Nonlinear Dynamics and Systems Theory, **13** (2) (2013) 203–216

On Stability Conditions of Singularly Perturbed Nonlinear Lur'e Discrete-Time Systems

B. Sfaihi [1*], M. Benrejeb [1] and P. Borne [2]

[1] *LARA Automatique, Université de Tunis El Manar, Ecole Nationale d'Ingénieurs de Tunis, B.P. 37, 1002 Tunis Le Belvédère, Tunisia.*
[2] *LAGIS, Ecole Centrale de Lille, BP 48, 59651 Villeneuve d'Ascq Cedex, France*

Received: April 5, 2012 ; Revised: March 19, 2013

Abstract: This paper deals with stability of discrete-time nonlinear Lur'e-type systems. Through the singular perturbations technique, the original system is reduced to a block-diagonal form with slow and fast decoupled modes. Stability conditions of the two-time-scale decoupled model based on Borne-Gentina practical stability criterion and the use of matrices in the Benrejeb arrow form are developed and compared with those concerning the original discrete-time system. It is shown that these results are practical and less conservative then the existing ones. A third order system is introduced to illustrate the efficiency of the proposed approach.

Keywords: *discrete Lur'e systems; singular perturbations technique; two-time-scale systems; stability; arrow form matrix.*

Mathematics Subject Classification (2010): 34H15, 34K35.

1 Introduction

During the past several decades, the stability problem of dynamical systems has attracted an immense attention in the control society. A great majority of the encountered problems is concerned with the closed-loop behavior of feedback nonlinear systems. An important and typical class of such systems is Lur'e-type systems introduced by Lur'e and Postnikov [39], and described by combinations of a dynamic linear bloc and a feedback interconnected to a static nonlinearity, assumed to lie in a given sector. Since that,

* Corresponding author: mailto:boutheina.sfaihi@isetr.rnu.tn

Lur'e systems have become an attractive research subject and have received a series of results in many relevant nonlinear engineering applications, such as mechanical, electrical, economic and biological [55].

The original analysis was motivated by the need to understand the effect of nonlinearities on control systems due to elements such as imperfect actuators or sensors that have gain or amplification that can vary over time. Within this framework, the nonlinearities are most commonly modeled as gain bounded or sector bounded uncertainties and the absolute stability is analyzed via the formulation of finite system of quadratic equations.

Defined as a global asymptotic stability tolerating any nonlinear perturbations with special constraints [57], the absolute stability problem has been the subject of extensive research for continuous Lur'e systems [11, 15, 22, 24, 25, 31, 36, 44, 49, 50, 53]. One of the most main results related to absolute stability has been the Popov criterion [43], which is a graphical construction that provides a simple approach to maximize the nonlinear sector. Popov proved that the analysis can be done in the frequency domain and the stability is derived by Lyapunov's direct method. The circle criterion [21,29], dealing with time varying nonlinearity, analyzes the absolute stability via a suitable strict positive-realness condition on the linear part and a given sector condition on the nonlinear part. Recently, more results about the stability analysis for Lur'e systems with slope-restricted are introduced in [3, 33, 41, 42, 48, 55], and with time-delays and model uncertainties in [7, 17, 23, 26, 32, 51].

Because of their wide applications in many practical processes, a great number of results in control and system theory have been extended successfully to singular systems [13]. The two-time-scale nature of such systems is exploited to decompose the design problem into two lower-order design problems for the slow and fast modes. Some results on singular perturbed nonlinear Lur'e systems in continuous-time are developed in the field [13,52,54] where the stability criterion is deduced by mean of Lyapunov functional method. However, the stability investigation on Lur'e type discrete time systems is limited [31].

The paper is organized as follows. The class of discrete Lur'e-type systems will be introduced in Section 2. In Section 3 a two-time-scale decoupling procedure for the original Lur'e-type system based on singular perturbation technique is presented. In Section 4 stability conditions of original Lur'e-type system and decoupled model, are derived and compared. The synthesized results are formulated by the use of the Benrejeb arrow form matrix and the Borne-Gentina practical stability criterion. In Section 5 the proposed model decoupling strategy is applied to a nonlinear system of order three. Stability conditions of original system and reduced order subsystems are developed and discussed.

2 System Description and Problem Statement

Consider the Lur'e type discrete-time system described by state space representation (1). The model consists of a static nonlinearity in cascade with a dynamic linear time invariant system according to [11] and [29]:

$$S : \begin{cases} x_{k+1} = A_L x_k + B_L u_k, \\ u_k = h\left(\varepsilon_k\right)\varepsilon_k, \\ \varepsilon_k = r_k - C_L x_k, \end{cases} \tag{1}$$

where A_L, B_L and C_L are known matrices of appropriate dimensions, $x_k \in \Re^n$ denotes the state vector, $u_k \in \Re$ is the vector input, $r_k \in \Re$ is the reference input, $r_k = 0$ and $\varepsilon_k \in \Re$ is the control error of the closed-loop system, $h(.) : \Re \to \Re$ represents memoryless nonlinear matrix valued function.

The investigated Lur'e-type discrete-time system can be represented by the nonlinear regression equation:

$$\varepsilon_{k+n} + \sum_{i=1}^{n} g_i(\varepsilon_{k+n-i})\varepsilon_{k+n-i} = 0, \tag{2}$$

where the corresponding expression in terms of state space representation (1) becomes:

$$S : x_{k+1} = A(\varepsilon_k)\, x_k \tag{3}$$

with

$$A(\varepsilon_k) = A_L - B_L h(\varepsilon_k)\, C_L, \tag{4}$$

$A(\varepsilon_k)$ denotes the instantaneous characteristic matrix expressed in Frobenius form as:

$$A(\varepsilon_n) = \begin{bmatrix} 0 & \cdots & 0 & -g_n(\varepsilon_n) \\ 1 & 0 & \vdots & -g_{n-1}(\varepsilon_n) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -g_1(\varepsilon_n) \end{bmatrix}. \tag{5}$$

In the design of complex and/or large scale systems, models are usually of high order. Model reduction techniques can be used to obtain a low-order approximation of these models, allowing for efficient analysis or control design. Many order reduction techniques can be found in the literature: reduced order models synthesized via aggregation and dominant modes approaches neglect fast stable dynamics and some of the poorly controllable and observable slow dynamics. With the singular perturbation method [1, 14, 35, 38, 47], both slow and fast dynamics are retained; analysis and design problems are solved in two steps, first for the fast and then for the slow dynamics. These methods for model reduction of nonlinear systems have in common that the stability of the reduced-order model is not guaranteed. In the present work, model reduction procedure, based on singular perturbation technique, for discrete Lur'e-type systems is presented, and conditions to ensure asymptotic stability of the fast and reduced-order decoupled subsystem as well as the original system (1) are given.

3 Two-Time-Scale Decoupling

By reordering and/or rescaling of states, let the nonlinear discrete system be structured in the two-time-scale model:

$$\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix}, \tag{6}$$

where x^1_k and x^2_k are n_1 and n_2 dimensional state vectors, respectively, and the overall system is of dimension $n = n_1 + n_2$, $x^1_k \in \Re^{n_1}$ and $x^2_k \in \Re^{n_2}$. This system is assumed to possess a two-time-scale property, which means that the n eigenvalues of the system can be separated into n_1 slow modes and n_2 stable fast modes related to x^1_k, and x^2_k, respectively. The fast subsystem x_k^2, assumed to be stable, is active only during a short initial period, after, it is negligible and the characterization of the system can be described by its slow subsystem x_k^1.

An explicit two-time-scale property of this model can be introduced by assuming that:

$$A_{11}^* = \mu^{-1}\left(A_{11} - I_{n_1}\right), \tag{7}$$

$$A_{12}^* = \mu^{-1} A_{12}, \tag{8}$$

$$A_{21}^* = A_{21}, \tag{9}$$

$$A_{22}^* = A_{22}. \tag{10}$$

The transformed system is expressed in the standard singular perturbation system structure [38] and [27,28,34,37]:

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} I_{n1} + \mu A_{11}^* & \mu A_{12}^* \\ A_{21}^* & A_{22}^* \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}, \tag{11}$$

where μ is a small positive singular perturbation parameter and $\det\left(I_{n_2} - A_{22}^*\right) \neq 0$ [47].

As $\mu \to 0$, the eigenvalues of (11) cluster into two groups and, the original system (6) can be decoupled in slow subsystem S_s and fast subsystem S_f candidates:

$$S_s: \quad x_{k+1}^s = (I_{n_1} + \mu A_s) x_k^s, \tag{12}$$

$$S_f: \quad x_{k+1}^f = A_{22}^* x_k^f, \tag{13}$$

with

$$A_s = A_{11}^* + A_{12}^*\left(I_{n_2} - A_{22}^*\right)^{-1} A_{21}^*, \tag{14}$$

where $x_s \in \Re^{n_1}$ and $x_f \in \Re^{n_2}$ are, respectively, the slow and the fast subsystems state vectors defined using a decoupling transformation [12,40,47], if it exists.

The slow subsystem is defined by neglecting the fast stable dynamics, which is equivalent to replace the second equation of (11) by its steady-state algebraic equation. The fast subsystem, supposed to be stable, is derived by assuming that slow variables are constant during fast transients and $\mu = 0$.

4 Main Results

By considering the instantaneous characteristic polynomial $P_S(.,\lambda)$ of (1), (2) or (3):

$$P(\,.\,,\lambda) = \lambda^n + \sum_{i=1}^{n} g_i(.)\lambda^{n-i} \tag{15}$$

and distinct arbitrary constant parameters α_j, $j = 1, 2, \cdots, n-1$, $\alpha_i \neq \alpha_j$, $\forall i \neq j$, it comes the following notations:

$$\beta_j = \prod_{\substack{k=1 \\ k \neq j}}^{n-1} (\alpha_j - \alpha_k)^{-1}, \forall j = 1, 2, \cdots, n-1, \tag{16}$$

$$\gamma_j(.) = -P(\,.\,, \alpha_j), \forall j = 1, 2, \cdots, n-1, \tag{17}$$

$$\delta_n(.) \;=\; -g_1(.) - \sum_{i=1}^{n-1} \alpha_i. \tag{18}$$

Let S be a Lur'e- type system of the form (1)-(3) and let S_s be the decoupled Lur'e-type subsystem (12). By applying the Borne-Gentina practical stability criterion [8,9,20] to the discrete Lur'e type systems characterized by the Benrejeb arrow form matrix [2–6, 10, 18], we obtain the following theorems and corollary.

Theorem 4.1 *The discrete nonlinear system S of the form (1) is asymptotically stable, if there exist constant parameters $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j, \forall i \neq j$, such that*

$$|\alpha_i| < 1 \qquad \forall i = 1, \cdots, n-1 \tag{19}$$

and

$$1 - \left|\delta_n(\cdot)\right| - \sum_{j=1}^{n-1} \left|\gamma_j(\cdot)\right| |\beta_j| \left(1 - |\alpha_j|\right)^{-1} > 0. \tag{20}$$

Theorem 4.2 *For chosen stable fast subsystem, i.e., $|\alpha_i| < 1$ $\forall i = n_1, \cdots, n-1$, the discrete nonlinear decoupled system (12) is asymptotically stable if there exist arbitrary constant parameters $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j, \forall i \neq j$, such that the following conditions are satisfied and*

$$|\alpha_i| < 1 \;\forall i = 1, ..., n_1 - 1, \tag{21}$$

$$1 - \left|\delta_n(\cdot) + \sum_{j=n_1}^{n-1} \gamma_j(\cdot)\beta_j \left(1 - \alpha_j\right)^{-1}\right| - \sum_{j=1}^{n_1-1} \left|\gamma_j(\cdot)\right| |\beta_j| \left(1 - |\alpha_j|\right)^{-1} > 0. \tag{22}$$

Corollary 4.1 *For chosen stable fast subsystem, i.e., $|\alpha_i| < 1$ $\forall i = n_1, \cdots, n-1$, the discrete nonlinear decoupled subsystem (12) (respectively the original system (1)) is asymptotically stable if the original system (1) (respectively decoupled subsystem (12)) is asymptotically stable and, if there exists constant parameter $\alpha_i \in \Re$; $\alpha_i \neq \alpha_j$, $\forall i \neq j$, such that the following conditions are satisfied*

$$\begin{cases} \alpha_j > 0, & \forall j = 1, ..., n_1 - 1, \\ \sum\limits_{i=1}^{n-1} \alpha_i > -g_1(.), & \\ \gamma_j(.)\beta_j > 0, & \forall j = 1, ..., n_1 - 1. \end{cases} \tag{23}$$

Proof. (Theorem 4.1) Let us consider the Lur'e-type system S of the form (1)-3). A change of coordinate defined by:

$$y_k = Tx_k \tag{24}$$

with $y_k \in \Re^n$ and

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_{n-2} & \alpha_{n-2}^2 & \cdots & \alpha_{n-2}^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \end{bmatrix}. \tag{25}$$

leads to the following state space description

$$y_{k+1} = G\,(.)\,y_k. \tag{26}$$

Allowing the synthesis of sufficient stability conditions easy to test, the new instantaneous characteristic matrix $G\,(.)$ is chosen to be in the arrow form [2–6,10,18], Appendix 2, as follows

$$G\,(.) = T\,A(.)\,T^{-1} = \begin{bmatrix} \delta_n\,(\cdot) & \beta_1 & \cdots & \beta_{n-1} \\ \gamma_1\,(\cdot) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n-1}\,(\cdot) & & & \alpha_{n-1} \end{bmatrix}, \tag{27}$$

where β_i, γ_i, δ_n and α_i, $\forall i = 1, 2, \cdots, n-1$ are defined (16)–(18).

A pseudo-overvaluing matrix $M\,(G\,(\cdot))$ of the system (26), corresponding to the use of the vector norm (Appendix 1):

$$p\,(y) = \left[|y_1|\,,\, |y_2|\,,\, \ldots\,,\, |y_n|\right]^T, \tag{28}$$

$y = [y_1,\, y_2,\, \ldots\,,\, y_n]^T$, for the stability study, can be obtained from the inequality:

$$p(y_{k+1}) \leq M\,(G\,(\cdot))\,p(y_k) \tag{29}$$

satisfied for each corresponding component; that leads to the following comparison system

$$z_{k+1} = M\,(G\,(\cdot))\,z_k \tag{30}$$

with

$$M\,(G\,(.)) = \begin{bmatrix} |\delta_n\,(\cdot)| & |\beta_1| & \cdots & |\beta_n| \\ |\gamma_1\,(\cdot)| & |\alpha_1| & & \\ \vdots & & \ddots & \\ |\gamma_n\,(\cdot)| & & & |\alpha_n| \end{bmatrix} \tag{31}$$

such as: $z_0 = p\,(y_0)$.

If the nonlinearities of the comparison nonlinear system (30) are isolated in one row of $M\,(G\,(\cdot))$, the verification of the Kotelyanski condition (Appendix 1) enables to conclude about the stability of the original system characterized by $G\,(\cdot)$ [3,9,10].

It comes the following sufficient asymptotic stability condition of original system:

$$(I_n - M\,(G\,(\cdot)))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} > 0 \quad \forall j = 1, \ldots, n. \tag{32}$$

This ends the proof of Theorem 4.1.

Proof. (Theorem 4.2) Note that the satisfaction of the condition (19), i.e. $|\alpha_i| < 1, i = 1, \cdots, n-1$, means that the fast system characterized by a diagonal matrix $\{\alpha_i\}$, $i = n_1, \cdots, n-1$ is stable. Conditions $|\alpha_i| < 1, i = 1, \cdots, n_1 - 1$ are necessary to satisfy for the reduced slow subsystem stability.

In order to synthesize the stability conditions of the two-time-scale decoupled system S, we first, reorder the transformed nonlinear system states (3). Resulting A_{11}, A_{12},

A_{21} and A_{22} matrices are then in the form (33), where the matrix A_{11} is candidate to characterize the slow subsystem of (6) and A_{22} the fast one:

$$A_{11} = \begin{bmatrix} \delta_n(\cdot) & \beta_1 & \cdots & \beta_{n_1-1} \\ \gamma_1(\cdot) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n_1-1}(\cdot) & & & \alpha_{n_1-1} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \beta_{n_1} & \cdots & \beta_{n-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} \gamma_{n_1}(\cdot) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \gamma_{n-1}(\cdot) & 0 & \cdots & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \alpha_{n_1} & & \\ & \ddots & \\ & & \alpha_{n-1} \end{bmatrix}. \tag{33}$$

Arbitrary constant parameters α_i, $i = n_1, \cdots, n-1$, are chosen in concordance with the estimation of the dynamics that we consider physically fast for the studied system.

Substituting the relations (33), (7)-(10) and (14) into (12) and (13), yields the following discrete slow and fast subsystems, respectively:

$$\begin{aligned} x^s_{k+1} &= A_s(.)\, x^s_k, \\ x^f_{k+1} &= A_f x^f_k, \end{aligned} \tag{34}$$

and then comparison systems, respectively:

$$y^s_{k+1} = M(A_s(\cdot))\, y^s_k, \tag{35}$$

$$y^f_{k+1} = M(A_f)\, y^f_k, \tag{36}$$

where $A_s \in \Re^{n_1 x n_1}$ and $A_f \in \Re^{n_2 x n_2}$ are given by

$$A_s = \begin{bmatrix} \delta_n(.) + \sum\limits_{j=n_1}^{n-1} \frac{\gamma_j(.)\beta_j}{(1-\alpha_j)} & \beta_1 & \cdots & \beta_{n_1-1} \\ \gamma_1(.) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n_1-1}(.) & & & \alpha_{n_1-1} \end{bmatrix}, \tag{37}$$

$$A_f = \begin{bmatrix} \alpha_{n_1} & & \\ & \ddots & \\ & & \alpha_{n-1} \end{bmatrix}, \tag{38}$$

and $M(A_s(\cdot))$ and $M(A_f(\cdot))$ are respectively the pseudo-overvaluing matrices of the slow and fast subsystems (12) and (13), corresponding to the use of the vector norm (28). By applying the practical Borne-Gentina criterion [3, 9, 10, 16] to the comparison systems (35) and (36) of (34), we deduce the stability conditions of the decoupled discrete systems. Theorem 4.2 is then proved.

Proof. (Corollary 4.1) The proof can be easily obtained by substituting the relations (23) in (22).

5 Illustrative Example

To show the effectiveness of the derived theorems, a numerical example is studied below. Consider the discrete nonlinear Lur'e system described by means of the following block-oriented nonlinear model (Figure 1), where $f(.) : \Re \to \Re$ is a nonlinear function, $B_0(s) =$

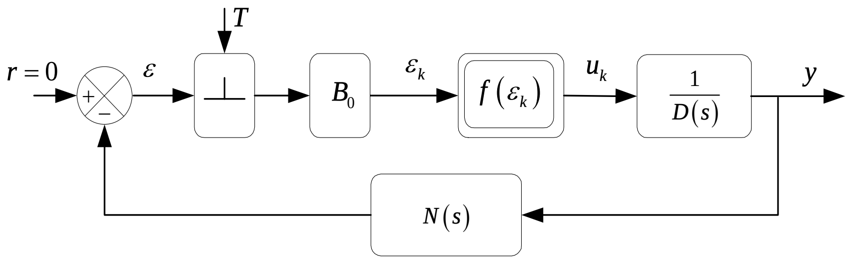

Figure 1: Lur'e systems.

$\frac{1-e^{-Ts}}{s}$ is a zero order holder, $T = 0.2s$ the sampling time, and $D(s)$ and $N(s)$ are polynomials defined by:

$$D(s) = s(1 + \tau_1 s)(1 + \tau_2 s), \tag{39}$$

$$N(s) = \lambda_2 s^2 + \lambda_1 s + \lambda_0. \tag{40}$$

A state space representation (3) synthesized in the canonical Frobenius form gives:

$$A(\varepsilon_k) = \begin{bmatrix} 0 & 0 & -1,19.10^{-6} f(\varepsilon_k) \\ 1 & 0 & -0,13 + 0,23.10^{-1} f(\varepsilon_k) \\ 0 & 1 & 1,13 - 1,92 f(\varepsilon_k) \end{bmatrix}. \tag{41}$$

By choosing $\alpha_1 = 0.9$ and $\alpha_2 = 0.1$ satisfying (19), the synthesized transformed state space representation in the arrow form is defined by:

$$N(\varepsilon_k) = \begin{bmatrix} 0,14 - 0,19 f(\varepsilon_k) & 1,20 & -1,20 \\ 0,69.10^{-1} - 0,14 f(\varepsilon_k) & 0,90 & 0 \\ -0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k) & 0 & 0,10 \end{bmatrix}. \tag{42}$$

Furthermore, by taking $\mu = 0.1$, the decoupled slow and the fast subsystems are given respectively by

$$N_s = \begin{bmatrix} 0,14 - 0,19 f(\varepsilon_k) & 1,20 \\ 0,69.10^{-1} - 0,14 f(\varepsilon_k) & 0,90 \end{bmatrix}, \quad N_f = 0,10. \tag{43}$$

The stability conditions of the original system deduced from Theorem 4.1, are, for chosen α_1 and α_2:

$$1 - |0,14 - 0,19 f(\varepsilon_k)| - 12 \times \left|0,69.10^{-1} - 0,14 f(\varepsilon_k)\right| - 1.33 \times \left|-0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k)\right| > 0$$

or

$$-0.01 < f(\varepsilon_k) < 1.05. \tag{44}$$

Now, by applying Theorem 4.2, the stability conditions of the decoupled nonlinear system (43) are:

$$1 - |0,14 - 0,19 f(\varepsilon_k)| - 12 \times \left|0,69.10^{-1} - 0,14 f(\varepsilon_k)\right| > 0$$

or

$$-0.01 < f(\varepsilon_k) < 1.05. \tag{45}$$

Furthermore, according to the corollary, if we impose the synthesized conditions (23)

$$\begin{cases} 0,14 - 0,19 f(\varepsilon_k) > 0, \\ -0,32.10^{-2} - 0,37.10^{-3} f(\varepsilon_k) < 0, \end{cases} \tag{46}$$

we obtain

$$-8.64 < f(\varepsilon_k) < 0.73. \tag{47}$$

Consequently, the original Lur'e discrete-time system (42) and the decoupled system (43) are asymptomatically stable for the common stability domain:

$$-0.01 < f(\varepsilon_k) < 0.73. \tag{48}$$

Stability domain (D1) of the original system (42) and the common stability domain (D2) are introduced in Figure 2.

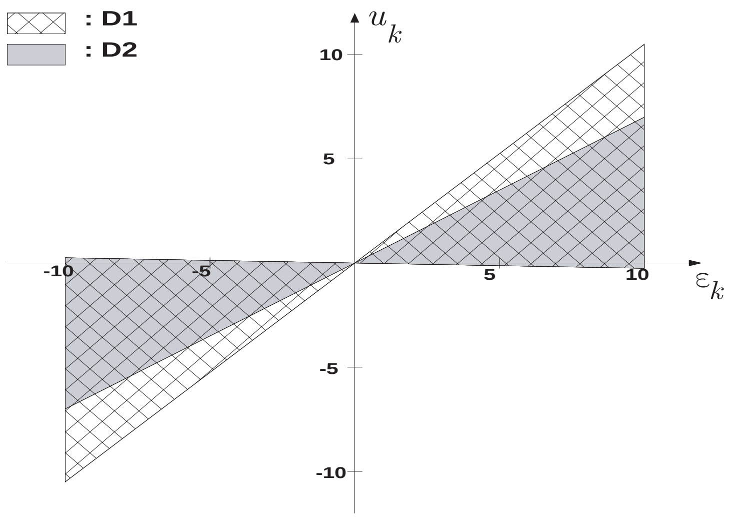

Figure 2: Stability domains.

6 Conclusion

The problem of singular perturbed nonlinear Lur'e discrete-time systems is addressed and a model reduction procedure based on the singular perturbation technique is introduced. Sufficient conditions for stability of the decoupled system as well as the original nonlinear Lur'e type discrete system(1) are then derived. Supplementary stability conditions are synthesized to ensure a common stability domain for the original and the decoupled system. An example is studied to illustrate the efficiency of the proposed results.

Appendix 1

Definition 6.1 (**Vector Norm** [45, 46]) Let $E = \Re^n$ be a vector space and $E_1, E_2, \cdots, E_k$ be subspaces of E which verify: $E = E_1 \cup E_2 \cup \cdots \cup E_k$. Let $x \in E$ be an n vector defined on E with a projection in the subspace E_i denoted by x_i, $x_i = P_i x$, where P_i is a projection operator from E into E_i, p_i is a scalar norm $(i = 1, \cdots, k)$ defined on the subspace E_i and p denotes the vector norm of dimension k and with i^{th} component, $p_i(x) = p_i(x_i)$, $p_i(x) : R^n \to R^k_+$, where $p_i(x_i)$ is a scalar norm of x_i.

Lemma 6.1 *(**Kotelyanski** [19, 30]) The real parts of the eigenvalues of matrix A, with non negative off diagonal elements, are less than a real number μ if and only if all those of matrix $M = \mu I_n - A$ are positive, with I_n being the n identity matrix.*

When successive principal minors of matrix $(-A)$ are positive, Kotelyanski lemma permits to conclude on stability property of the system characterized by A.

Appendix 2

Let us consider the observable nonlinear system:

$$z_{k+1} = A\,(.)\,z_k,$$

$$A\,(.) = \begin{bmatrix} 0 & \cdots & 0 & -a_n\,(.) \\ 1 & 0 & \vdots & -a_{n-1}\,(.) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -a_1\,(.) \end{bmatrix},$$

where $a_i(.)$ are the instantaneous characteristic polynomial $P_A(.,\lambda)$ coefficients of $A\,(.)$, such that:

$$P_A(\,.\,,\lambda) = \lambda^n + \sum_{i=1}^{n} a_i(.)\,\lambda^{n-i}.$$

A change of base defined by:

$$\hat{z}_k = T z_k,$$

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \alpha_{n-1} & \alpha_{n-1}^2 & \cdots & \alpha_{n-1}^{n-1} \\ 1 & \alpha_{n-2} & \alpha_{n-2}^2 & \cdots & \alpha_{n-2}^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \end{bmatrix},$$

where α_j, $j = 1, 2, \cdots, n-1$ are distinct arbitrary constant parameters, allows the new state matrix, denoted by $F\,(.)$, to be in arrow form [2–6, 10, 18]:

$$F(.) = T\,A(.)\,T^{-1} = \begin{bmatrix} \delta_n(.) & \beta_1 & \cdots & \beta_{n-1} \\ \gamma_1(.) & \alpha_1 & & \\ \vdots & & \ddots & \\ \gamma_{n-1}(.) & & & \alpha_{n-1} \end{bmatrix}$$

with

$$\beta_j = \prod_{\substack{k=1\\k\neq j}}^{n-1} (\alpha_j - \alpha_k)^{-1}, \forall j = 1, 2, \ldots, n-1,$$

$$\delta_j(.) = -P_A(\,.\,, \alpha_j), \forall j = 1, 2, \ldots, n-1,$$

$$\delta_n(.) = -a_1(.) - \sum_{i=1}^{n-1} \alpha_i.$$

References

[1] Aliyu, M.D.S., Perrier, M., and Baron, L. $\mathcal{H}_\infty$ Filtering for Discrete-time Nonlinear Singularly-Perturbed Systems. *Nonlinear Dynamics and Systems Theory* **12**(1) (2012) 19–36.

[2] Benrejeb, M. *Sur l'analyse et la synthèse de processus complexes hiérarchisés. Application aux systèmes singulièrement perturbés.* Doctor Science Thesis, USTL, Lille, France, 1980.

[3] Benrejeb, M. and Borne, P. On an algebraic stability criterion for non-linear processes. Interpretation in the frequency domain. *Proceedings of the Measurement and Control International Symposium MECO'78*, Athens, Greece, 1978, 678–682.

[4] Benrejeb, M. and Gasmi, M. On the use of an arrow form matrix for modelling and stability analysis of singularly perturbed nonlinear systems. *Systems Analysis Modelling Simulation* **40** (2001) 509–525.

[5] Benrejeb, M., Gasmi, M. and Borne, P. New stability conditions for TS fuzzy continuous nonlinear models. *Nonlinear Dynamics and Systems Theory* **5**(4) (2005) 369–379.

[6] Benrejeb, M., Soudani, D., Sakly, A. and Borne, P. New Discrete Tanaka Sugeno Kang Fuzzy Systems Characterization and Stability Domain. *International Journal of Computers, Communications & Control* **1**(4) (2006) 9–19.

[7] Bilman, P. Lyapunov-Krasovskii functionals and frequency model: delay-independent absolute stability criteria for delay systems. *International Journal of Robust and Nonlinear Control* **11**(8) (2001) 771–788.

[8] Borne P. and Benrejeb, M. On the representation and the stability study of large scale systems. *International Journal of Computers, Communications and Control* **3** (Suppl. Issue – ICCC 2008) (2008) 55–66.

[9] Borne, P. Non-linear systems stability : vector norm approach. *Systems and Control Encyclopedia.* T.5. Pergamon Press, 1987, 3402–3406.

[10] Borne, P., Vanheeghe, P. and Duflos, E. *Automatisation des processus dans l'espace d'état*, Editions Technip, Paris, 2007.

[11] Boyd, S., Ghaoui, L.E., Feron, E. and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia PA, 1994.

[12] Chow J. and Kokotovic, P. A decomposition of near-optimum regulators for systems with slow and fast modes. *IEEE Transactions on Automatic Control* **21**(5) (1976) 701–705.

[13] Ding, Y., Zhong, S. and Che, W. A delay-range-dependent uniformly asymptotic stability criterion for a class of nonlinear singular system. *Nonlinear Analysis: Real World Applications* **12**(2) (2011) 1152–1162.

[14] Doan, T.S., Kalauch, A. and Siegmund, S. Exponential Stability of Linear Time-Invariant Systems on Time Scales. *Nonlinear Dynamics and Systems Theory* **9**(1) (2009) 37–50.

[15] Ellouze, I., Ben Abdallah, A. and Hammami, M. A. On the Absolute Stabilization of Dynamical-Delay Systems. *Nonlinear Dynamics and Systems Theory* **10**(3) (2010) 225–234.

[16] El Kamel, A., Borne, P., Ksouri-Lahmari, M. and Benrejeb, M. On the stability of nonlinear multimodel systems. *SACTA* **2**(1-2) (1999) 40–52.

[17] El Madssia, S., Saadaoui, K. and Benrejeb, M. New stability conditions for neutral and retarded time delay systems. *International Review of Automatic and Control* **4** (2011) 204–210.

[18] Filali, R.L., Hammami, S., Benrejeb, M. and Borne, P. On synchronization, anti-synchronization and hybrid synchronization of 3D discrete generalized Hénon map. *Nonlinear Dynamics and Systems Theory* **12**(1) (2012) 81–96.

[19] Gentina, J.C., Borne, P. and Laurent, F. Stabilité des systèmes continus non linéaires de grande dimension. *RAIRO Revue jaune J3* (1972) 69–77.

[20] Grujic, L.T., Richard, J.P., Borne, P. and Gentina, J.C. Stability Domains, *Chapman and Hall*, London, 2004.

[21] Haddad, W.M. and Bernstein, D.S. Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle and Popov theorems and their application to robust stability part II: Discrete-time theory. *International Journal of Robust and Nonlinear Control* **4**(2) (1994) 249–265.

[22] Halanay, A. and Rasvan, V. Absolute stability of feedback systems with several differentiable nonlinearities. *International Journal of Systems Science* **22**(10) (1991) 1911–1927.

[23] Han, Q.L. Absolute stability of time-delay systems with sector-bounded nonlinearity. *Automatica* **41**(12) (2005) 2171–2176.

[24] Hu, T., Huang, B. and Lin, Z. Absolute stability with a generalized sector condition. *IEEE Transactions on Automatic Control* **49**(4) (2004) 535–548.

[25] Iervolino, R., Iannelli, L. and Vasca, F. Absolute stability of Lur'e systems: a complementarity and passivity approach. *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, USA, (2010) 2420–2425.

[26] Impram, S.T. and Munro, N. A note on absolute stability of uncertain systems. *Automatica* **37**(4) (2001) 605–610.

[27] Kafri, W.S. and Abed, E.H. Stability analysis of discrete-time singularly perturbed systems. *IEEE Transactions on Circuits and Systems* **43**(10) (1996) 848–850.

[28] Kando, H. and Iwazumi, T. Multirate digital control design of an optimal regulator via singular perturbation theory. *International Journal of Control* **44**(6) (1986) 1555–1578.

[29] Khalil, H.K. *Nonlinear Systems*. Prentice Hall, Upper Saddle River, New Jersey, 2002.

[30] Kotelyanski, D.M. Some properties of matrices with positive elements. *Matematicheskii Sbornik* **31**(73) (1952) 497–505. [in Russian]

[31] Lee, L. and Chen, J.L. Strictly positive real lemma and absolute stability for discrete-time descriptor systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **50**(6) (2003) 788–794.

[32] Lee, S.M. and Park, J.H. Delay-dependent criteria for absolute stability of uncertain time-delayed Lur'e dynamical system. *Journal of the Franklin Institute* **3**47(1) (2010) 146–153.

[33] Lee, S.M., Park, J.H. and Kwon, O.M. Improved asymptotic stability analysis for Lur'e systems with sector and slope restricted. *Nonlinearities Physics Letters A* **362** (5) (2007) 348–351.

[34] Li, T.H.S. and Li, J.H. Stabilization bound of discrete two-time-scale systems. *Systems & Control Letters* **18**(6) (1992) 479–489.

[35] Li, T.H.S., Chiou, J.S. and Kung, F.C. Stability bounds of singularly perturbed discrete systems. *IEEE Transactions on Automatic Control* **44**(10) (1999) 1934–1938.

[36] Lin, G., Balachandran B. and Abed E.H. Absolute stability of second-order systems with asymmetric sector boundaries. *IEEE Transactions on Automatic Control* **55**(2) (2010) 458–463.

[37] Litkouhi, B. and Khalil, H.K. Infinite-time regulators for singularly perturbed difference equations. *International Journal of Control* **39**(3) (1984) 587–598.

[38] Litkouhi, B. and Khalil, H.K. Multirate and composite control of two-time-scale discrete-time systems. *IEEE Transactions on Automatic Control* **30**(7) (1985) 645–651.

[39] Lur'e, A.I. and Postnikov, V.N. On the theory of stability of control systems. *Applied Mathematics and Mechanics* **8**(3) (1944) 246–248. [in Russian]

[40] Mahmoud, M.S. Design of Observer-based Controllers For a Class of Discrete Systems. *Automatica* **18**(3) (1982) 323–328.

[41] Park, P. Stability criteria of sector- and slope-restricted Lur'e systems. *IEEE Transactions on Automatic Control* **47**(2) (2002) 308–313.

[42] Park, P., Banjerdpongchai, D. and Kailath, T. The asymptotic stability of nonlinear (Lur'e) systems with multiple slope restrictions. *IEEE Transactions on Automatic Control* **43**(7) (1998) 979–982.

[43] Popov, V.M. Absolute stability of nonlinear systems of automatic control. *Automation and Remote Control* **22**(8) (1962) 857–875.

[44] Rasvan, V. New results and applications of the frequency domain criteria to absolute stability of nonlinear systems. *Qualitative Theory of Differential Equations* **53** (1988) 577–594.

[45] Robert, F. Recherche d'une M-matrice parmi les minorantes d'un opérateur linéaire. *Numerische Mathematik* **9**(3) (1966) 188–199.

[46] Robert, F. Normes vectorielles de vecteurs et de matrices. *Revue Française de Traitement de l'Information*, Chiffres **7**(4) (1964) 261–299.

[47] Saksena, V.R., O'Reilly J. and Kokotovic P.V. Singular perturbations and time-scale methods in control theory: Survey 1976-1983. *Automatica* **20**(3) (1984) 273–293.

[48] Singh, V. A new approach to the absolute stability of nonlinear feedback systems with slope-restricted nonlinearity. *IEEE Transactions on Circuits and Systems* **32**(4) (1985) 402–403.

[49] Van de Wouw, N., Doris, A., de Bruin, J.C.A., Heemels, W.P.M.H. and Nijmeijer H. Output-feedback control of Lur'e-type systems with set-valued nonlinearities: a Popov-criterion approach. *Proceedings of the American Control Conference, Seattle, USA*, (2008) 2316–2321.

[50] Vidyasagar, M. *Nonlinear System Analysis*. Prentice Hall, Englewood Cliffs, NJ, 1993.

[51] Wada, T., Ikeda, M., Ohta, Y. and Siljak, D.D. Parametric absolute stability of multivariable Lur'e systems. *Automatica* **36**(9) (2000) 1365–1372.

[52] Wang, H.J., Xue, A.K. and Lu, R.Q. Absolute stability criteria for a class of nonlinear singular system with time delay. *Nonlinear Analysis* **70**(2) (2009) 621–630.

[53] Yang, C., Zhang, Q. and Zhou, L. Strongly absolute stability problem of descriptor systems: Circle criterion. *Journal of the Franklin Institute* **345**(5) (2008) 437–451.

[54] Yang, C., Zhang, Q. and Zhou, L. Strongly absolute stability of Lur'e descriptor systems: Popov-type criteria. *International Journal of Robust and Nonlinear Control* **19**(7) (2009) 786–806.

[55] Yin, F., Yang, C. and Zhai, D. Strongly absolute stability of Lur'e-type discrete-time descriptorsystems with sector and slope restricted nonlinearities. In: *Proceedings of the Chinese Control and Decision Conference*. Mianyang, China, 2011, pp. 2704–2709.

[56] Zames, G. On the input-output stability of time-varying nonlinear feedback systems-Part II. *IEEE Transactions on Automatic Control* **11**(3) (1966) 465–476.

[57] Zhang, B., Lamb, J., Xu, S. and Shu, Z. Absolute exponential stability criteria for a class of nonlinear time-delay system. *Nonlinear Analysis: Real World Applications* **11**(3) (2010) 1963–1976.