Nonlinear Dynamics and Systems Theory, **13** (4) (2013) 332–343

Stability Analysis for a Class of Nonlinear Nonstationary Systems via Averaging

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Received: February 17, 2013; Revised: October 10, 2013

Abstract: A class of nonlinear nonstationary systems of Persidskii type is studied. The right-hand sides of the systems are represented in the form of linear combinations of sector nonlinearities with time-varying coefficients. It is assumed that the coefficients possess mean values. By means of the Lyapunov direct method, it is proved that if the investigated systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables, then the asymptotic stability of the zero solutions of the corresponding averaged systems implies the local uniform asymptotic stability of the zero solutions for original nonstationary systems. We treat both cases of delay free and time delay systems. Furthermore, it is shown that the proposed approaches can be used as well for the stability analysis of some classes of nonlinear systems with nontrivial linear approximation.

Keywords: *asymptotic stability; Lyapunov function; averaging technique; nonstationary systems; time delay.*

Mathematics Subject Classification (2010): 34D20, 39B72, 34C29.

1 Introduction

A general approach for the stability analysis of nonlinear systems is the Lyapunov direct method (the Lyapunov functions method). By means of this approach, the stability conditions for many types of systems were obtained, see, for example, [9, 11, 17–19, 26] and the references cited therein. However, it should be noted that until now there are no general constructive methods for the finding of Lyapunov functions for nonlinear systems.

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This problem is especially complicated for nonstationary systems [8, 10, 11, 17, 26]. An effective approach for the investigation of dynamical properties of such systems is the averaging technique [10, 11, 13, 17]. This technique allows to reduce stability analysis of time-varying differential equations to the analysis of time-invariant differential equations, possibly resulting in an important simplification. However, it is worth mentioning that the application of the averaging technique is well developed only in the case when original systems are fast time-varying.

In [1, 2], nonlinear nonstationary systems with homogeneous with respect to phase variables right-hand sides have been studied. For such systems, the approach for the Lyapunov functions constructing was proposed. Its application permits to show that if the order of the homogeneity of right-hand sides of the considered time-varying system is greater than one, then the asymptotic stability of the zero solution of the corresponding averaged system implies the same property for the zero solution of the original system. These results have got a further development in [3, 21, 23, 24, 27]. In particular, in [27], a modification of the approach for the Lyapunov functions constructing was suggested. Another techniques for the determination of similar asymptotic stability conditions for time-varying homogeneous systems have been developed in [21, 23]. Recently, these approaches have been extended to nonlinear nonstationary systems with time delay [4–6]. The delay independent asymptotic stability conditions were found on the basis of the stability analysis of corresponding averaged delay free systems.

The principal novelty of the results of the papers [1–6, 21, 23, 24, 27], as compared to the known stability conditions obtained by the application of averaging technique, is that, to guarantee the asymptotic stability for a nonstationary homogeneous system, right-hand sides of the system need not be fast time-varying. It is shown that in the averaging technique, instead of a small parameter providing the fast time-variation of a vector field, the orders of homogeneity can be used.

In the present paper, a class of nonlinear nonstationary systems of Persidskii type [16] is studied. The right-hand sides of the systems are represented in the form of linear combinations of sector nonlinearities with time-varying coefficients. It is assumed that the coefficients possess mean values. By means of the Lyapunov direct method, it is proved that if the investigated systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables, then the asymptotic stability of the zero solutions of the corresponding averaged systems implies the local uniform asymptotic stability of the zero solutions for original nonstationary systems. We treat both cases of delay free and time delay systems. Furthermore, it is shown that the proposed approaches can be used as well for the stability analysis of some classes of nonlinear systems with nontrivial linear approximation.

2 Statement of the Problem

Consider the ordinary differential equations system

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{1}$$

Here the functions $f_j(x_j)$ are continuous for $|x_j| < H$, $0 < H \leq +\infty$, and belong to a sector-like constrained set defined as follows: $x_j f_j(x_j) > 0$ for $x_j \neq 0$; the coefficients $p_{ij}(t)$ are continuous and bounded for $t \geq 0$. Such systems are widely used in both automatic control [9, 16, 17] and neural networks [15, 16].

We assume that the functions $p_{ij}(t)$ possess mean values $\bar{p}_{ij}$, and the tendencies

$$\frac{1}{T}\int_t^{t+T} p_{ij}(s)ds \to \bar{p}_{ij} \quad \text{as} \quad T \to +\infty, \quad i,j = 1,\dots,n,$$

are uniform with respect to $t \geq 0$. Hence, the coefficients $p_{ij}(t)$ can be represented in the form $p_{ij}(t) = \bar{p}_{ij} + \tilde{p}_{ij}(t)$, with the mean values of the functions $\tilde{p}_{ij}(t)$ equal to zero, $i,j = 1,\dots,n$.

Thus,

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} f_j(x_j(t)), \qquad i = 1,\dots,n, \tag{2}$$

is the averaged system for (1).

It follows from the properties of functions $f_1(x_1),\dots,f_n(x_n)$ that systems (1) and (2) admit the zero solution. We will look for the conditions under which the asymptotic stability of the zero solution of the averaged system implies the same property for the zero solution of original system.

In what follows, we impose some additional restrictions on the right-hand sides in (1).

Assumption 2.1 The matrix $\overline{\mathbf{P}} = \{\bar{p}_{ij}\}_{i,j=1}^{n}$ is diagonally stable [16], i.e. there exist positive numbers $\lambda_1,\dots,\lambda_n$ such that the quadratic form

$$W(\mathbf{x}) = \mathbf{x}^T \left(\overline{\mathbf{P}}^T \mathbf{\Lambda} + \mathbf{\Lambda}\overline{\mathbf{P}}\right) \mathbf{x}$$

is negative definite. Here $\mathbf{x} = (x_1,\dots,x_n)^T$, $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\}$.

Remark 2.1 The problem of matrix diagonal stability is well investigated, see, for example, [16] and references therein.

Remark 2.2 If Assumption 2.1 is fulfilled, then the zero solution of (2) is asymptotically stable, and, for this system, a Lapunov function can be chosen in the form

$$V(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \int_0^{x_i} f_i(s)ds. \tag{3}$$

Remark 2.3 It is well known [28] that if system (1) is linear ($f_j(x_j) = x_j$, $j = 1,\dots,n$), it may be unstable, despite of the asymptotic stability of the corresponding averaged system.

In view of Remark 2.3, hereinafter we consider only the case when the following assumption is fulfilled.

Assumption 2.2 Functions $f_j(x_j)$ can be represented in the form

$$f_j(x_j) = \beta_j x_j^{\mu_j} + g_j(x_j), \quad j = 1,\dots,n,$$

where β_j are positive constants, $\mu_j > 1$ are rational numbers with odd numerators and denominators, and $g_j(x_j)/x_j^{\mu_j} \to 0$ as $x_j \to 0$.

Remark 2.4 Without loss of generality, we assume that $\beta_j = 1$, $j = 1, \ldots, n$, and $\mu_1 \le \ldots \le \mu_n$.

Thus, the investigated equations are essentially nonlinear, and the systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} \left(\bar{p}_{ij} + \tilde{p}_{ij}(t)\right) x_j^{\mu_j}(t), \qquad i = 1, \ldots, n, \tag{4}$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} x_j^{\mu_j}(t), \qquad i = 1, \ldots, n, \tag{5}$$

can be considered as systems of the first, in a broad sense, approximation for (1) and (2) respectively.

Let Assumption 2.1 be fulfilled. Then the zero solution of (5) is globally asymptotically stable, and, for this system, the Lyapunov function (3) takes the form

$$V(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}}{\mu_i + 1}.$$

First, we will show that the zero solution of (4) is locally asymptotically stable. Next, we will determine the stability conditions for a perturbed system, and, on the basis of these conditions, the asymptotic stability of the zero solution of (1) will be proved. Furthermore, along with (1), we will consider the corresponding time-delay system

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_{ij}(t) f_j(x_j(t-\tau)), \quad i = 1, \ldots, n, \quad \tau = \text{const} \ge 0. \tag{6}$$

By the usage of the Lyapunov direct method and the Razumikhin approach [25], for (6), delay independent stability conditions will be found.

3 Sufficient Conditions of Asymptotic Stability

In [3], it was shown that if Assumption 2.1 is fulfilled, and the integrals

$$\int_0^t \tilde{p}_{ij}(s)ds, \quad i, j = 1, \ldots, n, \tag{7}$$

are bounded for $t \in [0, +\infty)$, then the zero solution of (4) is asymptotically stable.

In the present paper, we consider the case when

$$\frac{1}{T} \int_t^{t+T} \tilde{p}_{ij}(s)ds \to 0 \quad \text{as} \quad T \to +\infty, \quad i, j = 1, \ldots, n,$$

uniformly with respect to $t \ge 0$. It is well known [11], that, in this case, integrals (7) may be unbounded.

Theorem 3.1 *Let Assumption 2.1 be fulfilled. Then the zero solution of (4) is uniformly asymptotically stable.*

Proof. By means of the approaches proposed in [1, 2, 27], construct a Lyapunov function for (4) in the form

$$\widetilde{V}(t,\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \frac{x_i^{\mu_i+1}}{\mu_i+1} - \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j}. \tag{8}$$

Here positive numbers $\lambda_1,\ldots,\lambda_n$ are chosen in accordance with Assumption 2.1,

$$L_{ij}(t,\varepsilon) = \int_0^t \exp(\varepsilon(s-t))\, \tilde{p}_{ij}(s)ds, \quad i,j=1,\ldots,n,$$

and ε is a positive parameter.

Differentiating $\widetilde{V}(t,\mathbf{x})$ with respect to system (4), we obtain

$$\dot{\widetilde{V}}\big|_{(4)} = \sum_{i,j=1}^{n} \lambda_i \bar{p}_{ij} x_i^{\mu_i} x_j^{\mu_j} + \varepsilon \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j}$$
$$- \sum_{i,j=1}^{n} \lambda_i \mu_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i-1} x_j^{\mu_j} \sum_{k=1}^{n} p_{ik}(t) x_k^{\mu_k}$$
$$- \sum_{i,j=1}^{n} \lambda_i \mu_j L_{ij}(t,\varepsilon)\, x_i^{\mu_i} x_j^{\mu_j-1} \sum_{k=1}^{n} p_{jk}(t) x_k^{\mu_k}.$$

Hence, the estimates

$$a_1 \sum_{i=1}^{n} x_i^{\mu_i+1} - \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i} \le \widetilde{V}(t,\mathbf{x}) \le a_2 \sum_{i=1}^{n} x_i^{\mu_i+1} + \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i},$$

$$\dot{\widetilde{V}}\big|_{(4)} \le -a_4 \sum_{i=1}^{n} x_i^{2\mu_i} + a_5 \psi(t,\varepsilon) \sum_{i=1}^{n} x_i^{2\mu_i} + \frac{a_6}{\varepsilon} \sum_{i,j=1}^{n} x_i^{2\mu_i} x_j^{\mu_j-1}$$

are valid for $t \ge 0$, $\mathbf{x} \in \mathbb{R}^n$. Here $a_1,\ldots,a_6$ are positive constants independent of chosen value of ε, and

$$\psi(t,\varepsilon) = \max_{i,j=1,\ldots,n} \varepsilon \left| L_{ij}(t,\varepsilon) \right|. \tag{9}$$

With the aid of the results of [10], it is easy to verify that $\psi(t,\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $t \ge 0$. Therefore, we can find and fix $\varepsilon > 0$ such that $a_5\psi(t,\varepsilon) < a_4/3$.

Then, for chosen ε and sufficiently small values of $\delta > 0$, the inequalities

$$\frac{a_1}{2} \sum_{i=1}^{n} x_i^{\mu_i+1} \le \widetilde{V}(t,\mathbf{x}) \le 2a_2 \sum_{i=1}^{n} x_i^{\mu_i+1}, \qquad \dot{\widetilde{V}}\big|_{(4)} \le -\frac{a_4}{2} \sum_{i=1}^{n} x_i^{2\mu_i}$$

hold for $t \ge 0$ and $\|\mathbf{x}\| < \delta$ (hereinafter $\|\cdot\|$ denotes the Euclidean norm of a vector). Thus, the Lyapunov function (8) satisfies all the assumptions of the Lyapunov asymptotic stability theorem [9, 26]. This completes the proof. □

Consider now, along with (4), the perturbed system

$$\dot{x}_i(t) = \sum_{j=1}^{n} (\bar{p}_{ij} + \tilde{p}_{ij}(t)) x_j^{\mu_j}(t) + q_i(t,\mathbf{x}(t)), \qquad i=1,\ldots,n. \tag{10}$$

Here functions $q_1(t,\mathbf{x}),\ldots,q_n(t,\mathbf{x})$ are defined and continuous in the region $t \geq 0$, $\|\mathbf{x}\| < H$, and, for any $\tilde{H} \in (0,H)$, the estimates

$$|q_i(t,\mathbf{x})| \leq c(\tilde{H}) \sum_{j=1}^{n} |x_j|^{\mu_j}, \quad i = 1,\ldots,n,$$

are valid for $t \geq 0$, $\|\mathbf{x}\| < \tilde{H}$, with $c(\tilde{H}) \to 0$ as $\tilde{H} \to 0$. Thus, system (10) admits the solution $\mathbf{x}(t) \equiv \mathbf{0}$, as well.

Theorem 3.2 *Let Assumption 2.1 be fulfilled. Then the zero solution of (10) is uniformly asymptotically stable.*

Proof. Consider the derivative of the Lyapunov function (8) with respect to the perturbed equations. We obtain

$$\dot{\tilde{V}}\big|_{(10)} \leq -\bar{a}_1 \sum_{i=1}^{n} x_i^{2\mu_i} + \bar{a}_2 \left(\psi(t,\varepsilon) + \frac{c(\tilde{H})}{\varepsilon}\right) \sum_{i=1}^{n} x_i^{2\mu_i} + \frac{\bar{a}_3}{\varepsilon}(1 + c(\tilde{H})) \sum_{i,j=1}^{n} x_i^{2\mu_i} x_j^{\mu_j - 1}$$

for $t \geq 0$, $\|\mathbf{x}\| < \tilde{H}$. Here $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are positive constants independent of chosen values of ε and $\tilde{H}$, and the function $\psi(t,\varepsilon)$ is determined by the formula (9).

In a similar way as in the proof of Theorem 3.1, it is easy to show that if ε and $\tilde{H}$ are sufficiently small, then the estimate

$$\dot{\tilde{V}}\big|_{(10)} \leq -\frac{\bar{a}_1}{2} \sum_{i=1}^{n} x_i^{2\mu_i}$$

holds for $t \geq 0$ and $\|\mathbf{x}\| < \tilde{H}$. This completes the proof. □

Corollary 3.1 *Let Assumptions 2.1 and 2.2 be fulfilled. Then the zero solution of (1) is uniformly asymptotically stable.*

4 Delay-Independent Stability Conditions

In this section, we will show that the results of Section 3 can be extended to the case of time-delay systems.

Consider the system (6), where $\tau \geq 0$ is a constant delay. Let $PC([-\tau,0],\mathbb{R}^n)$ be the space of piece-wise continuous functions $\varphi(\theta) : [-\tau,0] \to \mathbb{R}^n$ with the uniform (supremum) norm $\|\varphi\|_\tau = \sup_{\theta\in[-\tau,0]} \|\varphi(\theta)\|$, and Ω_H be the set of functions $\varphi(\theta) \in PC([-\tau,0],\mathbb{R}^n)$ satisfying the inequality $\|\varphi\|_\tau < H$.

By $\mathbf{x}(t,t_0,\varphi)$ we denote a solution of system (6) with the initial conditions $t_0 \geq 0$, $\varphi(\theta) \in \Omega_H$, while $\mathbf{x}_t(t_0,\varphi)$ is the restriction of the solution to the segment $[t-\tau,t]$, i.e. $\mathbf{x}_t(t_0,\varphi) : \theta \to \mathbf{x}(t+\theta,t_0,\varphi)$, $\theta \in [-\tau,0]$. In some cases, when the initial conditions are not important, or well defined from the context, we write $\mathbf{x}(t)$ and $\mathbf{x}_t$, instead of $\mathbf{x}(t,t_0,\varphi)$ and $\mathbf{x}_t(t_0,\varphi)$, respectively. We will study the impact of delay on the stability of the zero solution of (6).

Consider the averaged system

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} f_j(x_j(t-\tau)), \quad i = 1,\ldots,n. \tag{11}$$

Under Assumption 2.1, the zero solution of the corresponding delay free system (2) is asymptotically stable. In [4], it was proved that if no additional restrictions are imposed on the right-hand sides of (11), then an arbitrary small delay may destroy the stability.

In many applications, it is important to have stability conditions under which a system remains stable for any nonnegative value of delay [14, 22]. Such conditions are known as delay-independent ones.

Let Assumption 2.2 be fulfilled. Then the systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} \left(\bar{p}_{ij} + \tilde{p}_{ij}(t)\right) x_j^{\mu_j}(t-\tau), \qquad i = 1, \dots, n, \tag{12}$$

$$\dot{x}_i(t) = \sum_{j=1}^{n} \bar{p}_{ij} x_j^{\mu_j}(t-\tau), \qquad i = 1, \dots, n, \tag{13}$$

are the systems of the first approximation for (6) and (11) respectively.

Delay-independent stability conditions for systems (12) and (13) have been studied in [4]. It was shown that, under Assumption 2.1, the zero solution of (12) is asymptotically stable for any $\tau \geq 0$. Furthermore, if, in addition to Assumption 2.1, the integrals (7) are bounded for $t \in [0, +\infty)$, then the zero solution of (13) is asymptotically stable for any $\tau \geq 0$ as well.

As it was mentioned in Section 3, in the present paper, we consider the case when integrals (7) may be unbounded.

Theorem 4.1 *Let Assumption 2.1 be fulfilled. Then the zero solution of (12) is uniformly asymptotically stable for any* $\tau \geq 0$.

Proof. Choose a Lyapunov function for (12) in the form (8) where positive coefficients $\lambda_1, \dots, \lambda_n$ are determined in accordance with Assumption 2.1.

Consider the derivative of the function with respect to system (12). We obtain

$$\begin{aligned} \dot{\widetilde{V}}\big|_{(12)} &= \sum_{i,j=1}^{n} \lambda_i \bar{p}_{ij} x_i^{\mu_i}(t) x_j^{\mu_j}(t) + \varepsilon \sum_{i,j=1}^{n} \lambda_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i}(t) x_j^{\mu_j}(t) \\ &- \sum_{i,j=1}^{n} \lambda_i \mu_i L_{ij}(t,\varepsilon)\, x_i^{\mu_i - 1}(t) x_j^{\mu_j}(t) \sum_{k=1}^{n} p_{ik}(t) x_k^{\mu_k}(t-\tau) \\ &- \sum_{i,j=1}^{n} \lambda_i \mu_j L_{ij}(t,\varepsilon)\, x_i^{\mu_i}(t) x_j^{\mu_j - 1}(t) \sum_{k=1}^{n} p_{jk}(t) x_k^{\mu_k}(t-\tau) \\ &+ \sum_{i,j=1}^{n} \lambda_i p_{ij}(t)\, x_i^{\mu_i}(t) \left(x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t) \right). \end{aligned}$$

Hence, if a solution $\mathbf{x}(t)$ of (12) is defined on an interval $[t_0, \hat{t}]$, $0 \leq t_0 < \hat{t}$, then the estimates

$$a_1 \sum_{i=1}^{n} x_i^{\mu_i + 1}(t) - \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i}(t) \leq \widetilde{V}(t, \mathbf{x}(t)) \leq a_2 \sum_{i=1}^{n} x_i^{\mu_i + 1}(t) + \frac{a_3}{\varepsilon} \sum_{i=1}^{n} x_i^{2\mu_i}(t),$$

$$\dot{\tilde{V}}\big|_{(12)} \le -a_4 \sum_{i=1}^{n} x_i^{2\mu_i}(t) + a_5\psi(t,\varepsilon)\sum_{i=1}^{n} x_i^{2\mu_i}(t)$$

$$+\frac{a_6}{\varepsilon}\sum_{i,j,k=1}^{n}\left|x_i^{\mu_i}(t)x_j^{\mu_j-1}(t)x_k^{\mu_k}(t-\tau)\right| + a_7\sum_{i,j=1}^{n}|x_i^{\mu_i}(t)|\left|x_j^{\mu_j}(t-\tau)-x_j^{\mu_j}(t)\right|$$

hold for $t \in [t_0, \hat{t}]$. Here $a_1, \ldots, a_7$ are positive constants independent of the value of ε, and the function $\psi(t,\varepsilon)$ is defined by the formula (9).

Choose and fix $\varepsilon > 0$ for which the inequality $a_5\psi(t,\varepsilon) < a_4/3$ is valid. Let us prove that, for such ε, the Lyapunov function (8) satisfies all the conditions of Theorem 4.2 in [14].

Assume that, for a solution $\mathbf{x}(t)$ of (12), the estimate $\|\mathbf{x}(\xi)\| < \delta$, and the Razumikhin condition $\widetilde{V}(\xi,\mathbf{x}(\xi)) \le 2\widetilde{V}(t,\mathbf{x}(t))$ are fulfilled for $\xi \in [t-(m+1)\tau, t]$. Here $\delta = \text{const} > 0$, and m is a positive integer such that

$$\frac{(m(\mu_1-1)+\mu_1)(\mu_n+1)}{(\mu_1+1)\mu_n} > 1.$$

If the value of δ is sufficiently small, then

$$x_i^{\mu_i+1}(\xi) < 8\frac{a_2}{a_1}\sum_{j=1}^{n} x_j^{\mu_j+1}(t), \qquad i = 1,\ldots,n, \tag{14}$$

for $\xi \in [t-(m+1)\tau, t]$.

With the aid of inequalities (14), it is easy to show that

$$|x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t)| = \tau\mu_j\, x_j^{\mu_j-1}(t-\eta_j\tau)\left|\sum_{l=1}^{n} p_{jl}x_l^{\mu_l}(t-\eta_j\tau-\tau)\right|$$

$$\le b_1\left(\sum_{l=1}^{n} x_l^{\mu_l+1}(t)\right)^{\frac{\mu_j-1}{\mu_j+1}}\left(\sum_{l=1}^{n}|x_l^{\mu_l}(t)| + \sum_{l=1}^{n}|x_l^{\mu_l}(t-\eta_j\tau-\tau)-x_l^{\mu_l}(t)|\right)$$

$$\le b_2\left(\sum_{l=1}^{n}|x_l^{\mu_l}(t)|\right)^{\frac{(\mu_1-1)(\mu_n+1)}{(\mu_1+1)\mu_n}}\left(\sum_{l=1}^{n}|x_l^{\mu_l}(t)| + \sum_{l=1}^{n}|x_l^{\mu_l}(t-\eta_j\tau-\tau)-x_l^{\mu_l}(t)|\right),$$

where $b_1 > 0$, $b_2 > 0$, $0 < \eta_j < 1$, $j = 1,\ldots,n$.

Further, for the functions $|x_l^{\mu_l}(t-\eta_j\tau-\tau) - x_l^{\mu_l}(t)|$, $l = 1,\ldots,n$, the similar estimates can be found.

Successively applying this procedure m times, we obtain

$$\left|x_j^{\mu_j}(t-\tau) - x_j^{\mu_j}(t)\right|$$

$$\le b_3\left(\sum_{s=1}^{n}|x_s^{\mu_s}(t)|\right)^{1+\frac{(\mu_1-1)(\mu_n+1)}{(\mu_1+1)\mu_n}} + b_4\left(\sum_{s=1}^{n}|x_s^{\mu_s}(t)|\right)^{\frac{(m(\mu_1-1)+\mu_1)(\mu_n+1)}{(\mu_1+1)\mu_n}},$$

where b_3 and b_4 are positive constants, $j = 1,\ldots,n$.

Thus, for sufficiently small values of δ, the inequality

$$\dot{\tilde{V}}(t,\mathbf{x}(t)) \le -\frac{a_4}{2}\sum_{i=1}^{n} x_i^{2\mu_i}(t)$$

holds. Hence [14], the zero solution of (12) is uniformly asymptotically stable. This completes the proof. □

Consider now the perturbed system

$$\dot{x}_i(t) = \sum_{j=1}^{n}(\bar{p}_{ij} + \tilde{p}_{ij}(t))x_j^{\mu_j}(t-\tau) + q_i(t,\mathbf{x}(t),\mathbf{x}(t-\tau)), \qquad i = 1,\dots,n. \tag{15}$$

Here functions $q_1(t,\mathbf{x},\mathbf{y}),\dots,q_n(t,\mathbf{x},\mathbf{y})$ are defined and continuous in the region $t \ge 0$, $\|\mathbf{x}\| < H$, $\|\mathbf{y}\| < H$, and, for any $\tilde{H} \in (0,H)$, the estimates

$$|q_i(t,\mathbf{x},\mathbf{y})| \le c(\tilde{H})\sum_{j=1}^{n}\left(|x_j|^{\mu_j} + |y_j|^{\mu_j}\right), \quad i = 1,\dots,n,$$

are valid for $t \ge 0$, $\|\mathbf{x}\| < \tilde{H}$, $\|\mathbf{y}\| < \tilde{H}$, with $c(\tilde{H}) \to 0$ as $\tilde{H} \to 0$.

Theorem 4.2 *Let Assumption 2.1 be fulfilled. Then the zero solution of (15) is uniformly asymptotically stable for any* $\tau \ge 0$.

The proof of the theorem is similar to that of Theorem 4.1.

Corollary 4.1 *Let Assumptions 2.1 and 2.2 be fulfilled. Then the zero solution of (6) is uniformly asymptotically stable for any* $\tau \ge 0$.

5 Stability Conditions for an Automatic Control System

In Sections 3 and 4, it was assumed that the considered systems are essentially nonlinear, i.e. the right-hand sides of the systems do not contain linear terms with respect to phase variables. In this section, we will show that the approaches proposed in the present paper can be used as well for the stability analysis of some classes of nonlinear time-varying systems with nontrivial linear approximations. Right-hand sides of such systems may include linear terms, but linear approximations are critical in the Lyapunov sense [9, 17].

Let the dynamic nonlinear feedback system [17, 26]

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,f(\sigma(t)), \\ \dot{\sigma}(t) &= \mathbf{c}^T\mathbf{x}(t) - f(\sigma(t)), \end{cases} \tag{16}$$

be given. Here $\mathbf{x}(t) \in \mathbb{R}^n$ and $\sigma(t) \in \mathbb{R}$, $\mathbf{A}$ is a constant Hurwitz matrix, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors, $f(\sigma)$ is a sector nonlinearity, which is continuous for $|\sigma| < H$, $0 < H \le +\infty$, and satisfies the condition $\sigma f(\sigma) > 0$ for $\sigma \ne 0$.

Assume that, for system (16), there exists a Lyapunov function of the form

$$V(\mathbf{x},\sigma) = \mathbf{x}^T\mathbf{D}\mathbf{x} + \int_0^{\sigma} f(s)\,ds,$$

where $\mathbf{D}$ is a constant symmetric positive definite matrix, such that the estimate

$$\dot{V}\Big|_{(16)} \leq -b\left(\|\mathbf{x}(t)\|^2 + f^2(\sigma(t))\right), \quad b = \text{const} > 0,$$

holds. The conditions for the existence of the Lyapunov function are well known, see, for instance, [17, 26]. The fulfilment of this assumption implies the asymptotic stability of the zero solution of (16).

Consider now the case when the control law includes a delay and a nonstationary perturbation. Let the system be of the form

$$\begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}\,f(\sigma(t-\tau)), \\ \dot{\sigma}(t) &= \mathbf{c}^T\mathbf{x}(t) - (1+\tilde{p}(t))f(\sigma(t-\tau)). \end{cases} \tag{17}$$

Here $\tau \geq 0$ is a constant delay, while the perturbation $\tilde{p}(t)$ is continuous and bounded for $t \in [0, +\infty)$ function, such that

$$\frac{1}{T}\int_t^{t+T} \tilde{p}(s)ds \to 0 \quad \text{as} \quad T \to +\infty$$

uniformly with respect to $t \geq 0$.

Furthermore, we assume that the nonlinearity $f(\sigma)$ can be represented as follows $f(\sigma) = \beta\sigma^\mu + g(\sigma)$, where $\mu > 1$ is a rational number with odd numerator and denominator, β is a positive constant, and $g(\sigma)/\sigma^\mu \to 0$ as $\sigma \to 0$.

It is worth mentioning that essentially nonlinear control laws were considered in [7, 12, 20]. In particular, in [20], controls of such type were used for solving the problem of angular stabilization of an airplane, whereas, in [12], they were applied for the developing of seismic mitigation devices.

Theorem 5.1 *The zero solution of (17) is uniformly asymptotically stable for any value of $\tau \geq 0$.*

Proof. Construct a Lyapunov function for (17) in the form

$$\widetilde{V}(t, \mathbf{x}, \sigma) = \mathbf{x}^T\mathbf{D}\mathbf{x} + \beta\frac{\sigma^{\mu+1}}{\mu+1} + \beta^2\sigma^{2\mu}\int_0^t \exp(\varepsilon(s-t))\,\tilde{p}(s)ds,$$

where ε is a positive parameter. With the aid of this function the subsequent proof is similar to that of Theorem 4.1. □

6 Conclusion

In this paper, for a special class of nonlinear nonstationary systems, new sufficient asymptotic stability conditions of the trivial solution are obtained via the averaging technique. It is proved that, for the considered essentially nonlinear systems, this technique can be applied without requirement of fast time-varying vector field – typical for averaging results.

It is easy to verify that the results obtained for time delay systems remain valid when the systems delays are continuous nonnegative and bounded functions of the time variable. Moreover, these results can be extended to systems with distributed delays as well.

An important direction of future research is application of the developed approaches for the stability analysis of nonlinear nonstationary complex (multiconnected) systems.

Acknowledgment

The reported study was partially supported by the St. Petersburg State University, project no. 9.38.674.2013, and by the Russian Foundation of Basic Researches, grants no. 13-01-00347-a and 13-01-00376-a.

References

[1] Aleksandrov, A.Yu. Asymptotic stability of solutions to systems of nonstationary differential equations with homogeneous right-hand sides. *Doklady Mathematics* **54**(1) (1996) 534–535.

[2] Aleksandrov, A.Yu. The stability of equilibrium of non-stationary systems. *Journal of Applied Mathematics and Mechanics* **60**(2) (1996) 199–203.

[3] Aleksandrov, A.Yu. Certain conditions of the stability of solutions of nonlinear nonautonomous systems. *Vestnik St. Petersburg University. Ser. 1* (8) (1998) 3–6. [Russian]

[4] Aleksandrov, A.Yu. and Zhabko, A.P. On the stability of the solutions of a class of nonlinear delay systems. *Automation and Remote Control* **67**(9) (2006) 1355–1365.

[5] Aleksandrov, A.Yu. and Zhabko, A.P. Asymptotic stability of solutions of a class of systems of nonlinear differential equations with delay. *Russian Mathematics* **56**(5) (2012) 1–8.

[6] Aleksandrov, A.Yu. and Zhabko, A.P. On the asymptotic stability of solutions of nonlinear systems with delay. *Siberian Mathematical Journal* **53**(3) (2012) 393–403.

[7] Arcak, M. and Teel, A. Input-to-state stability for a class of Lurie systems. *Automatica* **38** (2002) 1945–1949.

[8] Balan, R. An extension of Barbashin-Krasovskii-LaSalle theorem to a class of nonautonomous systems. *Nonlinear Dynamics and Systems Theory* **8**(3) (2008) 255–268.

[9] Barbashin, E.A. *Introduction to the Theory of Stability.* Wolters-Noordhoff Publishing, Groningen, 1970.

[10] Bogoliubov, N.N. and Mitropolsky Y.A. *Asymptotic Methods in the Theory of Non-Linear Oscillations.* Gordon and Breach, New York, 1961.

[11] Demidovich, B.P. *Lectures on Mathematical Theory of Stability.* Nauka, Moscow, 1967. [Russian]

[12] Gendelman, O.V. and Lamarque, C.H. Dynamics of linear oscillator coupled to strongly nonlinear attachment with multiple states of equilibrium. *Chaos, Solitons and Fractals* **24** (2005) 501–509.

[13] Guckenheimer, J. and Holmes, P. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[14] Hale, J.K. and Verduyn Lunel, S.M. *Introduction to Functional Differential Equations.* Springer-Verlag, New York, 1993.

[15] Hopfield, J.J., Tank, D.W. Computing with neural circuits: a model. *Science* **233** (1986) 625–633.

[16] Kazkurewicz, E. and Bhaya, A. *Matrix Diagonal Stability in Systems and Computation.* Birkhauser, Boston, 1999.

[17] Khalil, H.K. *Nonlinear Systems* Prentice-Hall, Upper Saddle River NJ, 2002.

[18] Leonov, G. Strange attractors and classical stability theory. *Nonlinear Dynamics and Systems Theory* **8**(1) (2008) 49–96.

[19] Martynyuk, A.A. and Slyn'ko, V.I. A new approach for construction of the matrix-valued Liapunov functionals. *Nonlinear Dynamics and Systems Theory* **8**(4) (2008) 389–406.

[20] Mel'nikov, G.I. *Dynamics of Nonlinear Mechanical and Electromechanical Systems.* Mashinostroenie, Leningrad, 1975. [Russian]

[21] Moreau, L., Aeyels, D., Peuteman, J. and Sepulchre, R. A duality principle for homogeneous vectorfields with applications. *Systems & Control Letters* **47** (2002) 37–46.

[22] Niculescu, S. *Delay Effects on Stability: A Robust Control Approach. Lecture Notes in Control and Information Science*. Springer, Berlin, Heidelberg, 2001.

[23] Peuteman, J. and Aeyels, D. Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying. *SIAM J. Control and Optimization* **37**(4) (1999) 997–1010.

[24] Peuteman, J. and Aeyels, D. Averaging techniques without requiring a fast time-varying differential equation. *Automatica* **47** (2011) 192–200.

[25] Razumikhin, B.S. On stability of systems with delay. *Prikladnaja Matematika i Mekhanika* **20**(4) (1956) 500–512. [Russian]

[26] Rouche, N., Habets, P. and Laloy, M. *Stability Theory by Liapunov's Direct Method.* Springer, New York etc., 1977.

[27] Tikhomirov, O.G. Stability of homogeneous nonstationary odinary differential equations systems. *Vestnik St. Petersburg University. Ser. 10* (3) (2007) 123–130. [Russian]

[28] Vinograd, R.E. On a criterion for instability in the sense of Lyapunov of solutions of the linear system of ordinary differential equations. *Dokl. Akad. Nauk SSSR* **84**(2) (1952) 201–204. [Russian]