 2.2, F has a fixed point $u^*\in\overline{\Omega}$, since $f(t,0,0)\not\equiv0$, then when $0<\lambda\le\lambda^*$, the problem (1.1) − (1.2) has a nontrivial solution $u^*\in Y$. This completes the proof.

Theorem 2.2 *Suppose that $(A_1),(A_2),(A_3),(A_4)$ and (A_5) hold, and $f:[0,T]\times R^2\to(-\infty,0]$ or $f:[0,T]\times R^2\to[0,\infty)$ is ld-continuous, $f(t,0,0)\not\equiv0$, $t\in[0,T]$ and there exist nonnegative functions $p_1,q_1\in L^1[0,T]$ such that*

$$
|f(t,u_1,v_1)-f(t,u_2,v_2)|\le p_1(t)\varphi(|u_1-u_2|)+q_1(t)\varphi(|v_1-v_2|)
$$

and there exists $t_0 \in [0,T]$ *such that* $p_1(t_0) \neq 0$ *or* $q_1(t_0) \neq 0$. *Then there exists a constant* $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$, *the problem* $(1.1) - (1.2)$ *has a unique nontrivial solution* $u^* \in Y$.

Proof. If $u_2 = v_2 = 0$, then we have

$$|f(t,u_1,v_1)| \leq p_1(t)\varphi(|u_1|) + q_1(t)\varphi(|v_1|) + |f(t,0,0)|.$$

From Theorem 2.1, we know that the problem (1.1)–(1.2) has a nontrivial solution $u^* \in Y$.

Now, we shall use the Banach fixed theorem to show the uniqueness of nontrivial solution of the problem $(1.1) - (1.2)$. For $|Fu_1(t) - Fu_2(t)|$, we have

$$\begin{aligned}
&|\int_{\rho(0)}^{t}\frac{1}{h(s)}\varphi^{-1}(\int_{s}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r)\Delta s - \int_{\rho(0)}^{t}\frac{1}{h(s)}\varphi^{-1}(\int_{s}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)\Delta s\\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)\\
&+\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r))\\
&-\frac{1}{\alpha}C_0(\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r))|\\
&\leq \int_{\rho(0)}^{t}\frac{1}{h(s)}|\varphi^{-1}(\int_{s}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{s}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)|\Delta s\\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}|\varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)|\\
&+\frac{1}{\alpha}C_0(|\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)]|)\\
&\leq \int_{\rho(0)}^{T}\frac{1}{h(s)}|\varphi^{-1}(\int_{s}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{s}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)|\Delta s\\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}|\varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\rho(0)}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)|\\
&+\frac{A}{\alpha}\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}|\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r) - \varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)|\\
&\leq \int_{\rho(0)}^{T}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\rho(0)}^{T}|f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r)\Delta s\\
&+\frac{\beta}{\alpha}\frac{1}{h(\rho(0))}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\rho(0)}^{T}|f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r)\\
&+\frac{A}{\alpha}\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}\varphi^{-1}(\lambda)\varphi^{-1}(\int_{\xi_i}^{T}|f(r,u_1,u_1^{\triangle}) - f(r,u_2,u_2^{\triangle})|\nabla r)
\end{aligned}$$

$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)\Delta s$$
$$+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)$$
$$+\frac{A}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\rho(0))}^{T}(p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|))\nabla r)$$
$$\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\int_{\rho(0)}^{T}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)\Delta s$$
$$+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)$$
$$+\frac{A}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r)$$
$$=\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1(T-\rho(0))M_1+\frac{\beta}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1$$
$$+\frac{1}{\alpha}\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1A\sum_{i=1}^{m-2}\alpha_i.$$

Then

$$\|Fu_1-Fu_2\|\leq \frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1\frac{1}{\alpha}(\alpha(T-\rho(0))+\beta+K)M_1,$$

where $M_1=\varphi^{-1}(\int_{\rho(0)}^{T}(p_1(r)+q_1(r))\nabla r)$, A is a constant such that

$$C_0(|\textstyle\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r)-\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)]|)$$
$$\leq A|\sum_{i=1}^{m-2}\alpha_i\frac{1}{h(\xi_i)}[\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_1,u_1^{\triangle})\nabla r)-\varphi^{-1}(\int_{\xi_i}^{T}\lambda f(r,u_2,u_2^{\triangle})\nabla r)]|$$

and $K=A\sum_{i=1}^{m-2}\alpha_i$. For all $t\in[0,T]$, we have

$$|((Fu_1)^{\triangle}-(Fu_2)^{\triangle})(t)|\leq\frac{1}{h(t)}\varphi^{-1}(\int_{t}^{T}\lambda|f(r,u_1,u_1^{\triangle})-f(r,u_2,u_2^{\triangle})|\nabla r)$$
$$\leq\frac{1}{h(t)}\varphi^{-1}(\int_{\rho(0)}^{T}\lambda[p_1(r)\varphi(|u_1-u_2|)+q_1(r)\varphi(|u_1^{\triangle}-u_2^{\triangle}|)]\nabla r)$$
$$\leq\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\varphi^{-1}(\int_{\rho(0)}^{T}\varphi(\|u_1-u_2\|_1)(p_1(r)+q_1(r))\nabla r).$$

Then

$$\|(Fu_1)^{\triangle}-(Fu_2)^{\triangle}\|\leq\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1.$$

So, we get

$$\|Fu_1-Fu_2\|_1\leq\frac{\varphi^{-1}(\lambda)}{h(\rho(0))}\|u_1-u_2\|_1M_1^*,$$

where $M_1^* = \dfrac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M_1$.

Choose $\lambda^* = \varphi(\dfrac{h(\rho(0))}{2M_1^*})$. Then when $0 < \lambda \leq \lambda^*$, we have

$$\|Fu_1 - Fu_2\|_1 \leq \frac{1}{2}\|u_1 - u_2\|_1.$$

Thus the problem (1.1)–(1.2) has a unique solution for $0 < \lambda \leq \lambda^*$.

Corollary 2.1 *Suppose that* $(A_1), (A_2), (A_3), (A_5)$ *hold,* $f : [0, T] \times R^2 \to R$ *is ld-continuous,* $f(t, 0, 0) \not\equiv 0, t \in [0, T]$ *and*

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t \in [0,T]} \frac{|f(t, u, v)|}{\varphi(|u|) + \varphi(|v|)} < +\infty. \tag{5}$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$, *the problem* $(1.1)-(1.2)$ *has at least one nontrivial solution* $u^* \in Y$.

Proof. Let $\varepsilon > 0$. By (2.5), there exists $H > 0$ such that

$$|f(t, u, v)| \leq (l + \varepsilon)(\varphi(|u|) + \varphi(|v|)), \quad |u| + |v| \geq H, t \in [0, T].$$

Let $K = \max_{t \in [0,T], |u|+|v| \leq H} |f(t, u, v)|$. Then for $(t, u, v) \in [0, T] \times R^2$, we have

$$|f(t, u, v)| \leq (l + \varepsilon)\varphi(|u|) + (l + \varepsilon)\varphi(|v|) + K.$$

From Theorem 2.1, we know that the problem (1.1)–(1.2) has at least one nontrivial solution $u^* \in Y$.

Corollary 2.2 *Suppose that* $(A_1), (A_2), (A_3), (A_5)$ *hold,* $f : [0, T] \times R^2 \to R$ *is ld-continuous,* $f(t, 0, 0) \not\equiv 0, t \in [0, T]$ *and*

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t \in [0,T]} \frac{|f(t, u, v)|}{\varphi(|u|)} < +\infty,$$

or

$$0 \leq l = \limsup_{|u|+|v| \to +\infty} \max_{t \in [0,T]} \frac{|f(t, u, v)|}{\varphi(|v|)} < +\infty.$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$, *the problem* $(1.1)-(1.2)$ *has at least one nontrivial solution* $u^* \in Y$.

Corollary 2.3 *Suppose that* $(A_1), (A_2), (A_3), (A_4)$ *and* (A_5) *hold,* $f : [0, T] \times R^2 \to [0, \infty)$ *is ld-continuous,* $f(t, 0, 0) \not\equiv 0$, $t \in [0, T]$, $C_0(v)$ *satisfies the condition that there exists* $B > 0$ *such that* $Bv \leq C_0(v)$, *for all* $v \geq 0$ *and there exist nonnegative functions* $p_1, q_1 \in L^1[0, T]$ *such that*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq p_1(t)\varphi(|u_1 - u_2|) + q_1(t)\varphi(|v_1 - v_2|)$$

and there exists $t_0 \in [0, T]$ *such that* $p_1(t_0) \neq 0$ *or* $q_1(t_0) \neq 0$. *Then there exists a constant* $\lambda^* > 0$ *such that for any* $0 < \lambda \leq \lambda^*$, *the problem* $(1.1) - (1.2)$ *has a positive unique solution* $u^* \in Y$.

3 Examples

In this section, we will give some examples to illustrate our main results.

Example 3.1 Let $T = [0,1] \cup \{2\} \cup [3,5]$. We consider the following second order eigenvalue problem

$$(\varphi((t+4)u^{\triangle}(t)))^{\nabla} + \lambda(te^{-t}u^2 - (u^{\triangle})^2 t^2 \sin t + \cos t) = 0, \quad t \in [0,4], \tag{6}$$

$$u(0) = \frac{1}{2}|u^{\triangle}(1) + u^{\triangle}(2)|, \quad u^{\triangle}(4) = 0, \tag{7}$$

where $h(t) = t+4, \alpha = 1, \beta = 0, T = 4, \xi_1 = 1, \xi_2 = 2, \alpha_1 = \alpha_2 = \frac{1}{2}$,

$$\varphi(u) = \begin{cases} -u^2, & u < 0, \\ u^2, & u \geq 0, \end{cases}$$

and $C_0(x) = |x|$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus $K = A(\alpha_1 + \alpha_2) = 1$.

Noticing, for all $t \in [0,4]$, f satisfies

$$|f(t,u,v)| = |te^{-t}u^2 - v^2 t^2 \sin t + \cos t| \leq t|u|^2 + t^2|v|^2 + 1.$$

Then $|f(t,u,v)| \leq t\varphi(|u|) + t^2\varphi(|v|) + 1$. It is easy to see by calculating that

$$M = \varphi^{-1}(\int_0^4 (r^2 + r)\nabla r) = \sqrt{\frac{74}{3}}, \; M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M = 6\sqrt{\frac{74}{3}}.$$

So, we have $\lambda^* = \varphi(\frac{h(0)}{2M^*}) \approx 0.0045$. Then by Theorem 2.1, we know that the problem (3.6)–(3.7) has nontrivial solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

Example 3.2 Let $T = \{0\} \cup \{\frac{1}{n} : n \in N\} \cup [2,4]$. We consider the following second order eigenvalue problem

$$(\varphi(e^t u^{\triangle}(t)))^{\nabla} + \lambda(t^2 \sin u + t) = 0, \qquad t \in [0,3], \tag{8}$$

$$2u(0) - u^{\triangle}(0) = \frac{1}{3}|u^{\triangle}(\frac{1}{5}) + u^{\triangle}(\frac{12}{5}) + u^{\triangle}(\frac{14}{5})|, \quad u^{\triangle}(3) = 0, \tag{9}$$

where $\varphi(u) = u$, $h(t) = e^t, \alpha = 2, \beta = 1, T = 3, \xi_1 = \frac{1}{5}, \xi_2 = \frac{12}{5}, \xi_3 = \frac{14}{5}, \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and $C_0(x) = |x|$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus, $K = A(\alpha_1 + \alpha_2 + \alpha_3) = 1$.

Noticing, for all $t \in [0,3]$, f satisfies

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| = |t^2 \sin u_1 + t - t^2 \sin u_2 - t| \leq t^2|u_1 - u_2|.$$

It is easy to see by calculating that

$$M_1 = \int_0^3 r^2 \nabla r = \frac{\pi^2 + 38}{6} \text{ and } M_1^* = 5(\frac{\pi^2 + 38}{6}).$$

Thus, we have $\lambda^* = \frac{h(0)}{2M_1^*} \approx 0.0125$. Then by Theorem 2.2, we know that the problem (3.8) – (3.9) has a unique solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

In the following example we will take the p-Laplacian operator $\phi_4(u)$ such that $\phi_p(u) = |u|^{p-2}u$ for $p > 1$ and $(\phi_p)^{-1} = \phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$ which is the special case of φ.

Example 3.3 Let $T = [0,1] \cup [2,7]$. We consider the following second order eigenvalue problem

$$(\phi_4((t+2)u^{\triangle}(t)))^{\nabla} + \lambda(\arctan(u^2 + (u^{\triangle})^2) + t^2 \sinh t) = 0,\ t \in [0,4], \tag{10}$$

$$2u(0) = \frac{1}{3}u^{\triangle}(1) + \frac{2}{3}u^{\triangle}(3), \quad u^{\triangle}(4) = 0, \tag{11}$$

where $h(t) = t+2, \alpha = 2, \beta = 0, T = 4, \xi_1 = 1, \xi_2 = 3, \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$ and $C_0(x) = x$. Then we can take $A = 1$ so that $|C_0(x)| \leq A|x|$ for all $x \in R$. Thus $K = A(\alpha_1 + \alpha_2) = 1$. It is clear that

$$\limsup_{|u|+|v| \to +\infty} \max_{t \in [0,4]} \frac{|\arctan(u^2 + v^2) + t^2 \sinh t|}{\phi_4(|u|) + \phi_4(|v|)} = 0.$$

Choosing $\epsilon = \frac{1}{2}$, we get

$$M = \phi_q\left(\int_0^4 \left(\frac{1}{2} + \frac{1}{2}\right)\nabla r\right) = \sqrt[3]{4} \text{ and } M^* = \frac{1}{\alpha}(\alpha(T - \rho(0)) + \beta + K + \alpha)M = \frac{11}{2}\sqrt[3]{4}.$$

So, we have $\lambda^* = \phi_4(\frac{h(0)}{2M^*}) \approx 0.0015$. Then by Corollary 2.1, we know that the problem (3.10)–(3.11) has nontrivial solution $u^* \in Y$ for any $\lambda \in (0, \lambda^*]$.

References

[1] Agarwal, R.P., Bohner, M. and Rehak, P. Half-linear dynamic equations. In: *Nonlinear Analysis and Applications*: To V. Lakshmikantham on his 80th Birthday. Kluwer Academic Publishers, Dordrecht, 2003, 1–57.

[2] Anderson, D.R., Avery, R. and Henderson, J. Existence of solutions for a one-dimensional p-Laplacian on time scales. *J. Difference Equ. Appl.* **10** (2004) 889–896.

[3] Aulbach, B. and Neidhart, L. Integration on measure chain. In: *Proc. of the Sixth Int. Conf. on Difference Equations.* CRC, BocaRaton, FL (2004) 239–252.

[4] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhauser, Boston, Cambridge, MA, 2001.

[5] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales.* Birkhauser, Boston, Cambridge, MA, 2003.

[6] Guo, D. and Lakshmikantham, V. *Nonlinear Problems in Abstract Cones.* Academic Press, San Diego, 1988.

[7] He, Z.M. Triple positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales. *J. Math. Anal. Appl.* **321** (2006) 911–920.

[8] He, Z.M. Double positive solutions of three-point boundary value problems for p-Laplacian difference equations. *Z. Anal. Anwendungen* **24** (2) (2005) 305–315.

[9] He, Z.M. Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales. *J. Comput. Appl. Math.* **182** (2005) 304–315.

[10] Shihuang Hong. Triple positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales. *J. Comput. Appl. Math.* **206** (2007) 967–976.

[11] Li, J. and Shen, J. Existence of three positive solutions for boundary value problems with p-Laplacian. *J. Math. Anal. Appl.* **311** (2005) 457–465.

[12] Lianga, S. and Zhanga, J. The existence of countably many positive solutions for nonlinear singular m-point boundary value problems on time scales. *Journal of Computational and Applied Mathematics* **223** (2009) 291–303.

[13] Liu, B. Positive solutions three-points boundary value problems for one-dimensional p-Laplacian with infinitely many singularities. *Appl. Math. Lett.* **17** (2004) 655–661.

[14] Sun, H.R. and Li, W.T. Positive solutions for nonlinear m-point boundary value problems on time scales. *Acta Math. Sinica* **49** (2006) 369–380. (in Chinese)

[15] Yao, Q. Positive solutions to a special singular second-order boundary value problem. *Math. Comput. Model.* **47** (11–12) (2008) 1284–1291.

[16] Yolcu, N. and Topal, S. Existence of positive solutions of a nonlinear third-order m-point boundary value problems for p-Laplacian dynamic equations on time scales. *Nonlinear Dynamics and Systems Theory* **12** (3) (2012) 311–323.