Nonlinear Dynamics and Systems Theory, **13** (4) (2013) 367–388

Formal Trigonometric Series, Almost Periodicity and Oscillatory Functions

C. Corduneanu

University of Texas, Arlington and The Romanian Academy

Received: January 7, 2013; Revised: October 30, 2013

Abstract: This paper is the second, in a cycle dedicated to the new approach in constructing new oscillatory functions spaces, taking as primary object the formal trigonometric series and their generalizations, whose terms are of the form $\exp if(t)$, with $f(t)$ functions that belong to various classes. The linear case being considered in the first part of the paper leads to the classical cases of periodicity and almost periodicity, while the generalized case is aimed to obtain more general spaces of oscillatory functions, including those already known, due to V.F. Osipov and Ch. Zhang.

Keywords: *almost periodicity; formal trigonometric series; oscillatory functions spaces.*

Mathematics Subject Classification (2010): 42A75.

1 Introduction

The periodic and, more general, the oscillatory functions/motions appeared in Science and Engineering and other fields of knowledge, have conducted to the development of classical Fourier Analysis of periodic functions and their associated series. While the first traces of this branch of classical analysis can be found in the Mathematics of the XVIII-th century (Euler, for instance), it is the XIX-th century that contains significant results, which stimulated substantially the birth of new theories, contributing vigorously to the new concepts of Modern Analysis (Set Theory, Real variables including Measure and Integral). The Fourier Analysis, as developed until the third decade of the XX-th century, has known a strong impulse due to the emerging of the concept of Almost Periodicity, due to H. Bohr (1923-25), and successfully continued to the present day.

* Corresponding author: mailto:corduneanu0726@gmail.com

It is also true that the topics of classical Fourier Analysis have also kept the attention of many leading mathematicians, after the birth of almost periodic functions.

The well known treatises of N.K. Bary (Pergamon, 1964) and A. Zygmund (Cambridge Univ. Press, 2002) contain a wealth of results and information about the periodic functions and their Fourier series, specially obtained before the introduction of the methods of Functional Analysis. More recent publications, due to J.P. Kahane [20], R.E. Edwards [16], G. Folland [19], have brought new ideas and results from this classical, but prolific field.

The concept of *almost periodicity* had several leading contributors to its beginning period. In his famous treatise *Nouvelles Méthodes de la Mécanique Céleste* (1893), H. Poincaré considered the problem of developing a function in a series of sine functions, namely

$$f(t) = \sum_{k=1}^{\infty} f_k \sin \lambda_k t, \quad t \in R, \tag{1}$$

where λ_k are arbitrary real numbers, not necessarily like $\lambda_k = k\omega$, $k \in N$, $\omega > 0$. Poincaré has succeeded to obtain the coefficients f_k, $k \geq 1$, simultaneously introducing the mean value of a function on the whole real line.

Using the complex notations, which became common with the new concept of almost periodicity, formula (1) can be rewritten as

$$f(t) = \sum_{k=1}^{\infty} f_k \exp(i\lambda_k t), \quad t \in R, \tag{2}$$

with $f_k \in \mathcal{C}$ and $\lambda_k \in R$, $k \geq 1$. The coefficients f_k are determined by the formulas

$$f_k = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t) \exp(-i\lambda_k t) dt, \tag{3}$$

in which the Poincaré's mean value (i.e., on an infinite interval) appears:

$$M(g) = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} g(t) dt, \tag{4}$$

$g \in L_{\mathrm{loc}}(R_+, R)$, under the assumption that the limit exists as a finite number.

It is known that most concepts related to almost periodicity, including the Fourier exponents and coefficients (see (3) above) are based on the *mean value* defined in (4).

Other early contributors, preceding the period initiated by H. Bohr, include P. Bohl (1893) and E. Esclangon (1919) who dealt with what was later called *quasiperiodic functions*, a special case of almost periodicity. They have investigated oscillatory functions with a *finite* number of frequencies, the periodic case being concerned with only one basic frequency $(2\pi/\omega)$, ω-period. Some methods encountered to P. Bohl, but particularly to E. Esclangon, have been adapted to the general case of almost periodicity by H. Bohr.

H. Bohr (1887–1951) was the first to create a theory of almost periodicity, in a series of papers (1922–1925) which contained most of the fundamental results of the new theory (Generalized Fourier Analysis). The new theory is marking the beginning of a *second stage* in the study of oscillatory functions, aiming at global behavior of its elements. The theory of almost periodic functions has attracted, in short time, the interest of many mathematicians, including V.V. Stepanov (1925), H. Weyl (1926), A.S.

Besicovitch (1926-1932), S. Bochner (1925-), J. Favard (1926-), J. von Neumann (1934-), B.M. Lewitan (1939-), N.N. Bogoliubov (1930-).

The definition of H. Bohr, for *almost periodic* functions, is showing the fact that these new functions are direct generalizations of the periodic ones:

A continuous function $f : R \to R$ (or $\mathcal{C}$) is called *almost periodic* if the following property holds: to each $\varepsilon > 0$, there corresponds a number $\ell = \ell(\varepsilon) > 0$, such that each interval $(a, a + \ell) \subset R$ contains a number τ with $|f(t + \tau) - f(t)| < \varepsilon$, $t \in R$.

The number τ is called an ε-*almost period* of the functions f and one says that all numbers τ, with the above property, form a *relatively dense set* on R.

This terminology has been present in all the generalizations the almost periodic functions have known so far.

The following two properties of almost periodic functions, in the sense of Bohr, have been readily discovered by Bohr himself, Bochner and Bogoliubov.

A. Approximation property: for each $\varepsilon > 0$, there exists a complex trigonometric polynomial

$$T(t) = T_\varepsilon(t) = \sum_{j=1}^{n} a_j \exp(i\lambda_j t), \quad t \in R, \tag{5}$$

with $\lambda_j \in R,\ a_j \in \mathcal{C}$, such that

$$|f(t) - T_\varepsilon(t)| < \varepsilon, \quad t \in R. \tag{6}$$

Rephrasing the above property, one may say that any almost periodic function (Bohr) can be uniformly approximated on R by trigonometric polynomials of the form (5).

B. Bochner property: the set of translates of an almost periodic function (Bohr), say $\mathcal{F} = \{f(t+h);\ h \in R\}$, is relatively compact in the sense of uniform convergence on R.

Each of properties A and B can be taken as definition for the almost periodic functions in the sense of Bohr. Bogoliubov has given a direct proof of the equivalence between the definition of Bohr and the approximation property, making possible the constructive presentation of the theory.

In what follows, by $AP(R, R)$ or $AP(R, \mathcal{C})$, we will understand the almost periodic set of functions in the sense of Bohr. These sets are actually Banach function spaces, the norm being given by the formula $|f|_{AP} = \sup\{|f(t)|;\ t \in R\}$, which makes sense for each almost periodic function (Bohr), because each function in $AP(R, R)$ or $AP(R, \mathcal{C})$ is bounded on R and uniformly continuous.

The three equivalent properties for the space of almost periodic functions, i.e., the Bohr's definition and A, B, constitute the core of the classical theory and numerous applications to various types of functional equations. See the books by H. Bohr [6], A.S. Besicovitch [5], J. Favard [17], B.M. Levitan [21], C. Corduneanu [9,10], L. Amerio and G. Prouse [2], A.M. Fink [18], S. Zaidman [32], Ch. Zhang [33], W. Maak [23], B.M. Levitan and V.V. Zhikov [22], for most of the evolution of the theory of almost periodic functions, until recently. These references contain a large number of sources in the field, with varied applications in Mathematics and other areas.

Currently, we assist at the beginning of a *third stage* in the development of mathematical concepts and theories to advance the study of various types of vibratory motions, encountered in the description of phenomena examined in Science or Engineering.

We shall touch partially this aspect in the following pages of this paper. It has been realized, by both users and designers of the new tools for investigation of oscillatory phenomena, that periodicity (first stage) and almost periodicity (seconde stage) cannot describe the wide variety of oscillatory or wave-like phenomena that one encounters in science or in the real world.

2 A Remark and Its Consequences

A new approach to build up spaces/classes of oscillatory functions, applicable also to the classical ones (periodic or almost periodic) consists in starting with formal/generic trigonometric series of the form:

$$\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad t \in R, \tag{7}$$

where $a_k \in \mathcal{C}$, $\lambda_k \in R$, $k \geq 1$, the assumption $\lambda_k \neq \lambda_j$ for $k \neq j$, $k, j \geq 1$, being accepted throughout the paper.

The *main idea* leading to the new approach, in this paper, partially illustrated in our previous paper [11], is to start with formal trigonometric series, of the form (7), as primary material, and identify conditions on the two sequences $\{a_k;\ k \geq 1\} \subset \mathcal{C}$ and $\{\lambda_k;\ k \geq 1\} \subset R$, such that (7) "characterizes" a certain type of oscillatory function, either in the classical category (periodic or almost periodic), or in the new classes of oscillatory functions (e.g., pseudo-almost periodic, to begin with in the third stage of development, or new types, as those investigated by Ch. Zhang [33, 34, 36]).

As we shall see, this new approach works for classes/spaces of classical type, but as well for introducing new spaces of oscillatory (or vibrating?) functions. The answer is not always simple, and to illustrate the situation we will start with the question:

Under what conditions does the series (7) *characterize the space* $AP(R, \mathcal{C})$ *of Bohr almost periodic function?*

Based on the theory of almost periodic functions, the answer has a simple formulation, which is:

Theorem 2.1 *The necessary and sufficient condition, for the series* (7)*, to characterize an almost periodic function of the space* $AP(R, \mathcal{C})$ *is the summability of this series, in the sense of Cesaró-Fejér-Bochner, with respect to the uniform convergence on* R.

Proof. The condition is necessary, because it is well known (see, for instance, Corduneanu [9], [10]) that for a function $f \in AP(R, \mathcal{C})$, whose Fourier series has the form (7), the sequence of trigonometric polynomials

$$\sigma_m(t) = \sum_{k=1}^{n} a_k r_{k,m} \exp(i\lambda_k t), \quad t \in R, \tag{8}$$

$n = n(m)$, with $r_{k,m}$ rationals depending on λ_k and m, but independent of $\{a_k;\ k \geq 1\}$, *converges uniformly* on R to $f(t)$.

The summability condition is also sufficient, because if (7) is summable with respect to the uniform convergence on R, the limit function will belong to $AP(R, \mathcal{C})$.

Let us point out that any linear method of summability, not necessarily the one described by (8), leads to the same conclusion. This ends the proof of Theorem 2.1, which characterizes the formal trigonometric series of the form (7), representing functions in $AP(R, \mathcal{C})$, the first space of almost periodic functions (Bohr).

Remark 2.1 Based on the uniqueness of the Fourier series corresponding to a function from $AP(R, \mathcal{C})$, in case of convergence on R of the series (7), there results that it is the Fourier series of its sum. This case takes place, obviously, when the convergence of (7) is uniform on R, and we can write

$$f(t) = \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad t \in R. \tag{9}$$

Otherwise, we have to be content with the relationship

$$\lim_{m \to \infty} \sigma_m(t) = f(t), \quad t \in R, \tag{10}$$

uniformly, the $\{\sigma_m(t),\ m \geq 1\}$ being the summability sequence consisting of trigonometric polynomials (e.g., like in (8)). Of course, any trigonometric polynomial $\sum_{k=1}^{n} a_k \exp(i\lambda_k t)$, when regarded as a formal series, is summable, hence Bohr's almost periodic.

In order to establish Theorem 2.1, we needed to rely on Bohr's properties of almost periodic functions.

What if we start with the new definition for $AP(R, \mathcal{C})$, a fact made possible by Theorem 2.1?

It turns out that the most basic properties can be routinely derived from the new definition. We shall list a few of them, leaving the task of proof to the reader.

a) An almost periodic function in Bohr's sense is bounded on R.

b) An almost periodic function in Bohr's sense is uniformly continuous on R.

c) If $f \in AP(R, \mathcal{C})$ and $c \in \mathcal{C}$, then $cf \in AP(R, \mathcal{C})$, as well as $\bar{f}$.

d) If $f, g \in AP(R, \mathcal{C})$, then $f + g \in AP(R, \mathcal{C})$ also fg.

e) If $f \in AP(R, \mathcal{C})$ and $h \in R$, then $f(t + h) = f_h(t)$ and $f(ht) = f^h(t)$ both belong to $AP(R, \mathcal{C})$.

More basic properties of Bohr's almost periodic functions can be "rediscovered" if we introduce a topology/convergence in the set of all formal trigonometric series (7). We shall not proceed on this way, preferring instead on relying on every fact in the existing theory of almost periodicity, as soon as essential connections are established.

Let us give one more example to illustrate the fact that, starting from trigonometric series, one can proceed successfully to the construction of various spaces of almost periodic functions. On behalf of Theorem 2.1 (and even of the new definition of AP-space), the approximation is assured by the summability assumption. This means that, for any $f \in AP(R, \mathcal{C})$, one can construct a sequence of trigonometric polynomials, say $\{f_n; n \geq 1\} \subset AP(R, \mathcal{C})$, such that $\lim f_n(t) = f(t)$, uniformly on R, as $n \to \infty$.

Starting from A, the space $AP(R, \mathcal{C})$ has been constructed by Bogoliubov in 1930's. This direct approach is discussed in detail in the book by Corduneanu [9].

The new approach, starting from trigonometric series as background material, instead of trigonometric polynomials, is not meant to be a substitute for other existing approaches. It has been shown in Corduneanu [9] or Shubin [27] that various applications make sense in this approach, and properties can be emphasized that were unknown before, in case of the spaces we have denoted by $AP_r(R,\mathcal{C})$, $1 \le r \le 2$, obtained by the procedure of *completion* of the linear space of trigonometric polynomials. Briefly, the space $AP_r(R,\mathcal{C})$ is defined as consisting of all series (7), satisfying the convergence condition

$$\sum_{k=1}^{\infty} |a_k|^r < \infty, \quad r \in [1,2],\ r \text{ fixed}. \tag{11}$$

This space is a linear space over $\mathcal{C}$, the norm being given by

$$\left| \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) \right|_r = \left(\sum_{k=1}^{\infty} |a_k|^r \right)^{1/r}, \tag{12}$$

the right hand side of (12) being known as Minkowski's norm.

The case $r = 1$ leads to the space of almost periodic functions with absolutely convergent series of Fourier coefficients. We have called this space Poincaré's space of almost periodic functions, and it is well known that it can be organized as a Banach algebra (see, for instance, Corduneanu [10]). It is denoted by $AP_1(R,\mathcal{C})$.

The other extreme, $r = 2$, leads to the Besicovitch space $B^2 = AP_2$ of almost periodic functions, the largest in which the Parseval's formula holds true:

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{13}$$

where $f(t)$ is the function associated to the series (7), in the manner we shall describe in subsequent lines. What appears in the right hand side of (13), according to (12) where $r = 2$, is actually the square of the *seminorm* of the function space $B^2(R,\mathcal{C})$

$$|f|_{B^2}^2 = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{14}$$

valid for all series/functions satisfying (11), for $r = 2$. As one sees from (14), the Poincaré's mean value on R is deeply involved in dealing with generalizations of Bohr's almost periodic functions.

The scale of spaces, of almost periodic functions, extended from the Poincaré's space $AP_1(R,\mathcal{C})$, to the Besicovitch space $B^2(R,\mathcal{C}) = AP_2(R,\mathcal{C})$, has been introduced and investigated in some detail in the recent paper by Corduneanu [11].

Applications of these spaces of almost periodic functions have been recently given in the papers by Corduneanu [11], Mahdavi [24] and Corduneanu and Li [14], concerning some classes (linear and nonlinear) of functional differential equations of the form

$$\dot{x}(t) = (Ax)(t) + (Fx)(t), \quad t \in R, \tag{15}$$

where A is a linear operator acting on an $AP_r(R,\mathcal{C})$ space, while $F : AP_r(R,\mathcal{C}) \to AP_r(R,\mathcal{C})$ is, generally, nonlinear. It is useful to notice that the operator A could involve convolution type terms, the convolution product being defined by the formula

$$(K * x)(t) \sim \sum_{j=1}^{\infty} \widetilde{x}_j \exp(i\lambda_j t), \quad t \in R, \tag{16}$$

with x represented by the series

$$x(t) \sim \sum_{k=1}^{\infty} x_k \exp(i\lambda_k t), \quad t \in R, \tag{17}$$

and

$$\widetilde{x}_k = x_k \int_R K(s) \exp(-i\lambda_k s) ds, \quad k \geq 1. \tag{18}$$

The sign $\sim$ will be used to mark the relationship between trigonometric series and its associated function, as in (16) and (17). In order for (18) to make sense, it will be assumed that $K \in L^1(R, \mathcal{C})$.

It can be easily checked that

$$|K * x|_r \leq |K|_{L^1} \cdot |x|_r, \quad r \in [1, 2], \tag{19}$$

for each $x \in AP_r(R, \mathcal{C})$. The inequality (19) is a replica of a similar one, namely

$$|f * g|_{L^p} \leq |f|_{L^1} \cdot |g|_{L^p}, \quad p \geq 1,$$

which is often used in convolution problems. Actually, the convolution product, in this generalized form, has been used in the above referenced papers by Corduneanu, Mahdavi and Li.

3 The Besicovitch Space $B^2(R, \mathcal{C})$

It is known that the space B^2 has properties that have been used in several applications, and presents various features making it more accessible to connections with other topics. Such a situation is not encountered when dealing with the Besicovitch space $B = B^1(R, \mathcal{C})$, even though this space is known as the largest for which the Fourier series can be associated to its elements. We will consider the space $B(R, \mathcal{C})$ in a subsequent section of this paper.

The construction of the space $B^2(R, \mathcal{C})$, starting from our approach (point of view), is rather simple. We know from the classical theory that, for each $f \in B^2(R, \mathcal{C})$, the Parseval formula

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{20}$$

where

$$f \sim \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \tag{21}$$

represents the connection between the function f and its Fourier series. Also, we know that for each sequence $\{a_k;\ k \geq 1\} \in \ell^2(N, \mathcal{C})$ =the complex Hilbert space, there exists $f \in B^2(R, \mathcal{C})$ such that (21) holds true.

Our basic assumption for constructing $B^2(R, \mathcal{C})$, starting from the set of trigonometric series of the form (7), will be

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty, \tag{22}$$

which is the same as $\{a_k;\ k \in N\} \in \ell^2(N, \mathcal{C})$.

Consider now a series like (21), and see what we can get if one searches its convergence in the norm derived from Poincaré's mean value on the real axis.

Why do we appeal to this type of convergence?

I think because it has proven to be a very important tool in Fourier Analysis (second stage), and hope to be also successful in the future. The procedure to be followed to define the space $B^2(R, \mathcal{C})$ and emphasize some of its properties has the origin in the theory of orthogonal functions. In this field of investigation, closely related to Fourier Analysis, there are numerous monographs and treatises. We send the reader to the classical references Alexits [1] and Sansone [26].

In order to apply this procedure to the case of almost periodic functions, the following elementary result is useful:

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp(i\lambda t)dt = \begin{cases} 1, & \text{for } \lambda = 0, \\ 0, & \text{for } \lambda \neq 0. \end{cases} \tag{23}$$

Equation (23) is an orthogonality relation, which clearly appears when one considers a sequence of complex exponentials $\{\exp(i\lambda_k t);\ k \geq 1\}$, with $\lambda_k \neq \lambda_j$ for $k \neq j$, and derive from (23) the relation

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\exp[i(\lambda_k - \lambda_j)t]dt = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{24}$$

Let us return to the assumption (22), and notice that

$$\begin{aligned} \left|\sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t)\right|^2 &= \left\langle \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t), \sum_{k=n+1}^{n+p} \bar{a}_k \exp(-i\lambda_k t)\right\rangle \\ &= \sum_{k=n+1}^{n+p} |a_k|^2 + \sum_{\substack{k,j=n+1 \\ k\neq j}}^{n+p} a_k \bar{a}_j \exp[(i(\lambda_k - \lambda_j)t]. \end{aligned}$$

If one takes (24) into account, and takes the Poincaré's mean value of both sides in the last equation, one obtains the relation

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\left|\sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t)\right|^2 dt = \sum_{k=n+1}^{n+p} |a_k|^2. \tag{25}$$

Now, taking into account our assumption (22), we see from (25) that the series converges on R, with respect to the seminorm $f \to |f|_{B^2}$, defined by

$$|f|_{B^2}^2 = \lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}|f(t)|^2 dt, \tag{26}$$

the right hand side of (26) being finite. Indeed, in the way we have obtained (25), one also obtains

$$\lim_{\ell\to\infty}(2\ell)^{-1}\int_{-\ell}^{\ell}\left|\sum_{k=1}^{n} a_k \exp(i\lambda_k t)\right|^2 dt = \sum_{k=1}^{n} |a_k|^2, \tag{27}$$

and letting $n \to \infty$, there results on behalf of (22) (the seminorm is continuous!) the formula (26), or

$$\sum_{k=1}^{\infty} |a_k|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt, \tag{$*$}$$

which is nothing else but Parseval's formula. See also formula (20).

A legitimate question is now whether the convergence, in the sense of the norm derived from Poincaré's mean value, defines a function belonging to $L^2_{\text{loc}}(R, \mathcal{C})$, such that (26) may have a meaning?

The answer to this question is positive and we shall dwell in getting it. If one denotes by $A > 0$ the sum of the series $\sum_{k=1}^{\infty} |a_k|^2$ in (22), then (27) allows us to write the inequality, valid when $n \geq 1$,

$$\int_{-\ell}^{\ell} \left| \sum_{k=1}^{n} a_k \exp(i\lambda_k t) \right|^2 dt < 2\ell(A + \varepsilon), \tag{28}$$

for $\ell \geq \ell(\varepsilon)$. Let us fix now ℓ as mentioned above, and read (28) as follows: the series in (21), under assumption (22), converges on the interval $[-\ell, \ell]$, in the space $L^2([-\ell, \ell], \mathcal{C})$. We assign now to $\ell \geq \ell(\varepsilon)$ a sequence of values $\{\ell_m;\ m \geq 1\}$, such that $\ell_m \nearrow \infty$ as $m \to \infty$. Since on each interval $[-\ell_m, \ell_m]$ the series in (21) is L^2-convergent, there results that we deal with convergence in $L^2_{\text{loc}}(R, \mathcal{C})$. The limit function, we have denoted by $f(t)$, satisfies the equation

$$f(t) = \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t), \quad a.e.\ \ t \in R, \tag{29}$$

the a.e. convergence being the consequence of the fact $f(t) \in L^2_{\text{loc}}(R, \mathcal{C})$. Therefore, we have the right to substitute (29) to (21), and we can now associate to each series, which satisfies (22), a function $f(t) \in L^2_{\text{loc}}(R, \mathcal{C})$. This function is exactly the sum of the series (21), which generates it in the way shown above when proving the convergence in $L^2_{\text{loc}}(R, \mathcal{C})$.

At this point in the discussion, it is very important to look more in detail at the correspondence from series to functions, as established above. The following remark is necessary. Namely, since the right hand side in (26) remains unchanged, when the integrand $|f(t)|^2$ is changed into $|f(t) - f_0(t)|^2$, with $f_0(t)$ such that

$$|f_0|_{B^2} = \left\{ \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_0(t)|^2 dt \right\}^{1/2} = 0, \tag{30}$$

it means that the correspondence from series to function is not one to one (as it happens in $AP(R, \mathcal{C})$). More precisely, to each series in (21), one associates a set of functions $f \in L^2_{\text{loc}}$, for which formula $(*)$ is verified. This set of functions is nothing else but the translation of the null space of Poincaré's functional, i.e., the space $\mathcal{N}_0$ of those functions for which (30) is satisfied. Let us notice that one of these functions is $f_0(t) = \exp(-|t|)$, $t \in R$.

Let us denote by $\mathcal{B}$ the set of all trigonometric series like (21), such that (22) holds true for each series. We shall denote by $\widetilde{\mathcal{B}}$ the space of all functions $f \in L^2_{\text{loc}}(R, \mathcal{C})$, corresponding to series from $\mathcal{B}$, by means of the procedure described above, that lead to the Parseval's formula $(*)$. See also the relation given by formula (29).

Before introducing the Besicovitch space of almost periodic functions, $B^2 = B^2(R, \mathcal{C})$, let us point out the fact that formula $(*)$ in this section is the vehicle that helps us to deal with either manner of constructing the space B^2. We shall prove, first, the following.

Lemma 3.1 *The set $\mathcal{B}$, organized as a linear seminormed space, is complete. Hence, it is isometric and isomorphic to a B-space* (*see Yosida* [31]).

Proof. Since the elements of $\mathcal{B}$ are series like (7), and the coefficients verify condition (22), it is to be expected that the Hilbert space $\ell^2(N, \mathcal{C})$ will play an important role in investigating properties of $\mathcal{B}$. Indeed, let us consider a Cauchy sequence $\{x_k;\ k \geq 1\} \subset \mathcal{B}$. This means that, for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, with the property

$$|x_n - x_m|_{\mathcal{B}} < \varepsilon \quad \text{for } n, m \in N(\varepsilon). \tag{31}$$

Since each $x_k \in \mathcal{B}$ can be regarded as an element in the Hilbert space $\ell^2(N, \mathcal{C})$, i.e., its representation in $\mathcal{B}$ is

$$x_k \sim \sum_{j=1}^{\infty} a_k^j \exp(i\lambda_j t), \tag{32}$$

with $\{a_k^j;\ j \geq 1\} \subset \ell^2(N, \mathcal{C})$, (31) takes the form

$$\sum_{j=1}^{\infty} |a_n^j - a_m^j|^2 < \varepsilon^2, \quad \text{for } n, m \geq N(\varepsilon). \tag{33}$$

Starting from (33), by a routine procedure (see for detailed discussion, for instance, V. Trénoguine [30]) one obtains the existence of an element/series in $\mathcal{B}$, say x, such that $x \sim \sum_{j=1}^{\infty} a^j \exp(i\lambda_j t)$. The coefficients a^j, $j \geq 1$, are limits for subsequences of the sequences $\{a_k^j;\ k \geq 1\}$, $j \in N$.

Remark 3.1 According to our notation, it appears that the set of λ_k's is common to all series involved in the representation of the elements x_k, $k \geq 1$. This is not a restriction, because the union of all such exponents to all x_k's $k \geq 1$, is a countable set. Therefore, one may have to add some terms, in the representations, whose coefficients are zero. In such a way, we can use the same complex exponentials for each $x_k \in \mathcal{B}$, $k \geq 1$.

Remark 3.2 In case we have two series in $\mathcal{B}$, say $\sum_{j=1}^{\infty} a_j \exp(i\lambda_j t)$ and $\sum_{j=1}^{\infty} b_j \exp(i\lambda_j t)$, the equation $(*)$ allows us to write

$$\sum_{j=1}^{\infty} |a_j - b_j|^2 = \lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - g(t)|^2 dt, \tag{34}$$

from which we derive

$$a_j = b_j,\ j \geq 1, \quad \text{iff} \quad f - g \in \mathcal{N}_0, \tag{35}$$

where $\mathcal{N}_0$ =the null space, has been defined above in this sections. In other words, two functions $f, g \in \widetilde{\mathcal{B}}$ generate the same series in $\mathcal{B}$, in case, and only in case $f - g \subset \mathcal{N}_0$.

Remark 3.3 From the relationship

$$\sum_{j=1}^{\infty} |a_n^j - a_m^j|^2 = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |x_n(t) - x_m(t)|^2 dt,$$

which results from Parseval equation, written for the difference $x_n(t) - x_m(t)$, in accordance with the representation (32), one derives the conclusion that the linear space $\widetilde{\mathcal{B}} \subset L^2_{\mathrm{loc}}(R, \mathcal{C})$ is also complete in the topology induced by the seminorm $|\cdot|_{B^2}$, as defined by (26).

To summarize the above discussion, we shall state the following.

Theorem 3.1 *The Banach space $\mathcal{B}$ of series like* (7)*, under assumption* (22)*, with the norm*

$$\left| \sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) \right|_{B^2} = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}, \tag{36}$$

is completely determined, as described above. First of its realizations is the model also described above, starting with the set $\mathcal{B}$, and endowing it until the Banach space $B^2 = B^2(R, \mathcal{C})$ is constructed. A second realization (isomorphism plus isometry), also described above, consists in the model starting with the set $\widetilde{\mathcal{B}} \subset L^2_{\mathrm{loc}}(R, \mathcal{C})$, which is isomorphic and isometric to $\mathcal{B}$, modulo $\mathcal{N}_0$ – the null space in $\widetilde{\mathcal{B}}$. The integral norm on the factor space $\widetilde{\mathcal{B}}/\mathcal{N}_0 = B^2$ is given by the formula (26).

The **proof**, to be complete, also requires to prove that $\mathcal{N}_0$ is a closed subspace of $\widetilde{\mathcal{B}}$, in the topology of the seminorm (26) on $\widetilde{\mathcal{B}}$.

Let $f_n \to f$ in $\widetilde{\mathcal{B}}$, as $n \to \infty$, and assume $\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_n(t)|^2 dt = 0$, $n \geq 1$. Let us show that $\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = 0$. This follows from the Minkowski's inequality

$$\left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right\}^{1/2} \leq \left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - f_n(t)|^2 dt \right\}^{1/2} + \left\{ (2\ell)^{-1} \int_{-\ell}^{\ell} |f_n(t)|^2 dt \right\}^{1/2},$$

which implies, as $\ell \to \infty$, $|f(t)|_{B^2} \leq |f(t) - f_n(t)|_{B^2}$. Now, letting $n \to \infty$, one obtains $|f(t)|_{B^2} = 0$, which means $f \in \mathcal{N}_0$.

For definitions and details concerning the factor space, see Yosida [31] and Swartz [29].

Finally, let us notice that Remark 3.3 to Lemma 3.1 proves the completeness of $\widetilde{\mathcal{B}}$, with respect to the seminorm (26), which is needed in obtaining the completeness, and hence the Banach type space for $B^2(R, \mathcal{C})$ – as a quotient or factor space.

With these considerations, related to the construction of the Besicovitch space $B^2(R, \mathcal{C})$, we end the proof of Theorem 3.1.

We have dealt with $B^2(R, \mathcal{C})$ in Corduneanu [10], when the notation $AP_2(R, \mathcal{C})$ has been used to stress its connection with the spaces $AP_r(R, \mathcal{C})$, $r \in [1, 2)$. But these spaces, all of them subsets of $B^2 = AP_2$, have different topologies, stronger than the topology of B^2. Moreover, the approximation property has been taken as definition, instead of starting with trigonometric series. Properties similar to A and B have been emphasized for the $AP_r(R, \mathcal{C})$-spaces.

In concluding this section, we shall recall the fact that in the book by Corduneanu [10], the construction of the space $\mathcal{B}^2(R,\mathcal{C}) = AP_2(R,\mathcal{C})$ is based on the approximation property applied in the Macinkiewicz' space $\mathcal{M}_2(R,\mathcal{C})$, taking the closure of the set of trigonometric polynomials.

4 The Besicovitch Space $B(R,\mathcal{C})$

In Besicovitch [4], one finds the construction of the spaces B^p for $p > 1$, the case $p = 1$ conducing to a more difficult treatment, with definitions for the upper and lower mean values. The difference with respect to the case $p > 1$ comes from the fact that Hölder inequality, in case $p = 1$ leads to the conjugate index $q = \infty$, while for L^∞ we don't have an integral norm. But, this tool is systematically used in building the theory of B^p-spaces when $p > 1$. It is known that the seminorm which plays the main role in constructing the spaces $B^p(R,\mathcal{C})$, $p < \infty$, is given by

$$|f|_{B^p}^p = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^p dt. \tag{37}$$

In the preceding section we have obtained and dealt with (37), in the case $p = 2$. But our approach was based on taking the trigonometric series as departing object and the condition (22) imposed on these series that characterize the B^2-functions.

The seminorm (37), for $p = 1$, which is the Poincaré's mean value of $|f(t)|$, will be also of great use in our approach to construct the space B.

Instead of starting from a condition similar to (22), which apparently does not exist, even though for $AP_r(R,\mathcal{C})$ spaces, $r \in (1,2]$, it has been helpful, we shall start from the space $AP(R,\mathcal{C})$ of Bohr, which has been characterized in our approach by Theorem 2.1.

In the space $AP(R,\mathcal{C})$, due to the summability of its associated series, the approximation property is valid. This means that the set of trigonometric polynomials, a fraction of the set $\mathcal{ST}$ of trigonometric series like (7) is everywhere dense in $AP(R,\mathcal{C})$, with respect to the uniform convergence on R. As it is well known (see, for instance, Lewitan [21] or Corduneanu [10]), once the approximation property is established, one can easily derive the existence of the mean value for each $f \in AP(R,\mathcal{C})$, starting from the obvious fact that the mean value exists for each trigonometric polynomial (equal to the term without complex exponential, if any, otherwise $= 0$).

The main properties of the mean value $M\{f\}$, $f \in AP(R,\mathcal{C})$, are

(a) $M\{\bar{f}\} = \overline{M\{f\}}$;

(b) $M\{\alpha f + \beta g\} = \alpha M\{f\} + \beta M\{g\}$, $\alpha,\beta \in \mathcal{C}$, $f,g \in AP(R,\mathcal{C})$;

(c) $f(t) \geq 0$ on R implies $M\{f\} \geq 0$, $f \in A(P,R)$ and $M\{f\} = 0$ implies $f \equiv 0$;

(d) $|M\{f\}| \leq M\{|f|\}$, $f \in AP(R,\mathcal{C})$.

Let us notice that the map $f \to M\{|f|\}$, from $AP(R,R)$ into R is a norm. Indeed, for $f,g \in AP(R,\mathcal{C})$, one has $|f+g| \leq |f|+|g|$, which leads to $M\{|f+g|\} \leq M\{|f|\}+M\{|g|\}$. Property (c) is a consequence of the uniqueness.

Lemma 4.1 *In the topology induced by the mean value norm, the space $AP(R,\mathcal{C})$ is always incomplete (denoted by $AP_M(R,\mathcal{C})$).*

Proof. The proof will be conducted on the principle of *reductio ad absurdum.* Hence, let us assume that the set of elements in $AP(R,\mathcal{C})$, with the norm $M\{|f|\}$, is complete. Therefore, it is a Banach space. Then the identity map, which is one-to-one, is a linear operator acting from the Banach space $AP(R,\mathcal{C})$, in its associate $AP_M(R,\mathcal{C})$, endowed with the mean-value norm $M\{|f|\}$. According to the Banach theorem on the continuity of the inverse operator, we derive that the identity map (which coincides with its inverse) is continuous from $AP_M(R,\mathcal{C})$ onto $AP(R,\mathcal{C})$. This fact implies the existence of a constant $C > 0$, such that

$$\sup\{|f(t)|;\ t \in R\} \leq CM\{|f|\}, \quad f \in AP(R,\mathcal{C}). \tag{38}$$

By an example, we shall prove now that (38) is impossible, and therefore our assumption that $AP_M(R,\mathcal{C})$ is complete is *false.*

Let us consider the sequence of periodic functions, defined by $f_n(t+1) = f_n(t)$, $t \in R$, $n \geq 2$, and for $t \in [0,1)$ by

$$f_n(t) = \begin{cases} 1 - nt, & 0 \leq t < n^{-1}, \\ 0, & n^{-1} \leq t < 1 - n^{-1}, \\ 1 - n + nt, & 1 - n^{-1} \leq t \leq 1. \end{cases} \tag{39}$$

Since periodic functions are almost periodic (Bohr), i.e. in $AP(R,R) \subset AP(R,\mathcal{C})$, we obtain $M\{f_n\} = n^{-1}$, $n \geq 2$, while $\sup f_n = 1$, $n \geq 2$. Hence, one should have $1 \leq C/n$, $n \geq 2$, which is obviously impossible. This ends the proof of Lemma 4.1.

Further, on our way to construct the space $B = B(R,\mathcal{C})$, we shall complete the space $AP_M(R,\mathcal{C})$, following the usual procedure (see, for instance, Trénoguine [30], or Yosida [31]).

Let us denote by $\mathcal{B}$ the linear complete space which is the (unique, up to isomorphism) completion of the space $AP_M(R,\mathcal{C})$. One has $AP_M(R,\mathcal{C}) \subset \mathcal{B}(R,\mathcal{C})$, more precisely $AP_M(R,\mathcal{C})$ can be identified with a set which is everywhere dense in $\mathcal{B}(R,\mathcal{C})$.

Applying the Hahn-Banach theorem on extension of functionals, from subspaces to a larger space, we can infer that the seminorm $M\{|f|\}$, which is defined on $AP_M(R,\mathcal{C})$, admits an extension to $\mathcal{B}(R,\mathcal{C})$, with preservation of its basic properties. If one denotes by $\widetilde{M}\{|f|\}$ the extension of M from $AP_M(R,\mathcal{C})$ to $\mathcal{B}(R,\mathcal{C})$, then $\widetilde{M}\{|f|\} = M\{|f|\}$ for each $f \in AP_M(R,\mathcal{C})$ and $\widetilde{M}$ satisfies on $\mathcal{B}$ the properties (a), (b), (c), (d), excepting the part of (c) which makes $AP_M(R,\mathcal{C})$ a normed space (not a seminormed one!).

A natural question arises at this point in our discussion. Namely, how do we know that the new elements in the completed space are functions locally integrable on$R$, so that $\widetilde{M}\{|f|\}$ makes sense.

The answer to this question results from the following considerations (also encountered when constructing $B^2(R,\mathcal{C})$, in the preceding section). If one considers a Cauchy sequence in $AP_M(R,\mathcal{C})$, say $\{f_k;\ k \geq 1\} \subset AP_M$, from $\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_n(t) - f_m(t)|dt < \varepsilon$, for $n, m \geq N(\varepsilon)$, one derives the inequality

$$\int_{-\ell}^{\ell} |f_n(t) - f_m(t)|dt < (2\ell + 1)\varepsilon, \tag{40}$$

for $n, m \geq N(\varepsilon)$ and $\ell \geq L(\varepsilon)$. As proceeded in the preceding section, one obtains that $F(t) = \lim f_m(t)$, as $m \to \infty$, in $L^1_{\text{loc}}(R,\mathcal{C})$. Hence, we are assured that in order to complete the normed space $AP_M(R,\mathcal{C})$, it is sufficient to add functions which are in $L^1_{\rm loc}(R,\mathcal{C})$. Of course, this situation takes place when the Cauchy sequence $\{f_k;\ k \geq 1\}$ does not have its limit in $AP_M(R,\mathcal{C})$.

So far, we have constructed a complete seminormed space, not a Banach space yet, denoted by $\mathcal{B}(R,\mathcal{C})$, the seminorm being the mean-value functional $f \to \widetilde{M}\{|f|\}$.

The last step to achieve the construction of the Besicovitch space $B(R,\mathcal{C})$, as a Banach space, is to take the factor space $\mathcal{B}/\mathcal{N}_0$, where $\mathcal{N}_0$ stands for the null space of the functional

$$\widetilde{M}\{|f|\} = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|dt. \tag{41}$$

For the construction of the factor space $\mathcal{B}/\mathcal{N}_0$, in order to obtain by means of this procedure a normed complete space (Banach), we need to show that $\mathcal{N}_0$ is a closed subspace of $\mathcal{B}$. Indeed, assume that $\{f_k;\ k \geq 1\} \subset \mathcal{B}$ is such that $\widetilde{M}(|f_k|) = 0$, $k \geq 1$, and

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f_k(t) - f(t)|dt = 0. \tag{42}$$

We need to prove that $f \in \mathcal{N}_0$, i.e., $\widetilde{M}(|f|) = 0$. Taking into account the relationship $|f(t)| \leq |f(t) - f_k(t)| + |f_k(t)|$, we obtain

$$\widetilde{M}(|f|) \leq \widetilde{M}(|f - f_k|) + \widetilde{M}(|f_k|), \quad k \geq 1, \tag{43}$$

and since $\widetilde{M}(|f_k|) = 0,\ k \geq 1$, while $\widetilde{M}(|f - f_k|) \to 0$ as $k \to \infty$, there results $\widetilde{M}(|f|) = 0$. This means $f \in \mathcal{N}_0$, and this is what we wanted to prove. Summarizing the discussion about the construction of the space $B = B(R,\mathcal{C})$, carried out above, we can formulate the following

Theorem 4.1 *The Besicovitch space $B = B(R,\mathcal{C})$ is constructed by the following procedure:*

1) *One starts with the Bohr space of almost periodic functions $AP(R,\mathcal{C})$ (see Theorem 2.1 above), which generates the incomplete normed space $AP_M(R,\mathcal{C})$, according to Lemma 4.1.*

2) *The (unique) completion of $AP_M(R,\mathcal{C})$, denoted by $\mathcal{B} = \mathcal{B}(R,\mathcal{C})$, is a seminormed complete space, with the seminorm $f \to \widetilde{M}(|f|)$ =the extended mean value/norm in $AP_M(R,\mathcal{C})$, defined by (41).*

3) *The Banach space $B = B(R,\mathcal{C})$ is the factor space $\mathcal{B}/\mathcal{N}_0$, with $\mathcal{N}_0$ the null space of the seminorm $\widetilde{M}\{|f|\}$, $f \in B$.*

The **proof** of Theorem 4.1 has been completed above, in this section, while the construction procedure is motivated by the known results on *completion* of seminormed spaces, as well as on the construction of the *factor space*. For details in this regard, see Yosida [31] and Swartz [29].

In concluding this section, we shall briefly discuss some properties of the space B, including its relationships with other spaces of almost periodic functions.

From the construction of the space $B(R,\mathcal{C})$ described above, there results several properties that we shall consider below.

First, let us notice the fact that the approximation property is valid, in the norm of the space $B(R,\mathcal{C})$. This means that for $f \in B$ and each $\varepsilon > 0$, one can determine a trigonometric polynomial of the form (5), such that

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - T_\varepsilon(t)| dt < \varepsilon, \tag{44}$$

is satisfied.

Second, the mean value of any function $g \in B(R,\mathcal{C})$ exists, being given by (4).

The proof of this statement can be found in Besicovitch [4] or Corduneanu [11].

Third, the mean value $f \to \widetilde{M}\{f\}$ satisfies conditions (a), (b), (d) mentioned above in this section, while in (c) only the first statement remains true.

Indeed, $\widetilde{M}(|f|) = 0$ does not imply $f = 0$, but only $f \in \mathcal{N}_0$. One has to take into account that $|\widetilde{M}(t)| \le \widetilde{M}(|f|)$, which is an obvious property. The property also shows that $f \to \widetilde{M}(f)$ is a continuous functional on B (or $\mathcal{B}$).

Fourth, once established the existence of the mean value $\widetilde{M}(f)$, for each $f \in B(R,\mathcal{C})$, one can find the Fourier series associated to $f \in B(R,\mathcal{C})$, which represents the trigonometric series of the form (7), characterizing not only f (as an individual function), but the equivalence class which contains f, i.e., any other g for which

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t) - g(t)| dt = 0.$$

Fifth, besides the spaces $AP(R,\mathcal{C})$ and $B^2(R,\mathcal{C})$, $B(R,\mathcal{C})$ is also containing the Stepanov's space of almost periodic functions, $S = S(R,\mathcal{C})$, which is defined as the set of all $f \in L^1_{\rm loc}(R,\mathcal{C})$, such that

$$\sup\left\{\int_t^{t+1} |f(s)| ds;\ t \in R\right\} = |f|_S < \infty. \tag{45}$$

Since for large $\ell > 0$ we can write for $f \in S$

$$\ell^{-1}\int_0^{\ell} |f(s)|ds \le \ell^{-1}\left(\int_0^1 |f(s)|ds + \int_1^2 |f(s)|ds + ... + \int_{[\ell]}^{[\ell]+1} |f(s)|ds\right) \le \ell^{-1}([\ell]+1)|f|_S,$$

one obtains, as $\ell \to \infty$, the inequality

$$|f|_B \le |f|_S, \quad f \in S(R,\mathcal{C}), \tag{46}$$

which tells us that $S \subset B$.

We took into account that one has

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|ds = \lim_{\ell\to\infty} \ell^{-1} \int_0^{\ell} |f(s)|ds = \lim_{\ell\to\infty} \ell^{-1} \int_{-\ell}^{0} |f(s)|ds,$$

which can be found in most books on almost periodic functions (for instance, Corduneanu [11]).

As far as the inclusion $B^2 \subset B$ is concerned, it follows from the inequality

$$(2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|ds \le \left[(2\ell)^{-1} \int_{-\ell}^{\ell} |f(s)|^2 ds\right]^{1/2},$$

valid for $\ell > 0$ and each $f \in B^2(R, \mathcal{C}) \subset L^2_{\rm loc}(R, \mathcal{C})$, on behalf of Cauchy's integral inequality (special case of Hölder's inequality).

Sixth, because the approximation property by trigonometric polynomials is assured for functions in $B(R, \mathcal{C})$, there results that the property B, mentioned in Introduction, is valid. As it is known, the Bochner's property (i.e., relative compactness) of the family of translates if f, $\mathcal{F} = \{f(t+h);\ h \in R\}$, implies Bohr's property. Of course, all these properties are meant in the sense of the norm of the space $B(R, \mathcal{C})$. More precisely, for $f \in B(R, \mathcal{C})$, to any $\varepsilon > 0$ there corresponds $\ell = \ell(\varepsilon)$, such that each interval $(a, a+\ell) \ni \tau$, such that $|f(t+\tau) - f(t)|_B < \varepsilon$, $t \in R$.

Seventh, the space $B(R, \mathcal{C})$ has been already involved in work pertaining to the third stage of the development of the theory of oscillatory functions. See the book by Ch. Zhang [33], which contains the theory of pseudo-almost periodic functions. When defining the space $PAP(R, \mathcal{C})$ of these functions, the B-norm is involved, together with that of BC-space (the supremum norm, on R). One has the inclusion $PAP(R, \mathcal{C}) \subset BC(R, \mathcal{C})$, but the pseudo-almost periodicity appears as perturbation of the classical case of Bohr. An example of the use of space $B(R, \mathcal{C})$ in proving existence of almost periodic solutions to certain functional equations is given in Corduneanu [9]. The solutions are in $B^2(R, \mathcal{C})$.

Eighth, the interest for oscillatory functions/solutions comes from their significance in the physical problems, and their frequent use. In the paper of Staffans [28], an example of a function belonging to the Weyl's space (see Besicovitch [4]) is provided, which does not present the oscillatory character. It is understood that the space $B(R, \mathcal{C})$ may contain functions whose behaviour may not be classified as oscillatory.

We shall make a final remark about the manner of introducing the space $B(R, \mathcal{C})$. Namely, if we start again from the set of trigonometric series, of the form (7), the Cauchy's type convergence condition

$$\lim_{\ell \to \infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n+1}^{n+p} a_k \exp(i\lambda_k t) \right| dt < \varepsilon, \tag{47}$$

for $n \geq N(\varepsilon)$, $p \geq 1$, is, very likely, leading to the space $B(R, \mathcal{C})$ after the operations used already (completion, factor space). We have used this approach in constructing the space $B^2(R, \mathcal{C})$. In that case, we have been essentially helped by condition (22) imposed on the coefficients of the candidate series. It is obvious that (47) is the condition guaranteeing the convergence of the series (7) in the space $\mathcal{B}$ or B (after factorization). The approach we have used in this section relies substantially on the facts known in the classical theory.

5 Some Preliminaries for Oscillatory Functions Spaces

Both classes of oscillatory functions, amply investigated during the last two centuries, are representable by means of series of the form (7). It does not mean that the series are convergent in the usual sense, but the procedure that can be associated to them, in various ways, allow the construction of corresponding functions (e.g., by summability methods or by convergence in certain nonclassical norms, usually inducing a weaker type of convergence than the sup norm). They are useful, because they permit the construction of the function, in a manner that leads to results that can be used in applications.

We have in mind the Fourier Analysis in the classical framework, but also its extension to various classes of almost periodic functions, starting with the functions in $AP(R, \mathcal{C})$, or $AP(R, R)$.

Let us point out that the problem of convergence of Fourier series, which constitute a special form of series (7), has been in the attention of famous mathematicians for a long time. An example constructed by Kolmogorov (see the treatises by Bary and Zygmund, quoted in Introduction) shows that there exists Fourier series, in the classical sense, nowhere convergent on the interval $[-\pi, \pi]$. It is also worth mentioning the fact that the attention paid to the convergence of series of the form (7) is directed to their convergence on the finite interval $[-\pi, \pi]$, even though each term of the series is defined on the whole R. This feature is not, generally, agreeing with the needs of applications, when large interval of time can be involved, such as it happens in Celestial Mechanics or in other types (could be man made) of evolutionary systems.

Some of the latest example of oscillatory systems/functions led to the investigation of series of a much more general form than (7), namely

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)] \tag{48}$$

with $\{a_k;\ k \geq 1\} \subset \mathcal{C}$, and $\lambda_k(t),\ k \geq 1$, some real functions defined on R, and such that certain orthogonality conditions are verified.

We shall use again the Poincaré's mean value on R, and write these conditions in the form

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \exp[i(\lambda_k(t) - \lambda_j(t))]dt = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{49}$$

where $k, j \geq 1$, and $\lambda_k(t) \neq \lambda_j(t)$ for $k \neq j$, with $\lambda_k(t) \in L^1_{\text{loc}}(R, R),\ k \geq 1$, while $\{a_k;\, k \geq 1\} < \mathcal{C}$ satisfy (22).

The following assertion shows how a certain type of convergence, applied to the series (48), can help to associate a function or set of functions to it.

Lemma 5.1 *Consider the series* (48)*, under the above stated conditions for the functions* $\lambda_k(t)$*,* $k \geq 1$*, and* $\{a_k;\ k \geq 1\} \subset \ell^2(N, \mathcal{C})$*. Then the series* (48) *converges on* R*, with respect to the* B^2*-seminorm, i.e.,*

$$f \to \left[\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt\right]^{1/2}, \tag{50}$$

which implies convergence in $L^2_{\text{loc}}(R, R)$*.*

The **proof** of Lemma 5.1 is completely similar to that given in the section of this paper dedicated to the construction of the space $B^2(R, \mathcal{C})$, where $\lambda_k(t) = \lambda_k t,\ t \in R$, $\lambda_k \in R$, $k \geq 1$. As shown there, one can write

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{51}$$

the convergence (on R) being that of the space $L^2([-\ell, \ell], R)$, for each $\ell > 0$.

An important aspect in the development of the approach of constructing classes/ spaces of oscillatory functions, starting from series of the form (51), under condition (22) for the coefficients, is the finding/construction of sets consisting of function $\lambda(t) : R \to R$, from which we can recruit sequences satisfying the conditions stipulated in Lemma 5.1.

We owe to Ch. Zhang [34], [35], [36] the finding of such a set of functions (polynomials), which allowed him to construct spaces of oscillatory functions, called *strong limit power* functions. These functions are obtained by the uniform approximation procedure from a set of polynomials, forming a group, under usual addition. These polynomials, actually "generalized polynomials", are defined as follows:

$$\lambda(t) = \begin{cases} \sum_{j=1}^{m} c_j t^{\alpha_j}, & t \geq 0, \\ -\sum_{j=1}^{m} c_j (-t)^{\alpha_j}, & t < -0, \end{cases} \tag{52}$$

where $c_j \in \mathcal{C},\ j \geq 1$ and $\alpha_1 > \alpha_2 > ... > \alpha_m > 0$ are arbitrary positive numbers. Then, one considers generalized polynomials of the form

$$P(t) = \sum_{k=1}^{n} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{53}$$

with each $\lambda_k(t)$ as described in (52). It is obvious that each $\lambda(t)$ in (52) is an odd function (like $\sin t$), a property which plays an important role in existence of the mean value on R.

Then, the orthogonality conditions (49) are satisfied, and one can proceed to the construction of the space $SLP(R, \mathcal{C})$ – strong limit power – as follows: $f \in SLP(R, \mathcal{C})$ if for every $\varepsilon > 0$, there exists a generalized polynomial of the form(53), such that

$$|f(t) - P_\varepsilon(t)| < \varepsilon, \quad t \in R. \tag{54}$$

From (54) we read that sup-norm is the one for $SLP(R, \mathcal{C})$.

The SLP space defined above is a Banach space, and each $f \in SLP(R, \mathcal{C})$ can be related to a generalized Fourier series, such that

$$f(t) \sim \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{55}$$

which satisfies the Parseval equality

$$\sum_{k=1}^{\infty} |a_k|^2 = M\{|f|^2\} \tag{56}$$

with the coefficients

$$a_k = M\{f(t)e^{-i\lambda_k(t)}\}. \tag{57}$$

Many properties of $AP(R, \mathcal{C})$ can be adapted to the $SLP(R, \mathcal{C})$ space. We can say that the space $SLP(R, \mathcal{C})$ is a "copy" of the Bohr space, with considerable extension of the class of functions involved.

The mean value functional $M\{f\}$ is the Poincaré's mean value on R, and possesses other properties that appear in the case of the space $AP(R, \mathcal{C})$. See also the papers by Ch. Zhang and C. Meng [37], [38].

We can now proceed to construct a space of almost periodic functions, relying on Lemma 5.1, and using the same procedure as in case of the space $B^2(R, \mathcal{C})$. In this way, we shall obtain a larger space than $SLP(R, \mathcal{C})$, because we shall use the seminorm that appears in (50). This space will be richer than the space $SLP(R,\mathcal{C})$, possessing less properties, but still pertaining to the oscillatory type.

We will denote this space, to be constructed, by $B^2_\lambda(R,\mathcal{C})$, where the index λ designates the fact that only polynomials of the form (53) will be used as exponents for the complex exponentials involved.

The space $B^2_\lambda(R,\mathcal{C})$ will be a space of oscillatory functions, and as $SLP(R,\mathcal{C})$, will be part of the *third* period in the development of classical Fourier Analysis.

6 Construction of the Space $B^2_\lambda(R,\mathcal{C})$

The space $B^2_\lambda(R,\mathcal{C})$ will be constructed in the manner used in case of the Besicovitch space $B^2(R,\mathcal{C})$. The first step is to start with formal generalized series of the form

$$\sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad t \in R, \tag{58}$$

instead of the trigonometric series (7). The function $\lambda_k(t)$, $k \geq 1$, are generalized polynomials as those defined by the formula (52) and used in constructing the SLP-space of Ch. Zhang [35], [36]. By applying Lemma 5.1, we shall associate a function f in $L^2_{\text{loc}}(R,\mathcal{C})$, such that

$$f(t) = \sum_{k=1}^{\infty} a_k \exp[i\lambda_k(t)], \quad \text{a.e. on } R, \tag{59}$$

and following step by step the construction of the space $B^2(R,\mathcal{C})$ in a previous section, we find the equation

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} \left| \sum_{k=n}^{n+p} a_k \exp[i\lambda_k(t)] \right|^2 dt = \sum_{k=n}^{n+p} |a_k|^2, \ n \geq 1, \ p \geq 1, \tag{60}$$

which, on behalf of (22), assures the convergence of the series (59) in $L^2_{\text{loc}}(R,\mathcal{C})$. Hence, we can write the formula

$$\lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt = \sum_{k=1}^{\infty} |a_k|^2, \tag{61}$$

which is the same as (56).

Formula (61) is the Parseval equation for the function $f \in \widetilde{B}^2_\lambda(R,\mathcal{C})$, which is defined as the set of functions respresentable in the form (59), with $\{a_k;\ k \geq 1\} \in \ell^2(N,\mathcal{C})$, and convergence in $L^2_{\text{loc}}(R,\mathcal{C})$. The connection between $f \in \widetilde{B}^2_\lambda(R,\mathcal{C})$ and the coefficients a_k is given by (57), formulas easy to obtain from (59) and the above procedure.

The set of functions, we have denoted by $\widetilde{B}^2_\lambda(R,\mathcal{C})$, is naturally organized as a seminormed linear space, with the seminorm in the left hand side of (61), taken at power 1/2, i.e.,

$$f \to \left\{ \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} |f(t)|^2 dt \right\}^{1/2}. \tag{62}$$

In order to prove the *completeness* of this seminormed space, one needs to proceed again like in the case of construction of the Besicovitch space $B^2(R,\mathcal{C})$. The key condition is again the assumption (22) on the coefficients of complex exponentials, and the validity of Parseval's type formula (61). In other words, everything reduces to the structure of the space $\ell^2(N, \mathcal{C})$. See Remark 3.3 to Theorem 2.1.

The last step in constructing the space $B^2_\lambda(R, \mathcal{C})$ consists in taking the factor space of $\widetilde{B}^2_\lambda(R, \mathcal{C})$, modulo the subspace of zero-seminorm elements in this space.

If the subspace above, say $\mathcal{N}_{0\lambda}$ is closed in the topology induced by the seminorm (62), then the factor space is a Banach space. Apparently, this is the case, but it is to be seen if the argument used in case of Besicovitch space $B^2(R, \mathcal{C})$ is valid in this situation. Otherwise, the final result is a seminormed complete space, which is widely accepted in Functional Analysis (see, for instance, Yosida [31] or Swartz [29]).

In other words, the last step may not be necessary in the construction of $B^2_\lambda(R, \mathcal{C})$, the space $\widetilde{B}^2_\lambda(R, \mathcal{C})$ constituting the complete seminormed space, which can be useful in various applications.

A few final remarks, related to the content of this paper, may be in order to conclude it.

First, this paper (a continuation of Corduneanu [9]), pursues the idea of constructing spaces of oscillatory functions, generalizing those encountered in the study of periodic functions (classical Fourier Analysis), of almost periodic functions and, lately, of new spaces of oscillatory functions, taking as starting point the set (say $\mathcal{TS}$) of formal trigonometric series (in complex form). By imposing various conditions to the formal series, one obtains old or new classes/spaces of oscillatory functions, with properties that allow their use in applications (particularly, in Engineering, whose impulse has been felt in mathematical research). See references to Zhang [34].

Second, this approach in constructing new spaces of oscillatory functions led to various classes of almost periodic functions, as the $AP_r(R, \mathcal{C})$, $r \in [1, 2]$, allowing to obtain a scale of almost periodic function spaces, with a good potential of applications to the theory of functional equations and the introduction of new concepts, like the generalization of the convolution product (see Corduneanu [8], for instance).

Third, the series characterizing various classes, generally, are not convergent in the classical sense (i.e., uniformly or in Lebesque's spaces), and in order to have a better tool for investigation, it would be desirable to "descend" from these rather abstract functions, to more affordable ones, necessary in numerical analysis and in many applications. For instance, to each series in $AP_r(R, \mathcal{C})$ or in $AP_r(R, R)$, with $r \in (1, 2)$, one can attach the series (in $AP_1(R, \mathcal{C})$), $\sum_{k=1}^{\infty} |a_k|^r \exp(i\lambda_k t)$, i.e., an absolutely convergent series. Can we take some advantage from the investigation of the operator $T_r : AP_r \to AP_1$,

$$\sum_{k=1}^{\infty} a_k \exp(i\lambda_k t) \to \sum_{k=1}^{\infty} |a_k|^r \exp(i\lambda_k t)?$$

We have also formulated this problem in Corduneanu [9].

Fourth, the approach based on dealing with formal series in order to obtain new classes of oscillatory functions, appears to be adequate in advancing the study of more and more intricate functions occurring in applied fields. The work of Ch. Zhang [34–36] is highly illustrative in this regard. One has to note also the contribution of V.F. Osipov [25], who presented a special case of the oscillatory functions of Fresnel type (for instance, the type of oscillations corresponding to the $\sin t^2$), and who dedicated a whole volume to this kind of problems.

Fifth, the method of formal series must be used, in particular, for finding oscillatory solutions of various classes of functional equations. In order to be applicable to partial differential equations, a theory of oscillatory functions, with values in Hilbert or Banach spaces, appears necessary. We will finish soon a paper, dedicated to the existence of such solutions, in which hyperbolic equations are tested – these representing the natural type to possess such solutions (but not only).

Sixth, one problem of great importance in constructing new spaces of oscillatory functions is finding adequate systems $\{\lambda_k(t);\ k \geq 1\}$, satisfying the orthogonality condition (49).

References

[1] Alexits, G. *Convergence Problems for Orthogonal Series.* NY, Pergamon Press, 1961.

[2] Amerio, L. and Prouse, G. *Almost Periodic Functions and Functional Equations.* Van Nostrand, 1971.

[3] Andres, J., Bersani. A.M. and Grande, R.F. Hierarchy of almost periodic functions spaces. *Rend. Matematica*, Series VII, **26** (2006) 121–188.

[4] Besicovitch, A.S. *Almost Periodic Functions.* Cambridge University Press, Cambridge, 1932.

[5] Besicovitch, A.S. and Bohr, H. Almost Periodicity and General Trigonometric Series. *Acta Mathematica* **57** (1931) 203–292.

[6] Bohr, H. *Fastperiodische Funktionen.* Berlin, Springer, 1932.

[7] Bohr, H. and Følner, E. On some types of functional spaces. *Acta Mathematica* **16** (1944) 31–155. Appendix by E. Følner.

[8] Corduneanu, C. Almost periodicity – a new approach. To appear in the *Proceedings of the 7th International Congress of Romanian Mathematicians*, Braşov (2011).

[9] Corduneanu, C. A scale of almost periodic function spaces. *Differential and Integral Equations* **24** (2011) 1–27.

[10] Corduneanu, C. *Almost Periodic Function. Chelsea.* N.-Y., 1989. Distributed by AMS and Oxford Univ. Press.

[11] Corduneanu, C. *Almost Periodic Oscillations and Waves.* Berlin, Springer, 2009.

[12] Corduneanu, C. AP_r-almost periodic solutions to functional differential equations with deviated argument. *Functional Differential Equations* **19** (2012) 63–72.

[13] Corduneanu, C. Almost periodicity in semilinear systems. In: *Integral Methods in Science and Engineering* (Eds.: C. Constanda and P.J. Harris). Springer, 2011.

[14] Corduneanu, C. and Li, Yizeng. A neutral-convolution type functional equation. *Libertas Mathematica* **31** (2011).

[15] Corduneanu, C., Mahdavi, M. and Li, Yizeng. Special Topics in Functional Equations (in preparation).

[16] Edwards, R.E. *Fourier Series. A modern Introduction* (2nd ed.), Springer-Verlag, 1979.

[17] Favard, J. *Leçons sur les Fonctions Presque-Périodiques.* Gauthier-Villars, Paris, 1933.

[18] Fink, A.M. *Almost Periodic Differential Equations.* Lectures Notes Math., 377, Springer, 1974.

[19] Folland, G. *Fourier Analysis and Approximations.* Pacific Grove, CA, 1991.

[20] Kahane, J.P. *Séries de Fourier Absolument Convergentes.* Springer-Verlag, 1970.

[21] Levitan, B.M. *Almost Periodic Functions*. Moscow, 1953. [Russian]

[22] Levitan, B.M. and V.V. Zhikov, V.V. *Almost Periodic Functions and Differential Equations*. Cambridge Univ. Press, 1982.

[23] Maak, W. *Fastperiodische Funktionen*. Springer-Verlag, 1950.

[24] Mahdavi, M. AP_r-almost periodic solutions to the equation $\dot{x}(t) = Ax(t) + (k * x)(t) + f(t)$, *African Diaspora Math. J.* **12** (2012) 44–47.

[25] Osipov, V.F. *Almost Periodic Functions of Bohr-Fresnel*. Univ. of Sankt Petersburg Press, 1992. [Russian]

[26] Sansone, G. *Orthogonal Functions*. Dover Publ., 1991.

[27] Shubin. M.A. Almost periodic functions and partial differential operators. *Usp. Mat. Nauk* **33** (2) (1978) 3–47.

[28] Staffans, O. On almost periodicity of solutions of an integrodifferential equation. *J. Integral Equations* **8** (1985) 249–260.

[29] Swartz, C. *Elementary Functional Analysis*. World Scientific, Singapore, 2010.

[30] Trénoguine, V. *Analyse Fonctionnelle*. Editions Mir, Moscow, 1985.

[31] Yosida, K. *Functional Analysis*. 5th Ed., Springer, 1978.

[32] Zaidman, S. *Almost Periodic Functions in Abstract Spaces*. Pitman, London, 1985.

[33] Zhang, Ch. *Almost Periodic Type Functions and Ergodicity*. Science Press/Kluwer, 2003.

[34] Zhang, Ch. Strong limit power functions. *Journal of Fourier Analysis Appl.* **12** (2006) 291–307.

[35] Zhang, Ch. New limit power functions spaces. *IEEE Trans. AC* **49** (2004) 763–766.

[36] Zhang, Ch. Generalized Kronecker's theorem and strong limit power functions. In: *Alexandru Myller Mathematical Seminar Centennial Conference. AIP Conference Proceedings* **1329** (2011) 281–299.

[37] Zhang, Ch. and Meng, C. C^*-algebra of strong limit power functions. *IEEE Trans. AC* **51** (2006) 828–831.

[38] Zhang, Ch. and Meng, Chenhui. Two new spaces of vector-valued limit power functions. *Studia Sci. Math. Hungarica* **44** (4) (2007) 423–443.