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$\mathcal{F}$ Mixing and $\mathcal{F}$ Scattering

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Abstract: In this paper, we study the complexity of group actions from the viewpoint of Furstenberg families, we characterize the $\mathcal{F}$ uniform rigidity and $\mathcal{F}$ equicontinuity using topological sequence complexity function, and we establish the connection between $\mathcal{F}$ mixing and $\mathcal{F}$ scattering.

Keywords: *$\mathcal{F}$ uniform rigidity; $\mathcal{F}$ mixing; $\mathcal{F}$ scattering.*

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1 Introduction

Blanchard, Host and Maass used open covers to define a complexity function for a continuous map on a compact metric space, and discussed the equicontinuity and scattering properties. Subsequently, Yang discussed the relations of $\mathcal{F}$ mixing and $\mathcal{F}$ scattering of a continuous map(see [1–3]). We study the complexity of group actions from the viewpoint of Furstenberg families. The results are as follows: we characterize the $\mathcal{F}$ uniform rigidity and $\mathcal{F}$ equicontinuity using topological sequence complexity function, and we establish the connection between $\mathcal{F}$ mixing and $\mathcal{F}$ scattering.

Suppose (X,T) is a semi-dynamical system, where X is a compact metric space, T is a topological semigroup and contains the unit element.

• Suppose X is a topological space, T is a topological semigroup, if a map

$$\pi : X \times T \to X$$

satisfies

$$\pi(\pi(x,t),s) = \pi(x,ts), \forall x \in X,\ \forall t,s \in T,$$

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then we call π a right action of T on X. If the right action π is continuous, then (X, T, π) is called a semi-dynamical system (abbreviation: (X, T)). Often we write $\pi(x, t) = xt$.

• We denote by $\mathcal{P}$ the collection of all subsets of T. Subset $\mathcal{F}$ of $\mathcal{P}$ is called a family, if $\mathcal{P}$ has hereditary upward, i.e., if $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$, then $F_2 \in \mathcal{F}$. The family $\mathcal{F}$ is a proper family when it is a proper subset of $\mathcal{P}$, neither empty nor all of $\mathcal{P}$.

For a family $\mathcal{F}$, we define the dual family:

$$\begin{aligned} k\mathcal{F} &= \{F | F \cap F_1 \neq \emptyset, \text{ for all } F_1 \in \mathcal{F}\} \\ &= \{F | T \backslash F \notin \mathcal{F}\}. \end{aligned}$$

• For $t \in T$ define $g^t : T \to T$ by $g^t(s) = ts, \forall s \in T$, g^t is called a translation map. If for any $t \in T$ and any $F \in \mathcal{F}$, we have $(g^t)^{-1}(F) \in \mathcal{F}$, then a family $\mathcal{F}$ is called translation invariant. Write $\tau\mathcal{F} = \{F | \, (g^{t_1})^{-1}(F) \cap \cdots \cap (g^{t_k})^{-1}(F) \in \mathcal{F},$ for any finite subset $\{t_1, t_2, \cdots, t_k\}$ $of T$ $\}$. Let $\mathcal{B}$ be a family of infinite subset of T, if $k\mathcal{B} \cdot \mathcal{F} = \{A \cap F | A \in k\mathcal{B}, F \in \mathcal{F}\} \subset \mathcal{F}$, then a proper family $\mathcal{F}$ is called full.

• Assume that $\mathcal{F}$ is a family, $x \in X$. Write $\omega_{\mathcal{F}}(x) = \bigcap_{F \in k\mathcal{F}} \overline{xF}$, then $\omega_{\mathcal{F}}(x)$ is called a $\mathcal{F}$ limit set of x; $y \in \omega_{\mathcal{F}}(x)$, i.e., for any neighborhood U of $y, D(x, U) = \{t | xt \in U\} \in \mathcal{F}$, then y is called a $\mathcal{F}$ limit point of x. Recall that the continuous action π on X induces a continuous action π_* of T on $C^u(X, X)$ by $(\pi_t)_*(h) = \pi_t \circ h$. We call (X, T) $\mathcal{F}$ uniformly rigid, if $id \in \omega_{\mathcal{F}}(id)$,i.e., for any $\varepsilon > 0, \{t | d(\pi_t, id) < \varepsilon\} \in \mathcal{F}$ (where $d(\pi_t, id) = sup\{d(\pi_t(x), x) | x \in X\}$).

• Let $C = \{U_1, \cdots, U_k\}$ be an open cover of X. If S is a infinite subset of T, denote the set of all finite subsets of S by $F(S)$. For $A \in F(S)$, denote $C_0^A = \bigvee_{t \in A} (\pi_t)^{-1} C$. Let $r_S(T, C, A)$ denote the number of sets in a finite subcover of C_0^A with smallest cardinality. We get a map $r_S(T, C, \cdot) : F(S) \to Z^+, \ A \mapsto r_S(T, C, A)$. $r_S(T, C, \cdot)$ is said to be the topological complexity function of the cover C along S. Put $E = \{1, \cdots, k\}$. One defines a map $\omega : T \to E, t \mapsto \omega(t)$. If $x \in \bigcap_{t \in S} \pi_t^{-1} U_{\omega(t)}$, then ω is called a C_S-name of x. Denote $J^*(\omega) = \bigcap_{t \in T} \pi_t^{-1} U_{\omega(t)}$, $J_S^*(\omega) = \bigcap_{t \in S} \pi_t^{-1} U_{\omega(t)}$. If $\bigcup_{i \in I} J_S^*(\omega_i) = X$, then we say that the set of C_S-names ω_i covers X. Let $M(T, E)$ be the set of maps from T to E and $M(S, E)$ be the set of maps from S to E.

• For any open set U, V of X, if $D(U, V) = \{t \in T | U \cap \pi_t^{-1} V \neq \emptyset\} \in \mathcal{F}$, then (X, T) is called $\mathcal{F}$ transitive. If $(X \times X, T)$ is $\mathcal{F}$ transitive, then (X, T) is called $\mathcal{F}$ mixing; If for any $S \in \mathcal{F}$, and any finite cover C of X by non-dense open sets, we have $r_S(T, C, \cdot)$ is unbounded, then (X, T) is called $\mathcal{F}$ scattering.

2 $\mathcal{F}$ Uniformly Rigid, $\mathcal{F}$ Mixing and $\mathcal{F}$ Scattering

Lemma 2.1 *Suppose T is countable, a finite cover $C = (U_1, \cdots, U_k)$ has complexity bounded by m if and only if there exist $\omega_1, \cdots, \omega_m \in M(T, E)$ such that $\bigcup_{i=1}^m J^*(\omega_i) = X$.*

Proof. Since T is countable, suppose $T = \{t_1, t_2, \cdots, t_n, \cdots\}$. Take $A_n = \{t_1, \cdots, t_n\}$, then $r_T(T, C, A_n) \leq m$.

Denote by $H(n)$ the set of m-tuples $(\upsilon_1, \cdots, \upsilon_m)$ of elements of $M(T, E)$ such that $(J_{A_n}^*(\upsilon_1), \cdots, J_{A_n}^*(\upsilon_m))$ covers X, the set $H(n)$ is non-empty and a closed subset of $M(T, E)^m$. If $(J_{A_n}^*(\upsilon_1), \cdots, J_{A_n}^*(\upsilon_m))$ covers X, then $(J_{A_{n-1}}^*(\upsilon_1), \cdots, J_{A_{n-1}}^*(\upsilon_m))$ covers X too, hence $H(n) \subseteq H(n-1)$, the intersection $H = \cap_{n=0}^\infty H(n)$ is non-empty, so there is $\omega = (\omega_1, \cdots, \omega_m) \in H$. Obviously $\cup_{i=1}^m J^*(\omega_i) = \lim_{n \to \infty} \cup_{i=1}^m J_{A_n}^*(\omega_i) = X$.

Theorem 2.1 *Suppose T is a topological group satisfying the second axiom of countability. Then (X,T) is $\mathcal{F}$ uniformly rigid if and only if there is a set $S\in\mathcal{F}$ containing a unit element, for any finite cover C of X, $r_S(T,C,\cdot)$ is bounded and C_S-names ω_i covering X are k instant.*

Proof. $\Rightarrow$. Since (X,T) is $\mathcal{F}$ uniformly rigid, $id\in\omega_{\mathcal{F}}(id)$. Let ε be a Lebesgue number of C, then $S=\{t\in T|sup_{x\in X}d(\pi_t(x),x)<\frac{\varepsilon}{2}\}\in\mathcal{F}$. Let $x_1,\cdots,x_m\in X$ be such that the open balls $\{B(x_i,\frac{\varepsilon}{2})|i=1,2,\cdots,m\}$ cover X. For any $t\in S$, we have $B(x_i,\frac{\varepsilon}{2})t\subset B(x_i,\varepsilon)$, and for any $1\le i\le m$, there is $U_{l(i)}\in C$ such that $B(x_i,\varepsilon)\subset U_{l(i)}$. Then for any finite set A of S, we have $B(x_i,\frac{\varepsilon}{2})\subset\bigcap_{t\in A}\pi_t^{-1}U_{l(i)}$, suppose the number of $U_{l(i)}$ is k. Since $\{\bigcap_{t\in A}\pi_t^{-1}(U_{l(i)})|i=1,\cdots,m\}$ is a finite cover of $\bigvee_{t\in A}\pi_t^{-1}(C)$, then $r_S(T,C,A)\le k$. By Lemma 2.1, for a countable dense set D of S, we have $\bigcup_{i=1}^{k}\bigcap_{t\in D}\pi_t^{-1}(U_{l(i)})=X$. By the denseness of D, $\bigcup_{i=1}^{k}\bigcap_{t\in S}\pi_t^{-1}(U_{l(i)})=X$.

$\Leftarrow$. If (X,T) is not $\mathcal{F}$ uniformly rigid, then there is $\varepsilon>0$, such that $\{t|d(\pi_t,id)<\varepsilon\}\notin\mathcal{F}$, then $S'=\{t|d(\pi_t,id)\ge\varepsilon\}\in k\mathcal{F}$. Let $C=\{U_1,\cdots,U_m\}$ be a finite cover by open balls with radius $\frac{\varepsilon}{4}$. If there is $S\in\mathcal{F}$, for any finite set A of S, we have $r_S(T,C,A)\le k$ and C_S-names ω_i covering X are instant. Then by Lemma 2.1, there exists a closed cover $\{X_1,\cdots,X_k\}$ of X, where $X_i=\bigcap_{t\in S}\pi_t^{-1}(\overline{U_{i'}})$. Because of $S\cap S'\neq\emptyset$, take $t\in S\cap S'$, then $d(\pi_t,id)\ge\varepsilon$, that is there is $x_t\in X$ such that $d(x_tt,x_t)\ge\varepsilon$. Let $x_t\in X_i$, then $x_t\in\overline{U_{i'}}$ and for any $s\in S$ we have $x_ts\in\overline{U_{i'}}$ that is $d(x_ts,x_t)\le\frac{\varepsilon}{2}$, which contradicts the assumption $d(x_tt,x_t)\ge\varepsilon$.

Theorem 2.2 *(X,T) is $\mathcal{F}$ equicontinuous if and only if there is $F\in\mathcal{F}$, and for any finite open cover C, $r_F(T,C,\cdot)$ is bounded.*

Proof. The proof is similar to the proof of Proposition 2.2 in [4].

Remark 2.1 In the case $T=Z_+$, $\mathcal{F}$ is the family of infinite subsets. If X is represented as the unit circle in C, then $\tilde{\theta}^1$ is given by $\tilde{\theta}^1(Z):=\alpha z(z\in C,|z|=1)$ with $\alpha:=exp(2\pi i\theta)$, let θ be irrational, then $(X,Z_+,\tilde{\theta}^1)$ is $\mathcal{F}$ equicontinuous.

In the following we discuss the existence of $\mathcal{F}$ equicontinuous point, and the connection between $\mathcal{F}$ mixing and $\mathcal{F}$ scattering.

Lemma 2.2 *Assume $\mathcal{F}$ is a translation invariant proper family, (X,T) is not $k\mathcal{F}$ mixing if and only if there is a non-empty open set U,V of X and $S\in\mathcal{F}$, such that for any $t\in S$ either $\pi_t^{-1}U\cap U=\emptyset$ or $\pi_t^{-1}V\cap U=\emptyset$.*

Proof. The proof is similar to the proof of Lemma 3.1 of [2].

Theorem 2.3 *Assume that $\mathcal{F}$ is a translation invariant proper family, if there is $F\in\mathcal{F}$, such that there is a F equicontinuous point x, then (X,T) is not $k\mathcal{F}$ mixing.*

Proof. Take $y\in X$ and $y\neq x$, let $\varepsilon<d(y,x)$. Since x is a F equicontinuous point, there is δ, $0<\delta<\frac{\varepsilon}{4}$, if $d(x,z)<\delta$, we have $d(xt,zt)<\frac{\varepsilon}{4}$ $(\forall t\in F)$. Let $U=B(y,\delta),V=B(x,\delta)$, if there is $t\in F$ such that $\pi_t^{-1}U\cap V\neq\emptyset$, then $\pi_tV\cap U\neq\emptyset$, thus $\pi_tV\cap V=\emptyset$, that is $\pi_t^{-1}V\cap V=\emptyset$. By Lemma 2.2, (X,T) is not $k\mathcal{F}$ mixing.

Lemma 2.3 *If the family $\mathcal{F}$ is full, then (X,T) is $\mathcal{F}$ mixing if and only if (X,T) is $\tau\mathcal{F}$ transitive.*

Proof. The proof can be found in [4].

Theorem 2.4 *Assume that T is commutative, $\mathcal{F}$ is full, and (X,T) is $\mathcal{F}$ mixing, then (X,T) is $k\tau\mathcal{F}$ scattering.*

Proof. For any non-trivial closed cover $\alpha = (W_1, \cdots, W_n)$ of X. Let U_1, U_2, V_1, V_2 be non-empty open sets of X, since (X,T) is $\mathcal{F}$ mixing,

$$F = D(U_1, U_2) \cap D(V_1, V_2) \in \mathcal{F}.$$

Take $t \in F$, let $U = U_1 \cap \pi_t^{-1} U_2, V = V_1 \cap \pi_t^{-1} V_2$. By Lemma 2.3, (X,T) is $\tau\mathcal{F}$ transitive, then $D(U,V) \in \tau\mathcal{F}$. Because of $D(U,V) \subset D(U_1,U_2) \cap D(V_1,V_2)$, and $\tau\mathcal{F}$ is a family, then $D(U_1,U_2) \cap D(V_1,V_2) \in \tau\mathcal{F}$.

Now we take U, V such that U, V do not simultaneously belong to any element of α.

Let $S_1 = D(U,U) \cap D(U,V) \in \tau\mathcal{F}$, for any $S \in k\tau\mathcal{F}$ there are $t_1 \in S_1 \cap S$ and $x_1, x_1' \in U$ such that

$$x_1 t_1 \in U,\ x_1' t_1 \in V.$$

So one takes $A_1 = \{t_1\}$, then $r_S(T, \alpha, A_1) \geq 2$. By the continuity of π, there exists a neighbourhood $U_1 \subset U$ of x_1' such that $U_1 t_1 \subset V$. Let $S_2 = D(U_1, U) \cap D(U_1, V) \in \tau\mathcal{F}$, then there are $t_2 \in S_2 \cap S$ and $x_2, x_2' \in U_1$ such that

$$x_2 t_1 \in V,\ x_2' t_1 \in V,\ x_2 t_2 \in V,\ x_2' t_2 \in U.$$

Obviously $t_1 \neq t_2$. so we take $A_2 = \{t_1, t_2\}$ then $r_S(T, \alpha, A_2) \geq 3$. By the continuity of π, there exists a neighbourhood $U_2 \subset U_1$ of x_2' such that $U_2 t_1 \subset U_1$. Let $S_3 = D(U_2, U) \cap D(U_2, V) \in \tau\mathcal{F}$, then there are $t_3 \in S_3 \cap S$ and $x_3, x_3' \in U_2$ such that

$$x_3 t_1 \in V,\ x_3' t_1 \in V,\ x_3 t_2 \in V,\ x_3' t_2 \in U\ x_3 t_3 \in U, x_3' t_3 \in V.$$

so one takes $A_3 = \{t_1, t_2\}$ then $r_S(T, \alpha, A_3) \geq 4$.

Using similar arguments repeatedly, we can get an infinite sequence

$$\{x_1, x_2, \cdots, x_n, \cdots\} \text{ and } \{t_1, \cdots, t_n, \cdots\} \text{ satisfy}$$

$$x_n \in U,\ i = 1, 2, \cdots,$$

$$x_1 t_1 \in U,\ x_i t_1 \in V,\ i = 2, 3, \cdots,$$

$$x_2 t_2 \in V,\ x_i t_2 \in U,\ i = 3, 4, \cdots,$$

$$x_3 t_3 \in U, x_i t_3 \in V,\ i = 4, 5, \cdots,$$

$$\cdots$$

For any $N \geq 1$, take $A_N = \{t_1, t_2, \cdots, t_N\}$ then $r_S(T, \alpha, A_N) \geq N + 1$.

Example 2.1 In the case $T = Z_+$, $\mathcal{F}$ is the family of infinite subsets. Let S be a finite set with at least two elements, say $S = \{0, \cdots, s-1\}$ with $s \in N, s \geq 2$. Consider S as a finite discrete topological space and put $\Omega := S^{Z_+}$. Endowed with the product topology. Define a mapping $\sigma : \Omega \to \Omega$, $(x_0, x_1, x_2, \cdots) \mapsto (x_1, x_2, \cdots)$. Clearly (Ω, Z_+, σ) is $\mathcal{F}$ mixing, then (Ω, Z_+, σ) is $k\tau\mathcal{F}$ scattering.

3 Concluding Remarks

In this paper, we study the complexity of group actions. We characterize the $\mathcal{F}$ uniform rigidity and $\mathcal{F}$ equicontinuity using topological sequence complexity function, and we show that $\mathcal{F}$ mixing implies $k\tau\mathcal{F}$ scattering.

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