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Possibilistic Modeling of Dynamic Uncertain Processes

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Abstract: In the paper a new class of uncertain differential equations based on the possibility theory is introduced. It is argued that this class is well-suited for modeling uncertain dynamic processes when the uncertainty has a non-probabilistic nature, or when the available statistical information is not sufficient for constructing a reliable stochastic model. The problems of existence and uniqueness of solutions of the proposed equations are studied and a numerical method for their solution is provided.

Keywords: *possibility theory; dynamical system; possibilistic walk process; Cauchy problem.*

Mathematics Subject Classification (2010): 34A07, 34A12.

1 Introduction

The methods of (quantitative) possibility theory [7, 10, 11, 20] allow one to estimate the level of possibility of some event with respect to possibilities of other events on the basis of subjective opinions of experts. These methods are useful for reasoning about uncertain processes and phenomena in cases when the lack of statistical information does not allow one to apply probabilistic methods, or when uncertainty has a non-probabilistic nature. The applications such as prognostication of social-economic phenomena, medical diagnostics, modeling of human-machine systems, etc. often require differential equations with uncertainty in the structure and/or parameters. However, in these applications the available statistical information is often rather limited or unreliable (because of absence of repetitions of the studied phenomena under the same conditions). Therefore, it is reasonable to apply non-probabilistic uncertainty theories (e.g. possibility theory) in such cases [4, 20].

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However, to the best of our knowledge, in the context of possibility theory a theory of uncertain differential equations has not been developed in the literature. In contrast, in the context of L. Zadeh's fuzzy set theory, fuzzy differential equations were studied extensively [2, 3, 8, 13–16, 19]. Such studies often consider either ordinary differential equations with fuzzy parameters [15], or equations of the evolution of a membership function [2, 12, 14, 15]. Although these approaches sometimes provide an alternative to stochastic modeling, they have some drawbacks. Differential equations with fuzzy parameters do not allow one to describe uncertain dynamic changes in the law of evolution (right-hand side of equation), because fuzzy parameters do not depend on time. The equations of evolution of membership function are not direct generalizations of ordinary differential equations. In most applications differential equations describe an evolution of the state of a system, but it is not obvious how to convert a state equation into an equation describing evolution of a membership function.

In this paper we propose a different approach to modeling of uncertain dynamics, which is based on possibility theory. We argue that it addresses the disadvantages of fuzzy differential equations described above.

Our class of possibilistic differential equations is based on the notion of a possibilistic walk process. Such equations can be considered as possibilistic analogs of stochastic Ito equations which have a wide range of applications in stochastic modeling. We will study the problems of existence and uniqueness of solutions of these equations and provide a numerical method for their solution.

2 Preliminaries

We will use the following framework of (quantitative) possibility theory [4, 7]. Let X be a non-empty set of elementary events and $(X, \mathbf{A})$, $\mathbf{A} \subseteq 2^X$ be a measurable space. The elements of $\mathbf{A}$ are called (compound) events.

Definition 2.1 A possibility measure is a function $P : \mathbf{A} \to [0, 1]$ such that

$$P(\bigcup_{i\in I} A_i) = \sup_{i\in I} P(A_i)$$

for any collection $(A_i)_{i\in I}$ of elements of $\mathbf{A}$ such that $\bigcup_{i\in I} A_i \in \mathbf{A}$.

Definition 2.2 A necessity measure is a function $N : \mathbf{A} \to [0, 1]$ such that

$$N(\bigcap_{i\in I} A_i) = \inf_{i\in I} N(A_i) \tag{1}$$

for any collection $(A_i)_{i\in I}$ of elements of $\mathbf{A}$ such that $\bigcap_{i\in I} A_i \in \mathbf{A}$.

Definition 2.3 A possibility space is a tuple $(X, \mathbf{A}, P, N)$, where P and N are respectively a possibility and necessity measure on the measurable space $(X, \mathbf{A})$.

Definition 2.4 A possibility space $(X, \mathbf{A}, P, N)$ is called regular, if $P(X) = 1$, $N(X) = 1$, and $N(A) = 1 - P(\neg A)$ for all $A \subseteq X$ (where $\neg A$ denotes the complement of a set $A \subseteq X$).

Definition 2.5 A possibility space $(X, \mathbf{A}, P, N)$ is called complete, if $\mathbf{A} = 2^X$ (the power set of X).

The assumptions of the regular possibility space are rather standard and are used in many works on possibility theory [11, 20]. It was shown in the work [5] that a regular possibility space $(X, \mathbf{A}, P, N)$ can be embedded in some complete regular possibility space $(X, 2^X, P', N')$, where P' and N' are extensions of P and N. For this reason, in this article we will consider only complete regular possibility spaces.

Let us fix a complete regular possibility space $(X, 2^X, P, N)$ and denote

$$X_\alpha = \{x \in X \mid P(\{x\}) > \alpha\}$$

for each $\alpha \in [0, 1]$. In particular, X_0 is the set of elementary events which have non-zero possibility.

Let $\mathbb{R}_+ = [0, +\infty)$ and T be a finite or infinite interval in $\mathbb{R}_+$. Under our assumption of completeness of the possibility space we will use the following terminology:

- A *possibilistic variable* is a (total) function $\xi : X \to Y$; if $Y = \mathbb{R}$, then ξ is called a scalar possibilistic variable; if $Y = \mathbb{R}^d$ (where d is a natural number), then ξ is called a vector possibilistic variable.

- The *distribution* of a possibilistic variable $\xi : X \to Y$ is a mapping $\mu_\xi : Y \to [0, 1]$ such that $\mu_\xi(y) = P\{x \in X \mid \xi(x) = y\}$.

- Possibilistic variables $\xi_k : X \to Y_k$, $k = 1, 2, ..., m$ are called *non-interactive* (independent), if the distribution $\mu_{\xi_1,\xi_2,..,\xi_m}$ of the vector possibilistic variable $(\xi_1, \xi_2, ..., \xi_m)$ satisfies the condition

$$\mu_{\xi_1,\xi_2,..,\xi_m}(u_1, u_2, ..., u_m) = \min\{\mu_{\xi_1}(u_1), \mu_{\xi_2}(u_2), ..., \mu_{\xi_m}(u_m)\}$$

  for all $u_1 \in Y_1, u_2 \in Y_2, ..., u_m \in Y_m$.

- A *possibilistic process* is a (total) function $p : T \times X \to Y$; if $Y = \mathbb{R}$, then p is called a scalar process; if $Y = \mathbb{R}^d$, then p is called a vector process.

- A *trajectory* of a possibilistic process $p : T \times X \to Y$ is a mapping $t \mapsto p(t, x)$ for a fixed $x \in X$.

- the *distribution of a process* $p : T \times X \to Y$ is a function $F_p : 2^{T \to Y} \to [0, 1]$, where

$$F_p(q) = P(\{x \in X \mid \forall t \in T\, p(t, x) = q(t)\})$$

  for each function $q : T \to Y$, i.e. $F_p(q)$ is a possibility of the event "q is a trajectory of p".

- An *α-trajectory* of p (where $\alpha \in [0, 1)$) is a function $q : T \to Y$ such that $F_p(q) > \alpha$, i.e. q is a trajectory of p with a possibility level greater than α.

We will abbreviate $P(\{x \in X \mid pred(\xi(x))\})$ as $P\{pred(\xi)\}$, where *pred* is some predicate. For example, $P\{\xi = y\}$ will denote $P(\{x \in X \mid \xi(x) = y\})$. Also, we will usually omit the second argument (elementary event) of a possibilistic process. For example, $P\{p(t) = 1\}$ will denote $P(\{x \in X \mid p(t, x) = 1\})$.

We will denote by $||.||$ the Euclidean norm on $\mathbb{R}^d$.

Definition 2.6 [4]. A possibilistic variable $\xi : X \to \mathbb{R}^d$ is called normal, if its distribution has a form

$$\mu_\xi(y) = \varphi\left(\left\|\Xi^{-1/2}(y - y_0)\right\|^2\right),$$

where $\varphi : \mathbb{R}_+ \to [0,1]$ is a monotonically decreasing function such that $\lim_{u\to+\infty} \varphi(u) = 0$ and $\varphi(0) = 1$, y_0 is a constant vector (mean value), Ξ is a positive-definite matrix (covariance-like matrix).

Definition 2.7 [4,20]. A possibilistic walk process $w : \mathbb{R}_+ \times X \to \mathbb{R}^d$ is a possibilistic process such that:

1. w has non-interactive increments, i.e. for any time moments $0 \le t_1 < t_2 < ... < t_{n+1}$, the possibilistic variables $w(t_{i+1}) - w(t_i)$, $i = 1, 2, ..., n$ are non-interactive.

2. For each $t_0 \ge 0$, $t > t_0$, $y, y_0 \in \mathbb{R}^d$ the transition possibility has a form

$$P\{w(t) = y, w(t_0) = y_0\} = \varphi\left(\frac{\left\|\Xi^{-1/2}(y - y_0)\right\|^2}{t - t_0}\right),$$

   where Ξ (a covariance-like matrix of w) is a positive-definite matrix, and $\varphi : \mathbb{R}_+ \to [0,1]$ (a distribution function of w) is a monotonically decreasing function such that $\lim_{u\to+\infty} \varphi(u) = 0$ and $\varphi(0) = 1$.

3. $w(0, x) = 0$.

Possibilistic walk processes can be considered as analogs of stochastic Wiener processes. The existence of a possibilistic walk process was established in [6], where it was proved that for any φ such that $\lim_{u\to+\infty} \varphi(u) = 0$, $\varphi(0) = 1$ and for any positive-definite matrix Ξ there exists a possibility space and a possibilistic walk process w such that φ is a distribution function of w and Ξ is a covariance-like matrix of w.

3 Main Result

Let w be a scalar possibilistic walk process with $\Xi = 1$ and a distribution function $\varphi : \mathbb{R}_+ \to [0,1]$. Let D be a domain in $\mathbb{R}^d$ (where $d \ge 1$), and $a : \mathbb{R}_+ \times D \to \mathbb{R}^d$ and $b : \mathbb{R}_+ \times D \to \mathbb{R}^d$ be continuous mappings. Let $(t_0, y_0) \in \mathbb{R}_+ \times D$.

We will use the following lemma to construct our class of possibilistic differential equations:

Lemma 3.1 *[6]. For each $\alpha \in [0,1]$, $t \in \mathbb{R}_+$, and $x \in X_\alpha$, the trajectory $t \mapsto w(t,x)$ is locally absolutely continuous and satisfies the following inequality almost everywhere on $\mathbb{R}_+$ (with respect to Lebesgue measure):*

$$\left|\frac{\partial w(t,x)}{\partial t}\right| \le \sqrt{\varphi^{-1}(\alpha)}.$$

Consider the following initial-value problem with parameter $x \in X$:

$$dy(t,x) = a(t, y(t,x))dt + b(t, y(t,x))dw(t,x), \tag{2}$$

$$y(t_0, x) = y_0, \tag{3}$$

or the same problem in the integral form:

$$y(t,x) = y_0 + \int_{t_0}^{t} a(s, y(s,x))ds + \int_{t_0}^{t} b(s, y(s,x))dw(s,x). \tag{4}$$

Definition 3.1 An α-solution (where $\alpha \in [0,1)$) of the problem (2)-(3) (or the problem (4)) on an interval $I \subseteq \mathbb{R}_+$ is a possibilistic process $y : I \times X \to D$ such that for each $x \in X_\alpha$ the trajectory $t \mapsto y(t,x)$ is locally absolutely continuous and satisfies (2)-(3) almost everywhere on I (in the sense of Lebesgue measure).

A solution of the problem (2)-(3) is a 0-solution of this problem.

We will use a special notion of uniqueness of solutions:

Definition 3.2 The problem (2)-(3) (or the problem (4)) has a unique α-solution on I, if it has some α-solution, and each two α-solutions of (2)-(3) on I are equal on the set $I \times X_\alpha$. The problem (2)–(3) has a unique solution, if it has a unique 0-solution.

Let us denote by $B(y_0, r) = \{y \in \mathbb{R}^d \mid ||y - y_0|| \leq r\}$ a closed ball in $\mathbb{R}^d$.

Theorem 3.1 (About existence and uniqueness of α-solution) *Assume that the functions $a(t,y)$ and $b(t,y)$ are continuous on the set $C = I \times B(y_0, r)$, where $I = [t_0, t_0 + \Delta t]$, $\Delta t > 0$, $r > 0$, and satisfy Lipschitz condition with respect to y, i.e.*

$$||a(t,y) - a(t,z)|| \leq L||y - z||, \quad ||b(t,y) - b(t,z)|| \leq L||y - z||,$$

for some constant $L > 0$ and all $t \in I$, $y, z \in B(y_0, r)$.

Let $\alpha \in (0,1)$, $M_a = \max\limits_{(t,y)\in C} ||a(t,y)||$, $M_b = \max\limits_{(t,y)\in C} ||b(t,y)||$. Then the problem (2)-(3) has a unique α-solution on $[t_0, t_0 + h)$, where

$$h = \min\left\{\frac{1}{2L}, \frac{1}{\sqrt{2L}\sqrt[4]{\varphi^{-1}(\alpha)}}, \frac{r}{M_a + \sqrt{\varphi^{-1}(\alpha)}M_b}, \Delta t\right\}.$$

Proof. Let us fix a number $\epsilon \in (0,1)$ and denote $I_\epsilon = [t_0, t_0 + \epsilon h]$. Consider the space F_ϵ of all continuous functions $f : I_\epsilon \to B(y_0, r)$ such that $f(t_0) = y_0$. Let us define a uniform metric on this space:

$$\rho_\epsilon(f,g) = \max_{t\in I_\epsilon} ||f(t) - g(t)||.$$

Because $B_\epsilon(y_0, r)$ is closed, it is a complete subspace of $\mathbb{R}^d$. Then the space of all continuous (and bounded) functions $f : I_\epsilon \to \mathbb{R}^d$ with metric ρ_ϵ is complete. Thus F_ϵ is a (non-empty) complete metric space.

Let us fix an elementary event $x_0 \in X_\alpha$ and consider the mapping $\Phi_\epsilon : F_\epsilon \to (I_\epsilon \to \mathbb{R}^d)$ such that

$$\Phi_\epsilon(f)(t) = y_0 + \int_{t_0}^{t} a(s, f(s))ds + \int_{t_0}^{t} b(s, g(s))dw(s, x_0).$$

For each $f \in F_\epsilon$ the function $t \mapsto \Phi_\epsilon(f)(t)$ is defined and continuous I_ϵ, because $h \leq \Delta t$, $s \mapsto a(s, f(s))$ and $s \mapsto b(s, f(s))$ are continuous on I_ϵ, and the trajectory $s \mapsto w(s, x_0)$ is absolutely continuous on I_ϵ.

Also, we have $\Phi_\epsilon(f)(t_0) = y_0$ and for each $t \in I_\epsilon$,

$$||\Phi_\epsilon(f)(t) - y_0|| \le \sup_{t\in I_\epsilon}\left(\int\limits_{t_0}^{t} ||a(s,f(s))||\,ds + \int\limits_{t_0}^{t} ||b(s,f(s))||\left|\frac{\partial w(s,x_0)}{\partial s}\right| ds\right) \le$$

$$\le \epsilon h\left(\max_{(s,y)\in C}||a(s,y)|| + \sqrt{\varphi^{-1}(\alpha)}\max_{(s,y)\in C}||b(s,y)||\right) =$$

$$= \epsilon h(M_a + \sqrt{\varphi^{-1}(\alpha)}M_b) \le \epsilon r < r$$

by Lemma 3.1. Thus Φ_ϵ maps F_ϵ to itself. Let us prove that Φ_ϵ is a contracting mapping. The Lipschitz condition implies that

$$\rho_\epsilon(\Phi_\epsilon(f), \Phi_\epsilon(g)) \le \max_{t\in I_\epsilon}\left(\int\limits_{t_0}^{t} ||a(s,f(s)) - a(s,g(s))||ds + \right.$$

$$\left.\int\limits_{t_0}^{t} ||b(s,f(s)) - b(s,g(s))||\left|\frac{\partial w(s,x_0)}{\partial s}\right| ds\right) \le$$

$$\le \left(L + L\epsilon h\sqrt{\varphi^{-1}(\alpha)}\right)\epsilon h\rho(f,g) \le$$

$$\le 2L\max\left\{\epsilon h, \sqrt{\varphi^{-1}(\alpha)}(\epsilon h)^2\right\}\rho(f,g) \le \max\{\epsilon, \epsilon^2\}\rho(f,g),$$

because $h \le \min\left(\frac{1}{2L}, \frac{1}{\sqrt{2L}\sqrt[4]{\varphi^{-1}(\alpha)}}\right)$. Then the mapping Φ_ϵ is contracting, because $\epsilon \in (0,1)$. By the Banach fixed point theorem, Φ has a unique fixed point. Obviously, this fixed point is absolutely continuous and satisfies (2)-(3) almost everywhere on I_ϵ. On the other hand, it is easy to see that every absolutely continuous function which satisfies (2)-(3) almost everywhere on I_ϵ is a fixed point of Φ_ϵ. Then because $\epsilon \in (0,1)$ and $x_0 \in X_\alpha$ are arbitrary, it is straightforward to show that the problem (2)-(3) has a unique α-solution on $[t_0, t_0 + h)$ in sense of Definition 3.2. □

Theorem 3.2 (About global existence and uniqueness of solution) *Assume that the functions $a(t,y)$ and $b(t,y)$ are continuous on the set $C = [t_0, +\infty) \times \mathbb{R}^d$ and satisfy a local Lipschitz condition with respect to y: there exists a continuous function $L : (0, +\infty) \to (0, +\infty)$ such that*

$$||a(t,y) - a(t,z)|| \le L(r)||y - z||,$$

$$||b(t,y) - b(t,z)|| \le L(r)||y - z||,$$

for all $t \ge 0$, $r > 0$, and $y, z \in B(y_0, r)$. Assume that the functions $a(t,y)$, $b(t,y)$ satisfy the following growth conditions for some constant $K > 0$:

$$||a(t,y)||^2 \le K\left(1 + ||y||^2\right),$$

$$||b(t,y)||^2 \le K\left(1 + ||y||^2\right).$$

Then the problem (2)-(3) has a unique solution on $[t_0; +\infty)$.

Proof. Let us choose an arbitrary $x_0 \in X_0$. Then $x_0 \in X_\alpha$ for some $\alpha \in (0,1)$.

Assume that a function $t \mapsto y(t,x_0)$ is defined (and continuous) on some segment $I = [t_0, t_0+h]$, $h > 0$, and satisfies (4) on I. Then

$$||y(t,x_0) - y_0|| \le \int_{t_0}^{t} ||a(s,y(s,x_0))||ds + \int_{t_0}^{t} ||b(s,y(s,x_0))|| \left|\frac{\partial w(s,x_0)}{\partial s}\right| ds.$$

This inequality, the growth conditions, and Lemma 3.1 imply that

$$||y(t,x_0) - y_0|| \le \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) \int_{t_0}^{t} \left(1 + K||y(s,x_0)||^2\right)^{1/2} ds. \tag{5}$$

Then for all $t \in I$,

$$||y(t,x_0)|| \le R(t),$$

where a scalar function $R(t)$ satisfies the following Cauchy problem:

$$R(t) = ||y_0|| + \left(1 + \sqrt{\varphi^{-1}(\alpha)}\right) \int_{t_0}^{t} \left(1 + KR(t)^2\right)^{1/2} ds. \tag{6}$$

It is easy to check that (6) has the following solution defined for all $t \ge t_0$:

$$R(t) = \frac{1}{\sqrt{K}} \sinh\left(\sqrt{K}(1 + \sqrt{\varphi^{-1}(\alpha)})(t - t_0) + \sinh^{-1}(\sqrt{K}||y_0||)\right).$$

Thus any extension of $t \mapsto y(t,x_0)$ from $I = [t_0, t_0+h]$ to $[t_0, t_0+h']$, $h' > h$ which satisfies (4) has a norm bounded from above by the function $R(t)$.

For each $r > 0$ let us denote

$$h(r) = \min\left\{\frac{1}{2L(r)}, \frac{1}{\sqrt{2L(r)}\sqrt[4]{\varphi^{-1}(\alpha)}}, \frac{r}{M(r) + \sqrt{\varphi^{-1}(\alpha)}M(r)}\right\},$$

$$M(r) = \sqrt{K(1 + (r + ||y_0||)^2)}.$$

The growth conditions imply that

$$\max_{t\ge 0, y\in B(y_0,r)} ||a(t,y)|| \le M(r), \max_{t\ge 0, y\in B(y_0,r)} ||b(t,y)|| \le M(r).$$

Then from Theorem 3.1 we have that for each $t_0' \ge t_0$, $r' > 0$ and $y_0' \in \mathbb{R}^d$ such that $B(y_0', r') \subseteq B(y_0, r)$ the problem (2) together with initial condition $y(t_0') = y_0'$ has a unique α-solution $y_{r',t_0',y_0'}(t,x)$ on $[t_0', t_0' + h(r'))$ (because we can choose an arbitrary $\Delta t > 0$ in the statement of Theorem 3.1).

Let us fix an arbitrary $\tau > t_0$. Let us construct a finite or infinite sequences of trajectories $y_1(t), y_2(t), ...$, positive numbers $r_0, r_1, r_2, ...$ and time moments $t_1, t_2, ...$ (t_0 is defined as in the statement of this theorem) such that

- $r_0 = R(\tau) - ||y_0|| + 1$;
- if $n \ge 0$ and $t_n < \tau$, then

– $t_{n+1} = t_n + h(r_n)/2$,
– $y_{n+1}(t) = y_{r_n,t_n,y_n(t_n)}(t, x_0)$ for all $t \in [t_n, t_{n+1}]$ (here $y_0(.) = y_0$),
– $r_{n+1} = r_0 - (R(t_{n+1}) - ||y_0||)$.

The sequence (t_n) is increasing and (r_n) is decreasing (because the function R is strictly monotone). These sequences may be finite, if (t_n) reaches or becomes greater than τ. It is easy to check by induction on n that the functions $y_1, y_2, ..., y_n$ are indeed correctly defined and their concatenation is a trajectory which satisfies (2)-(3) on $[t_0, t_n]$ using the inclusion $B(y_n(t_n), r_n) \subseteq B(y_0, r_0)$ which follows from (5) and (6).

If we assume that the sequence (t_n) is infinite, then it is bounded from above (by τ) and the equation $t_{n+1} = t_n + h(r_0 - R(t_n) + ||y_0||)/2$ holds for all $n \geq 1$. Then because of continuity of the functions h and R, we have $h(r_0 - R(\lim_{n\to\infty} t_n) + ||y_0||) = h(1 + R(\tau) - R(\lim_{n\to\infty} t_n)) = 0$. But this is impossible when $\lim_{n\to\infty} t_n < \tau$. Thus the sequence (t_n) is finite or its elements tend to τ. This implies that there exists a trajectory $t \mapsto y(t, x_0)$ which satisfies (2)-(3) on $[t_0, \tau)$.

Because $\tau > t_0$, $\alpha \in (0, 1)$ and $x_0 \in X_\alpha$ are arbitrary, we conclude that the problem (2)-(3) has a unique solution on $[t_0; +\infty)$ in the sense of Definition 3.1. Uniqueness of this solution (in sense of Definition 3.2) easily follows from Theorem 3.1. □

4 Numerical Solution

Definition 4.1 *The (t, α)-cut of a solution of the equation (4) is the set*

$$Y(t, \alpha) = \{y(t, x) \,|\, x \in X_\alpha\},$$

where $t \in \mathbb{R}_+$, $\alpha \in [0, 1)$.

The (t, α)-cut contains all points which can be reached by α-trajectories of a solution at time t. The family of all cuts of the solution gives a complete description of its distribution.

Definition 4.2 An estimate of (t, α)-cut of the solution of the equation (4) is a set $\hat{Y}(t, \alpha) \subseteq \mathbb{R}^n$ such that the (t, α)-cut $Y(t, \alpha)$ is a dense subset of $\hat{Y}(t, \alpha)$.

By a *numerical solution* of the equation (4) we mean some numerical representation of a family of estimates of (t_i, α_i)-cuts for a finite set of pairs $\{(t_i, \alpha_i) \,|\, i \in I\}$. The numerical solution gives information about sets which can be reached by the solution of (4) with a given level of possibility.

Let us associate with the equation (4) the following dynamical system with scalar input control $u(t)$:

$$dz(t) = a(t, z(t))dt + b(t, z(t))u(t), \tag{7}$$

$$z(t_0) = y_0. \tag{8}$$

Let us denote by $BU(r)$ the set of all bounded measurable controls $u : \mathbb{R}_+ \to \mathbb{R}$ such that $\sup_t |u(t)| \leq r$.

Theorem 4.1 *[6]. The set $U(t, \alpha) \subseteq \mathbb{R}^n$ of all points which can be reached by the system (7), (8) at time t by means of controls $u \in BU\left(\sqrt{\varphi^{-1}(\alpha)}\right)$ is an estimate of (t, α)-cut of the solution of equation (4).*

This theorem reduces the problem of finding numerical solution of the equation (4) to the problem of finding reachable sets of the controlled system (7), (8). The problem of finding reachable sets is well studied [1] and can be solved numerically using existing tools such as dynamic programming.

5 Numeric Example

Let us consider how the results obtained above can be applied to the problem of modeling dynamics of epidemics. We start with a simple Ross epidemic model [18]. In this model the population of N individuals is divided into two groups:

- susceptible individuals, S;
- infective individuals, I.

It is assumed that the following statements hold:

(1) the population is homogeneous, there are no births, deaths, immigrations and emigrations;

(2) there is no latent period of the infection, recoveries from illness are not taken into account;

(3) the infection rate is proportional to the fraction of infectives.

The model is described by the following equations:

$$S(t) = N - I(t), \tag{9}$$

$$\frac{dI}{dt} = aI(t)(N - I(t)), \tag{10}$$

where $S(t)$ and $I(t)$ are the numbers of susceptible and infective individuals at time t, N is the total number of individuals (constant), a is a positive constant.

The model (9)-(10) can be improved by taking into account recovery and transmission of disease from external source as described in [9]. Let us denote by $y(t) = I(t)/N$ the *fraction* of infected individuals. The improved model has the form

$$y'(t) = ay(t)(1 - y(t)) - by(t) + c(1 - y(t)), \tag{11}$$

where

- $a > 0$ is the rate of transmission from individual to individual;
- $b > 0$ is the rate of recovery;
- $c > 0$ is the rate of transmission from external source.

Although the model (11) is more accurate than (9)-(10), it is still a rather rough simplification of the real dynamics of epidemics. To take into account inaccuracy of (9), following [9] let us add a dynamic uncertainty to this model:

$$dy(t, x) = ay(t, x)(1 - y(t, x))dt - by(t, x)dt + c(1 - y(t, x))dt + \sigma(y(t))dw(t, x), \tag{12}$$

where $\sigma(y)$ is a function of the form $\delta y(1-y)$, $\delta > 0$. This equation differs from a stochastic epidemic model proposed in [9] in that the uncertainty is modeled by a possibilistic walk process $w(t,x)$ instead of the Wiener process. This allows one to estimate the influence of uncertainties on propagation of epidemics on the basis of expert opinions [11, 20] instead of statistical data (the latter may be very limited for new or unfamiliar types of infections).

Then we obtain the final possibilistic epidemic model:

$$dy(t,x) = ay(t,x)(1-y(t,x))dt - by(t,x)dt+ \tag{13}$$

$$+c(1-y(t,x))dt + \delta y(t)(1-y(t,x))dw(t,x),$$

$$y(0,x) = y_0.$$

It is not difficult to check that this problem has a unique solution (to simplify this task it is sufficient to assume that y always takes values in [0,1], because it represents a fraction of infected individuals).

The solution $y(t,x)$ is a possibilistic process. Let us find an estimate of α-cut of $y(t,x)$. Let us apply the system (7)-(8) to the equation (13):

$$z'(t) = az(t)(1-z(t)) - bz(t)+ \tag{14}$$

$$+c(1-z(t)) + \delta z(t)(1-z(t))u(t),$$

$$z(0) = y_0.$$

Let us define $y_1(t)$, $y_2(t)$ as solutions of the following equations:

$$y_1'(t) = ay_1(t)(1-y_1(t)) - by_1(t) + c(1-y_1(t)) - \tag{15}$$

$$-\sqrt{\varphi^{-1}(\alpha)}|\delta y_1(t)(1-y_1(t))|, \quad y_1(0) = y_0,$$

$$y_2'(t) = ay_2(t)(1-y_2(t)) - by_2(t) + c(1-y_2(t)) + \tag{16}$$

$$+\sqrt{\varphi^{-1}(\alpha)}|\delta y_2(t)(1-y_2(t))|, \quad y_2(0) = y_0.$$

It is easy to verify that the segment $[y_1(t), y_2(t)]$ is a reachable set at time t for the system (14) with controls $u \in BU(\sqrt{\varphi^{-1}(\alpha)})$. So the set $[y_1(t), y_2(t)]$ is an estimate of α-cut of the solution of (14) by Theorem [6].

Assume that $\alpha > \varphi(a^2/\delta^2)$. Then non-negative stationary solutions of the equations (15), (16) are given by the following expressions:

$$\hat{y}_1(\alpha) = \frac{a-b-c+\delta C_\alpha + \sqrt{(a-b+c+\delta C_\alpha)^2 + 4bc}}{2(a+\delta C_\alpha)},$$

$$\hat{y}_2(\alpha) = \frac{a-b-c-\delta C_\alpha + \sqrt{(a-b+c-\delta C_\alpha)^2 + 4bc}}{2(a-\delta C_\alpha)},$$

where $C_\alpha = \sqrt{\varphi^{-1}(\alpha)}$.

Thus we can accept that for large t, the fraction of infected individuals belongs to the segment $[\hat{y}_1(\alpha), \hat{y}_2(\alpha)]$ with the level of possibility α.

Let us consider a numerical example.

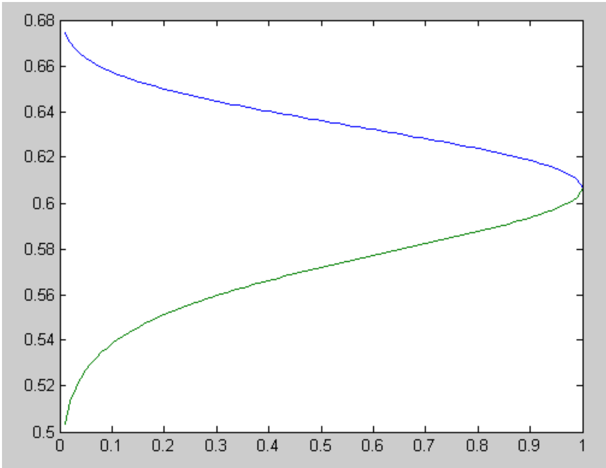

Figure 1: The lower and upper bound for the fraction of infected individuals when $a = 1$. The horizontal axis represents the possibility level α.

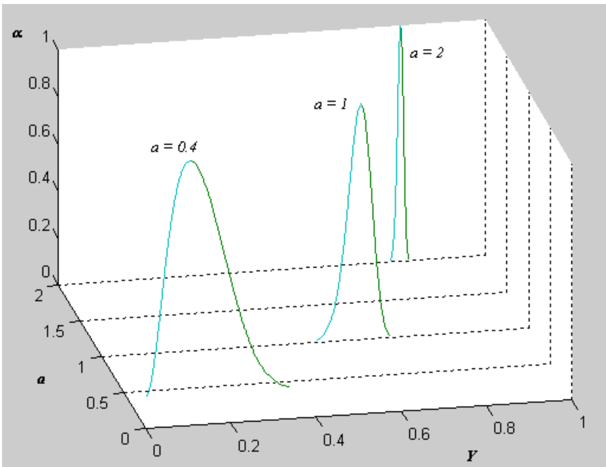

Figure 2: The lower and upper bounds for the fraction of infected individuals for different values of parameter a.

Example 5.1 Let $\varphi(x) = \exp(-x)$, $a = 1$, $b = 0.4$, $c = 0.01$, $\delta = 0.1$. Figure 1 shows the curves $\hat{y}_1$ (lower bound) and $\hat{y}_2$ (upper bound). The horizontal axis represents the possibility level α. Figure 2 shows the similar curves for different values of a (but the possibility level α is represented on vertical axis).

Figures 1 and 2 were produced by the following MATLAB [17] program:

```
function r = f(alpha,a,s)
  b = 0.4; c = 0.01;
  dC = s * 0.1 * sqrt(-log(alpha));
```

```
  r = (a-b-c+dC + sqrt((a-b+c+dC).^2+4*b*c))./(2*(a+dC));

I = 0.01:0.01:1; nul = zeros(length(I));
plot3(f(I,0.4,1),nul+0.4,I, f(I,0.4,-1),nul+0.4,I); hold on
plot3(f(I,1,1),nul+1,I, f(I,1,-1),nul+1,I);
plot3(f(I,2,1),nul+2,I, f(I,2,-1),nul+2,I);
```

6 Conclusion

In the paper we have studied the problem of modeling of uncertain dynamics using the methods of possibility theory. We have constructed a new class of uncertain differential equation with respect to a possibilistic walk process. We have studied the problems of existence and uniqueness of solutions of these equations and proposed a method which can be used to solve them numerically.

The obtained results can be used for modeling social-economic and ecological phenomena, medical diagnostic tasks, and other uncertain processes or phenomena in which available statistical information is not sufficient for constructing a reliable stochastic model, or the uncertainty has a non-probabilistic nature.

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