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On Parameterized Lyapunov and Control Lyapunov Functions for Discrete-Time Systems

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Abstract: This paper deals with the existence and synthesis of parameterized-(control) Lyapunov functions (p-(C)LFs) for discrete-time nonlinear systems that are possibly subject to constraints. A p-LF is obtained by associating a finite set of parameters to a standard LF. A set-valued map, which generates admissible sets of parameters, is defined such that the corresponding p-LF enjoys standard Lyapunov properties. It is demonstrated that the so-obtained p-LFs offer non-conservative stability analysis conditions, even when Lyapunov functions with a particular structure, such as quadratic forms, are considered. Furthermore, possible methods for synthesizing p-CLFs are discussed. These methods require solving on-line a low-complexity convex optimization problem.

Keywords: *difference equations; asymptotic stability; Lyapunov methods; convex optimization.*

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1 Introduction

The problems considered in this paper are stability analysis and stabilizing controller synthesis via Lyapunov methods for discrete-time nonlinear systems that are possibly subject to constraints. It is well known that such methods rely on the existence and construction of a proper Lyapunov function (LF) [8, 11, 12, 19] and control Lyapunov function (CLF) [1, 9, 24], respectively. Unfortunately, the construction of LFs for general nonlinear systems is a very challenging problem. In particular, even linear systems with hard state/input constraints pose a non-trivial challenge to finding a non-conservative LF. As such, it would be desirable to identify a non-conservative class of Lyapunov

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functions that leads to a tractable implementation for nonlinear systems. As our interest lies mainly within the discrete-time domain, the following brief account of advances in Lyapunov methods is restricted to discrete-time systems.

For unconstrained linear systems, existence of a LF with a fixed structure (e.g., quadratic or polyhedral) and parameter set (e.g., common weight matrix for all states) is necessary and sufficient for stability. However, when constraints are present, the unconstrained solution usually provides a conservative domain of attraction. Moreover, when other classes of nonlinear systems are considered, such as systems with polytopic uncertainty, piecewise affine (PWA) or switched systems, existence of a fixed LF with a common set of parameters is known to be conservative. For such relevant classes of nonlinear systems it was already shown that enriching the set of admissible parameters for the Lyapunov weight matrix leads to a less conservative LF, even when the structure of the LF is fixed. For example, parameter dependent quadratic Lyapunov functions were constructed in [6] for uncertain linear systems by parameterizing the weight matrix of a quadratic LF as a function of the uncertain parameter. This idea was further used to construct switched quadratic LFs for switched systems in [7]. For recent results on parameter dependent Lyapunov functions for uncertain linear systems we refer the interested reader to [23], [4] and the references therein. A different type of relaxation was developed for PWA systems in [13]. To deal with a switching law defined by a state-space partition, the weight matrix of a quadratic LF was allowed to have different values (within a finite set of admissible matrices), which yielded a piecewise quadratic (PWQ) LF. More recently, a method to synthesize trajectory-dependent time-variant CLFs defined using the infinity norm was proposed in [18].

This paper continues on this line of research and proposes a definition of a parameterized LF (p-LF), without a fixed structure, that is applicable to general discrete-time nonlinear systems. The term parameterized LF denotes the fact that the LF candidate is endowed with a set of parameters, not necessarily structured in a particular form (e.g., a matrix of certain dimensions), which can take multiple values within an admissible set that depends on each state. As such, the Lyapunov conditions for stability can be formulated in terms of the set valued map that generates an admissible set of parameters for each state. In contrast to the set-up of [18], the conditions that define a p-LF are time-invariant. The non-conservatism of the proposed p-LFs, even with a fixed structure, is indicated by a converse theorem, which establishes that exponentially stable nonlinear systems always admit a p-quadratic LF. A corresponding definition of a parameterized control Lyapunov function (p-CLF) is also provided, which leads to several possibilities for synthesizing trajectory-dependent stabilizing control laws. Furthermore, it is shown that for p-quadratic-CLFs and input affine nonlinear systems, a synthesis solution based on solving on-line a single low-complexity semi-definite program (SDP) can be obtained, under the assumption of recursive feasibility.

2 Preliminaries

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \mathbb{R} \cap \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, let $\operatorname{int}(\mathcal{S})$ denote the interior of $\mathcal{S}$. A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. A polytope is a closed and bounded polyhedron. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|\cdot\|$ denotes an arbitrary p-norm, $p \in \mathbb{Z}_{\geq 1} \cup \infty$. Let $\|x\|_\infty := \max_{i=1,\ldots,n} |[x]_i|$ and $\|x\|_2 := \sqrt{\sum_{i=1}^n |[x]_i|^2}$, where $|\cdot|$ denotes the absolute value. $I_n \in \mathbb{R}^{n\times n}$ denotes the n-th dimensional identity matrix. For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). Moreover, $*$ is used to denote the symmetric part of a matrix. For the definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions we refer the reader to [11].

Next, consider the discrete-time autonomous system

$$x(k+1) = \Phi(x(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at the discrete-time instant k and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary map with $\Phi(0) = 0$.

Definition 2.1 Let $\lambda \in \mathbb{R}_{[0,1]}$. We call a set $\mathbb{X} \subseteq \mathbb{R}^n$ λ-*contractive* (or shortly, contractive) for system (1) if for all $x \in \mathbb{X}$ it holds that $\Phi(x) \in \lambda\mathbb{X}$. When this property holds with $\lambda = 1$ we call $\mathbb{X}$ a *positively invariant (PI)* set.

Definition 2.2 Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. We call system (1) asymptotically stable in $\mathbb{X}$, or shortly, AS($\mathbb{X}$), if there exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that, for each $x(0) \in \mathbb{X}$ it holds that the corresponding state trajectory of (1) satisfies $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{Z}_+$. We call system (1) exponentially stable in $\mathbb{X}$, or shortly, ES($\mathbb{X}$), if $\beta(s,k) := \theta\mu^k s$ for some $\theta \in \mathbb{R}_{\geq 1}$, $\mu \in \mathbb{R}_{[0,1)}$.

Theorem 2.1 [11,14] *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with $0 \in \text{int}(\mathbb{X})$. Furthermore, let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$ and let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \ \forall x \in \mathbb{X}, \tag{2a}$$

$$V(\Phi(x)) \leq \rho V(x), \quad \forall x \in \mathbb{X}. \tag{2b}$$

Then system (1) *is AS($\mathbb{X}$).*

A function V that satisfies (2) is called a *Lyapunov function* and ρ is called the *rate of decrease* of V. Notice that in discrete-time, continuity of the dynamics or Lyapunov function is not necessary (except at the origin) for stability, as pointed out in [14]. As such, in what follows we do not explicitly require this property. However, as pointed out recently in [17], one must take additional precautions with respect to inherent robustness, when discontinuous Lyapunov functions are involved.

3 Parameterized Lyapunov Functions

Let $\mathbb{P}$ denote a set of parameter sets, where each parameter set (or element of $\mathbb{P}$) contains a finite number of parameters with an arbitrary structure, e.g., a parameter set or element in $\mathbb{P}$ can be a matrix of certain fixed dimensions. Let us now define a function $V : \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}_+$, which is zero at zero for all elements in $\mathbb{P}$. Next, let $(P_1, P_2) \in \mathbb{P} \times \mathbb{P} =: \mathbb{P}^2$ and consider the following inequalities for some $x \in \mathbb{X}$:

$$\alpha_1(\|x\|) \leq V(x, P_1) \leq \alpha_2(\|x\|), \tag{3a}$$

$$V(\Phi(x), P_2) \leq \rho V(x, P_1). \tag{3b}$$

Consider the set-valued map $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{P} \times \mathbb{P}$,

$$\mathcal{P}(x) := \{(P_1, P_2) \in \mathbb{P}^2 \mid \text{(3a) and (3b) hold}\}. \tag{4}$$

For any $x \in \mathbb{X}$, $\mathcal{P}(x) \neq \emptyset$ denotes the fact that there exists at least one pair $(P_1, P_2) \in \mathbb{P}^2$ that satisfies (3). To distinguish between the two outputs of $\mathcal{P}$ we will use $[\mathcal{P}(x)]_1$ and $[\mathcal{P}(x)]_2$ to denote the sets where the first and the second component of a pair $(P_1, P_2) \in \mathbb{P}^2$ that satisfies (3) take values, respectively. $[\mathcal{P}(x)]_\bullet$ denotes an element of $\mathcal{P}(x)$. With a slight abuse of notation we will use $P(x)$ to denote any $P_1 \in [\mathcal{P}(x)]_1$.

Definition 3.1 A function $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ is called a parameterized Lyapunov function (p-LF) in $\mathbb{X} \subseteq \mathbb{R}^n$ for system (1) if

$$\mathcal{P}(x) \neq \emptyset, \quad \forall x \in \mathbb{X}, \tag{5a}$$

$$[\mathcal{P}(x)]_2 \cap [\mathcal{P}(\Phi(x))]_1 \neq \emptyset, \quad \forall x \in \mathbb{X}. \tag{5b}$$

Theorem 3.1 *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with $0 \in \operatorname{int}(\mathbb{X})$. Suppose that system* (1) *admits a parameterized Lyapunov function in $\mathbb{X}$. Then system* (1) *is AS($\mathbb{X}$).*

Proof. The claim is proven using standard arguments [8,11,14]. From (5a) we obtain that for all $x \in \mathbb{X}$

$$\alpha_1(\|x\|) \leq V(x, P(x)) \leq \alpha_2(\|x\|), \ \forall P(x) \in [\mathcal{P}(x)]_1.$$

From (5b) we obtain that for all $x \in \mathbb{X}$ there exists at least one pair $(P(x), P_2) \in \mathcal{P}(x)$ such that $P_2 \in [\mathcal{P}(\Phi(x))]_1$, which yields that

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) \leq 0, \quad \forall x \in \mathbb{X},$$

with $P(\Phi(x)) = P_2 \in [\mathcal{P}(\Phi(x))]_1$ and $P(x) \in [\mathcal{P}(x)]_1$. As $\mathbb{X}$ is a PI set, the above inequality can be applied recursively for any trajectory $\{x(k)\}_{k \in \mathbb{Z}_+}$ with $x(0) \in \mathbb{X}$, which yields:

$$\begin{aligned} \alpha_1(\|x(k+1)\|) &\leq V(\Phi(x(k)), P(\Phi(x(k)))) \\ &\leq \rho^{k+1} V(x(0), P(x(0))) \leq \rho^{k+1} \alpha_2(\|x(0)\|), \end{aligned}$$

for all $x(0) \in \mathbb{X}$. Hence, $\|x(k)\| \leq \beta(\|x(0)\|, k)$ for all $x(0) \in \mathbb{X}$, where $\beta(s, k) := \alpha_1^{-1}(\rho^k \alpha_2(s)) \in \mathcal{KL}$, which completes the proof. $\square$

To illustrate the relaxation with respect to existing approaches, consider the case when one adds a particular structure to the parameter set $\mathbb{P}$ and the candidate p-LF. As such, let us consider p-quadratic-LFs, defined as $V(x, P(x)) := x^\top P(x) x$, $P(x) \in [\mathcal{P}(x)]_1$, $\mathcal{P}(x) \subseteq \mathbb{P}^2$ for all x, where $\mathbb{P} \subseteq \mathbb{R}^{n \times n}$. Consider now the case of a PWA system with a fixed switching law defined by a partition of the state-space, i.e., $\{\Omega_j\}_{j \in \mathcal{S}}$, with $\mathcal{S}$ a finite set of indices, and let $\mathbb{P} := \{P_i\}_{i \in \mathcal{S}}$, $P_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathcal{S}$. If one sets $\mathcal{P}(x) := \{(P_i, P_j)\}$ for all $(x, \Phi(x)) \in \Omega_i \times \Omega_j$ and imposes (3a) for all $x \in \Omega_j$ and (3b) for all $(x, \Phi(x)) \in \Omega_i \times \Omega_j$, $(i, j) \in \mathcal{S} \times \mathcal{S}$, one obtains a PWQ Lyapunov function with an $\mathcal{S}$-procedure relaxation [13], as a particular case of a p-quadratic-LF. Similarly, it can be shown that quadratic periodic Lyapunov functions [3] are a particular case of p-quadratic-LFs. Moreover, it can be shown that parameter dependent Lyapunov functions form a particular type of p-LFs, by allowing the map $\mathcal{P}$ to depend on both the state and the uncertain parameter. It would be interesting to further relate p-LFs with polynomial Lyapunov functions, which can be obtained if $P(x)$ is allowed to be a particular polynomial of x. Then, the map $\mathcal{P}(x)$ would assign the coefficients of the polynomial. As the relation to polynomial LFs is beyond the scope of this paper, we will not pursue it any further.

The following converse result reveals the non-conservatism of p-LFs, even when a particular structure is imposed. We will consider two of the most popular type of structures for candidate LFs, i.e., p-quadratic-LFs and p-polyhedral-LFs, defined as $V(x, P(x)) := \|P(x)x\|_\infty$, $P(x) \in [\mathcal{P}(x)]_1$, $\mathcal{P}(x) \subseteq \mathbb{P}^2$ for all x, where $\mathbb{P} \subseteq \mathbb{R}^{r\times n}$ ($r \in \mathbb{Z}_{\geq n}$). Let $\mathcal{N}$ denote an arbitrary neighborhood of the origin (i.e., a bounded set with a non-empty interior that contains the origin in its interior).

Assumption 3.1 *There exists a positively invariant neighborhood of the origin $\mathcal{N}$ such that system* (1) *admits a p-quadratic-LF (p-polyhedral-LF) in $\mathcal{N}$.*

The above assumption is reasonable, as most nonlinear systems can be approximated around the origin by a linear system, PWA system or a polytopic difference inclusion and then one can use the above indicated results to obtain a local p-quadratic-LF. Next, suppose that system (1) is either ES($\mathbb{X}$) or AS($\mathbb{X}$) and, as such, by a standard converse theorem, see, e.g., Theorem 1 in [12] or Lemma 4 in [20] (AS) and Theorem 2 (ES) in [12], it admits a Lyapunov function in $\mathbb{X}$. Notice that the above-mentioned converse theorems require AS($\mathbb{R}^n$). In what follows we implicitly assume that these results can be applied to an invariant subset $\mathbb{X}$ of $\mathbb{R}^n$.

Let V_1 denote a LF established by a converse theorem and let $V_L(x, P_L(x))$ with $P_L(x) \in [\mathcal{P}_L(x)]_1$ for some $\mathcal{P}_L(x) \subseteq \mathbb{P}^2$ denote a p-quadratic-LF (or p-polyhedral-LF) in $\mathcal{N}$.

Theorem 3.2 *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a PI set for* (1) *with* $0 \in \text{int}(\mathbb{X})$.

(i) *Suppose that system* (1) *is ES($\mathbb{X}$). Then, there exists a p-quadratic-LF in $\mathbb{X}$ for system* (1).

(ii) *Suppose that system* (1) *is AS($\mathbb{X}$), Assumption 3.1 holds and there exists a $c \in \mathbb{R}_{(0,1]}$ such that $V_1(x) \geq cV_L(x, P_L(x))$ for all $x \in \mathcal{N}$ and all $P_L(x) \in [\mathcal{P}_L(x)]_1$. Then, there exists a p-quadratic-LF (p-polyhedral-LF) in $\mathbb{X}$ for system* (1).

Proof. Let us begin with the proof of *(i)*. As system (1) is ES($\mathbb{X}$), by Theorem 2 in [12] it admits a standard Lyapunov function V_1 that satisfies (2) for all $x \in \mathbb{X}$. Moreover, V_1 satisfies (2a) with $\alpha_1(s) := s^2$ and $\alpha_2(s) := ls^2$ for some $l \in \mathbb{R}_{\geq 1}$. Using this function define

$$\mathcal{P}(x) := \left\{ \left(\frac{V_1(x)}{\|x\|_2^2} I_n, \frac{V_1(\Phi(x))}{\|\Phi(x)\|_2^2} I_n \right) \right\}, \quad \forall x \in \mathbb{X}. \tag{6}$$

Note that $\Phi(x) \in \mathbb{X}$ for all $x \in \mathbb{X}$ and

$$l = \frac{\alpha_2(\|x\|)}{\|x\|_2^2} \geq \frac{V_1(x)}{\|x\|_2^2} \geq \frac{\alpha_1(\|x\|)}{\|x\|_2^2} = 1, \quad \forall x \in \mathbb{X}.$$

Thus, $\mathcal{P}(x)$ is well-defined for all $x \in \mathbb{X}$. Observing that the candidate p-quadratic-LF $V(x, P(x)) := x^\top P(x)x$ with $P(x) \in [\mathcal{P}(x)]_1$ satisfies $V(x, P(x)) = V_1(x)$ for all $x \in \mathbb{X}$ and V_1 is a LF in $\mathbb{X}$ for system (1) completes the proof.

Consider now hypothesis *(ii)*. As system (1) is AS($\mathbb{X}$), by Theorem 1 in [12] it admits a standard Lyapunov function V_1 that satisfies (2) for all $x \in \mathbb{X}$. Using this function define $\bar{P}(x) := \frac{V_1(x)}{\|x\|_2^2} I_n$ for a p-quadratic-LF, or $\bar{P}(x) := \frac{V_1(x)}{\|x\|_\infty} I_n$ for a p-polyhedral-LF. Note that, as $\frac{V_1(x)}{\|x\|_2^2} \leq \frac{\alpha_2(\|x\|)}{\|x\|_2^2}$ for all $x \in \mathbb{X}$ and $\mathcal{N}$ is bounded and contains the origin in its interior, $\frac{V_1(x)}{\|x\|_2^2}$ is well defined for all $x \in \mathbb{X} \setminus \mathcal{N} =: \mathbb{X}_{\mathcal{N}}$. Similarly, $\frac{V_1(x)}{\|x\|_\infty}$ is well defined for all $x \in \mathbb{X}_{\mathcal{N}}$. Next, consider the candidate p-quadratic-LF (p-polyhedral-LF) $V(x, P(x)) = x^\top P(x)x$ ($V(x, P(x)) = \|P(x)x\|_\infty$), where $P(x) \in [\mathcal{P}(x)]_1$ and

$$\mathcal{P}(x) := \begin{cases} c\mathcal{P}_L(x), & \forall (x, \Phi(x)) \in \mathcal{N}^2, \\ \{(\bar{P}(x), c[\mathcal{P}_L(\Phi(x))]_1)\}, & \forall (x, \Phi(x)) \in \mathbb{X}_{\mathcal{N}} \times \mathcal{N}, \\ \{(\bar{P}(x), \bar{P}(\Phi(x)))\}, & \forall (x, \Phi(x)) \in (\mathbb{X}_{\mathcal{N}})^2. \end{cases} \tag{7}$$

Then, for all $(x, \Phi(x)) \in (\mathbb{X}_{\mathcal{N}})^2$, as $V(x, P(x)) = V_1(x)$ for all $x \in \mathbb{X}_{\mathcal{N}}$, (2b) yields

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) = V_1(\Phi(x)) - \rho V_1(x) \leq 0.$$

Moreover, for all $(x, \Phi(x)) \in \mathbb{X}_{\mathcal{N}} \times \mathcal{N}$, (2b) also yields

$$\begin{aligned} V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)) & \\ &= cV_L(\Phi(x), P_L(\Phi(x))) - \rho V_1(x) \\ &\leq V_1(\Phi(x)) - \rho V_1(x) \leq 0, \end{aligned}$$

for all $P_L(\Phi(x)) \in [\mathcal{P}_L(\Phi(x))]_1$. As $\mathcal{N}$ is a PI set for system (1), the last case to be analyzed is when $(x, \Phi(x)) \in \mathcal{N}^2$. Then

$$\begin{aligned} V(\Phi(x), & P(\Phi(x))) - \rho V(x, P(x)) \\ &= c(V_L(\Phi(x), P_L(\Phi(x))) - \rho V_L(x, P_L(x))) \leq 0, \end{aligned}$$

where $P_L(x) \in [\mathcal{P}_L(x)]_1$ for all $x \in \mathcal{N}$. Thus, we conclude that $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ for all $x \in \mathbb{X}$ and $\mathcal{P}(x)$ as defined in (7) satisfies

$$V(\Phi(x), P(\Phi(x))) - \rho V(x, P(x)), \quad \forall x \in \mathbb{X}.$$

Observing that

$$\alpha_{1,L}(\|x\|) \leq V_L(x, P_L(x)) \leq \alpha_{2,L}(\|x\|), \, \forall x \in \mathcal{N},$$

for some $\alpha_{1,L}, \alpha_{2,L} \in \mathcal{K}_\infty$, yields that

$$\alpha_{1,p}(\|x\|) \leq V(x, P(x)) \leq \alpha_{2,p}(\|x\|), \, \forall x \in \mathbb{X},$$

where $\alpha_{1,p}(s) := \min(\alpha_1(s), c\alpha_{1,L}(s)) \in \mathcal{K}_\infty$ and $\alpha_{2,p}(s) := \alpha_2(s) \in \mathcal{K}_\infty$. This further implies that $V(x, P(x))$ (i.e., the constructed p-quadratic-LF or p-polyhedral-LF candidate) satisfies the conditions of Definition 3.1, which completes the proof. □

For clarity of exposition, in this section we have considered discrete-time systems of the form (1) that are described by a difference equation. However, all the developed results apply *mutatis mutandis* to the case when $\Phi(x)$ is a compact and non-empty set-valued map and yield *strong* asymptotic stability in $\mathbb{X}$ (i.e., AS($\mathbb{X}$) for all possible trajectories generated by the set-valued map). Then, the converse theorem in [10] should be used instead of the ones in [12]. In the next section we will deal with a difference equation that involves a set-valued control input and refer to the results established in this section, as in fact, these results hold for difference inclusions as well.

4 Parameterized Control Lyapunov Functions

Consider the discrete-time system

$$x(k+1) = \phi(x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{8}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input, $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary map with $\phi(0,0) = 0$ and $\mathbb{X}$, $\mathbb{U}$ contain the origin in their interior.

Definition 4.1 We call a set $\mathbb{X} \subseteq \mathbb{R}^n$ constrained control invariant with respect to $\mathbb{U}$ (CCI($\mathbb{X}$,$\mathbb{U}$)) for system (8) if for all $x \in \mathbb{X}$, $\exists u \in \mathbb{U}$ such that $\phi(x,u) \in \mathbb{X}$.

Assumption 4.1 *$\mathbb{X} \subseteq \mathbb{R}^n$ is a CCI($\mathbb{X}$,$\mathbb{U}$) set for the discrete-time system* (8)*.*

Notice that the above assumption is made only for ease of exposition. If $\mathbb{X}$ is not a CCI($\mathbb{X}$,$\mathbb{U}$), the results simply apply for the largest subset of $\mathbb{X}$ with this property.

Next, consider the following inequalities corresponding to (3) for some $x \in \mathbb{X}$:

$$\alpha_1(\|x\|) \leq V(x, P_1) \leq \alpha_2(\|x\|), \tag{9a}$$

$$V(\phi(x,u), P_2) \leq \rho V(x, P_1). \tag{9b}$$

Consider the set-valued map $\mathcal{P} : \mathbb{R}^n \rightrightarrows \mathbb{P} \times \mathbb{P}$,

$$\mathcal{P}(x) := \{(P_1, P_2) \in \mathbb{P}^2 \mid \exists u \in \mathbb{U} \text{ s.t. (9) holds}\}. \tag{10}$$

Furthermore, let $\pi : \mathbb{R}^n \times \mathbb{P} \times \mathbb{P} \rightrightarrows \mathbb{U}$ denote

$$\pi(x, [\mathcal{P}(x)]_\bullet) := \{u \in \mathbb{U} \mid \text{(9) holds for } [\mathcal{P}(x)]_\bullet\}.$$

Definition 4.2 A function $V(x, P(x))$ with $P(x) \in [\mathcal{P}(x)]_1$ is called a parameterized control Lyapunov function (p-CLF) in $\mathbb{X}$ for system (8) if

$$\mathcal{P}(x) \neq \emptyset, \quad \forall x \in \mathbb{X}, \tag{11a}$$

$$\exists [\mathcal{P}(x)]_\bullet \in \mathcal{P}(x), \exists u \in \pi(x, [\mathcal{P}(x)]_\bullet) \text{ s.t.}$$
$$[\mathcal{P}(x)]_2 \cap [\mathcal{P}(\phi(x,u))]_1 \neq \emptyset, \quad \forall x \in \mathbb{X}. \tag{11b}$$

In what follows we will focus on the synthesis of p-CLFs. Although these methods will also provide useful insights for stability analysis via synthesis of p-LFs, exploring this path further is beyond the scope of this paper.

Next, we will formulate an optimization problem to be solved on-line that yields a trajectory-dependent p-CLF (td-p-CLF). By trajectory-dependent we mean that the computed sequence of parameter sets $\{P(x(k))\}_{k \in \mathbb{Z}_+}$, with $P(x(k)) \in [\mathcal{P}(x(k))]_1$ for all $k \in \mathbb{Z}_+$, will only be valid along the trajectory $\{x(k)\}_{k \in \mathbb{Z}_+}$. The advantage of this approach is that the non-conservatism of a p-CLF is preserved. The challenge, which is common to all optimization based controllers, is to guarantee recursive feasibility. Unfortunately, the problem of constructing a set of *a priori* verifiable conditions for recursive feasibility is non-trivial and it is not solved in this paper. Instead, we propose a heuristic solution for attaining recursive feasibility, which requires minimization of the decrease rate of the td-p-CLF. Simulations conducted on several challenging case studies indicate that *not* enforcing a steep decrease of the p-CLF is beneficial in terms of recursive feasibility. This is in contrast with the classical CLF approach of [1], where the optimal decrease is required.

Let the structure, e.g., p-quadratic-CLF, the set of parameter sets $\mathbb{P}$, the functions α_1, α_2 and the rate of decrease $\rho \in \mathbb{R}_{[0,1)}$ of an arbitrary candidate p-CLF V be specified.

Problem 4.1 Let $x(k) \in \mathbb{X}$ be known at each $k \in \mathbb{Z}_+$. Let $x^+(k) := \phi(x(k), u(k))$ for all $k \in \mathbb{Z}_+$ and consider the following inequalities:

$$x^+(k) \in \mathbb{X},\ \ u(k) \in \mathbb{U}, \tag{12a}$$

$$\alpha_1(\|x(k)\|) \leq V(x(k), P(x(k))) \leq \alpha_2(\|x(k)\|), \tag{12b}$$

$$\alpha_1(\|x^+(k)\|) \leq V(x^+(k), P(x^+(k))) \leq \alpha_2(\|x^+(k)\|), \tag{12c}$$

$$V(x^+(k), P(x^+(k))) \leq \rho V(x(k), P(x(k))). \tag{12d}$$

If $k = 0$ find a $u(0) \in \mathbb{U}$ and a $(P(x^+(0)), P(x(0))) \in \mathbb{P}^2$ that satisfy (12). If $k \in \mathbb{Z}_{\geq 1}$ *set* $P(x(k)) = P(x^+(k-1))$ and find a $u(k) \in \mathbb{U}$ and a $P(x^+(k)) \in \mathbb{P}$ that satisfy (12a)-(12c)-(12d). □

In the above problem, $x^+(k)$ can be interpreted as the one-step ahead prediction calculated at time $k \in \mathbb{Z}_+$ using the measured state $x(k)$, the input $u(k)$ and the plant model $\phi(\cdot, \cdot)$. Obviously, in the ideal case $x^+(k-1) = x(k)$ and then the assignment $P(x(k)) = P(x^+(k-1))$ becomes redundant. However, this assignment plays a very important role if a perturbation $w \in \mathbb{R}^n$ acts on state $x^+(k-1)$, which yields $x(k) = x^+(k-1) + w$. In this case, by setting $P(x(k)) = P(x^+(k-1))$, one can exploit continuity of $V(\cdot, P(x^+(k-1)))$ in its first argument to establish inherent input to state stability [11, 17].

Next, let us propose a cost function that penalizes the decrease of the p-CLF. Let $x^+ := \phi(x, u)$ and let $\bar{\phi}(x) := \{\phi(x, u) \mid u \in \pi(x, [\mathcal{P}(x)]_\bullet),\ [\mathcal{P}(x)]_\bullet \in \mathcal{P}(x)\}$. Furthermore, suppose that we augment Problem 4.1 with the following cost function that guides the choice of $(u(k), P(x^+(k)))$. For any known $x(k) \in \mathbb{X}$ and $P(x(k)) = P(x^+(k-1))$, $k \in \mathbb{Z}_{\geq 1}$, consider the cost function

$$J(x(k), u(k), P(x^+(k))) := \rho V(x(k), P(x(k))) - V(x^+(k), P(x^+(k))) \tag{13}$$

and let

$$(u^*(k), P^*(x^+(k))) := \arg \inf_{u \in \pi(x(k), [\mathcal{P}(x(k))]_\bullet), P \in [\mathcal{P}(x(k))]_2} J(x(k), u, P)$$

denote the corresponding infimizer. Notice that due to (12d) J is bounded by zero from below and thus, the infimum is a minimum. For brevity we assume the minimum is attainable, which is true if V and ϕ are continuous in both arguments, respectively, and $\pi(x(k), [\mathcal{P}(x(k))]_\bullet) \subseteq \mathbb{U}$, $[\mathcal{P}(x(k))]_2 \subseteq \mathbb{P}$ are compact sets for all $k \in \mathbb{Z}_+$, and unique. Alternatively, one can always infimize J over $u \in \mathbb{U}$ and $P \in \mathbb{P}$, for some known compact sets $\mathbb{U}$ and $\mathbb{P}$. Several examples are presented in the next section, which illustrate the benefits of augmenting Problem 4.1 with the cost J, as defined in (13).

Remark 4.1 The p-CLFs defined in this section, along with the corresponding synthesis problem, i.e., Problem 4.1, bring a significant relaxation with respect to the trajectory dependent time-variant CLF construction proposed in [18]. The conditions imposed on the td-CLF therein translate into $P_1 = P_2$ for all $(P_1, P_2) \in \mathcal{P}(x)$, for all $x \in \mathbb{X}$. With respect to Problem 4.1 these conditions would require that for each $k \in \mathbb{Z}_+$, $P(x^+(k)) = P(x(k))$, which is obviously more conservative. It should be mentioned that the benefit of the conditions in [18] is that the corresponding Problem 4.1 can be rendered tractable for a polyhedral V as well. □

It is also worth to point out that the concept of a state-dependent Riccati equation [5] can be related to a particular setting of the proposed parameterized Lyapunov inequality, i.e., to a corresponding parameterized Lyapunov equation.

4.1 Synthesis of p-quadratic-CLFs

In what follows we will restrict our attention to input affine nonlinear systems, i.e.,

$$\phi(x, u) := f(x) + g(x)u \tag{14}$$

for some $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with $f(0) = 0$. Also, we will consider p-quadratic-CLF candidates of the form $V : \mathbb{R}^n \times \mathbb{P} \to \mathbb{R}_+$,

$$V(x, P(x)) = x^\top P(x)x,\ P(x) \in [\mathcal{P}(x)]_1,\ \mathcal{P}(x) \subseteq \mathbb{P}^2,$$

where $\mathbb{P} = \mathbb{R}^{n\times n}$. Notice that such a function satisfies $V(0, P) = 0$ for all $P \in \mathbb{P}$, but it does not already satisfy (9a). Next, we will present an LMI based formulation of Problem 4.1. Let $\gamma \in \mathbb{R}_{>0}$ and $\Gamma \in \mathbb{R}_{\geq\gamma}$ denote positive constants and suppose that $\mathbb{X}$ and $\mathbb{U}$ are polytopes. As such, constraint (12a) becomes a set of linear inequalities in $u(k)$ for each $x(k)$, $k \in \mathbb{Z}_+$. So, we will only focus on fulfillment of the inequalities (12b), (12c) and (12d). Consider now the following inequalities

$$\begin{aligned} x(k)^\top(P(x(k)) - \gamma I_n)x(k) &\geq 0,\\ x(k)^\top(\Gamma I_n - P(x(k)))x(k) &\geq 0, \end{aligned} \tag{15a}$$

$$Z(k) - \Gamma^{-1} I_n \succeq 0,\ \gamma^{-1} I_n - Z(k) \succeq 0, \tag{15b}$$

$$\begin{pmatrix} \rho x(k)^\top P(x(k))x(k) & * \\ f(x(k)) + g(x(k))u(k) & Z(k) \end{pmatrix} \succeq 0. \tag{15c}$$

Lemma 4.1 *Let $k \in \mathbb{Z}_+$ and let $x(k) \in \mathbb{X}$, γ, Γ and ρ be known. Suppose that $\{u(k), P(x(k)), Z(k)\}$ are a feasible solution of the LMI* (15)*. Then, $V(x(k), P(x(k))) = x(k)^\top P(x(k))x(k)$, $P(x^+(k)) = Z^{-1}(k)$ and $u(k)$ are a feasible solution of* (12b)*,* (12c) *and* (12d) *with $\alpha_1(s) := \gamma s^2$ and $\alpha_2(s) := \Gamma s^2$.*

Proof. Notice that (15a) is equivalent to (12b) for the specified α_1, α_2 and, by applying the Schur complement to (15c) one obtains (12d). (15b) yields that $\Gamma I_n \succeq P(x^+(k)) = Z^{-1}(k) \succeq \gamma I_n$. Thus, (12c) holds for the specified α_1, α_2, which completes the proof. □

The advantage of the solution of Lemma 4.1 is that (15c) offers a translation of the decreasing condition (12d) that does not introduce any conservatism. However, (15c) yields $P(x^+(k)) \succeq 0$, which is not necessary for (12c) to hold.

Notice that the resulting receding horizon control law is stabilizing only if the corresponding optimization problem is recursively feasible. In that respect, minimization of the cost (13) is advised. For example, using some non-trivial facts about positive semi-definite matrices it can be proven that by adding the LMI

$$\varepsilon(k)I_{n+1} - \begin{pmatrix} \rho x(k)^\top P(x(k))x(k) & * \\ f(x(k)) + g(x(k))u(k) & Z(k) \end{pmatrix} \succeq 0$$

to (15) and minimizing $\varepsilon(k)$ at each $k \in \mathbb{Z}_+$, minimization of the cost J in (13) is attained.

5 Illustrative Examples

In this section we present several examples of nonlinear systems that pose a non-trivial challenge to the problem of synthesizing a stabilizing control law. For each example we will provide a plot of the state trajectory and of the sub-level sets $\{z \in \mathbb{R}^n \mid V(z, P(x(k))) \leq 1\}_{k\in\mathbb{Z}_+}$ in Figure 1 and Figure 2, respectively.

Example 5.1 The first example consists of an uncertain linear system defined by

$$x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} \delta(k) \\ 1 \end{pmatrix} u(k), \quad k \in \mathbb{Z}_+,$$

where $\delta(k) \in \mathbb{R}_{[-c,c]}$ for all $k \in \mathbb{Z}_+$ ($c \in \mathbb{R}_{>0}$) is an unknown time-varying parameter. If $c \leq 1$, the system admits a quadratic CLF. However, for any $c > 1$, this no longer holds, i.e., *this system exhibits an infinite gap in the existence of a (robust) quadratic CLF.* However, the uncertain system does admit a parameter dependent quadratic CLF, which can be computed as shown in [6], but the implementation of the corresponding control law requires knowledge of $\delta(k)$, for all $k \in \mathbb{Z}_+$. To design a stabilizing controller for the above system with $c = 1.15$ we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.99$. In (15c) we made use of the extreme realizations $\delta(k) = 1.15$ and $\delta(k) = -1.15$ for all $k \in \mathbb{Z}_+$. Notice that this is sufficient for (15c) to hold for all $\delta(k) \in \mathbb{R}_{[-c,c]}$. To optimize convergence, we added the one-step cost $J_1 + J$ to (15), with $J_1(x(k), u(k)) := x^+(k)^\top Q x^+(k)$ ($Q = I_2$) and J defined as in (13), which still allows a conversion into a SDP. Only the extreme realizations of $\delta(k)$ were used to implement minimization of the above cost. A state trajectory plot obtained for $x(0) = [4 \ \ -4]^\top$ is given in Figure 1.

Example 5.2 The second example is taken from [14,18] and it consists of *a piecewise linear system that does not admit a common quadratic or PWQ CLF*. For brevity, we refer to the above references for the numerical details regarding the system. As shown in [14] the problem of computing such a CLF requires solving a bilinear matrix inequality. To design a stabilizing controller for this system we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.9$. The cost J as defined in (13) was added to (15). A state trajectory plot obtained for $x(0) = [5 \ \ -5]^\top$ is given in Figure 1.

Example 5.3 The third example is taken from [15] and it consists of a nonlinear system subject to state and input constraints. This system corresponds to (8)-(14) with $\mathbb{X} = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 5\}$, $\mathbb{U} = \{u \in \mathbb{R} \mid |u| \leq 1\}$ and

$$f(x) = \begin{pmatrix} [x]_1 + 0.7[x]_2 + ([x]_2)^2 \\ [x]_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0.245 + \sin([x]_2) \\ 0.7 \end{pmatrix}.$$

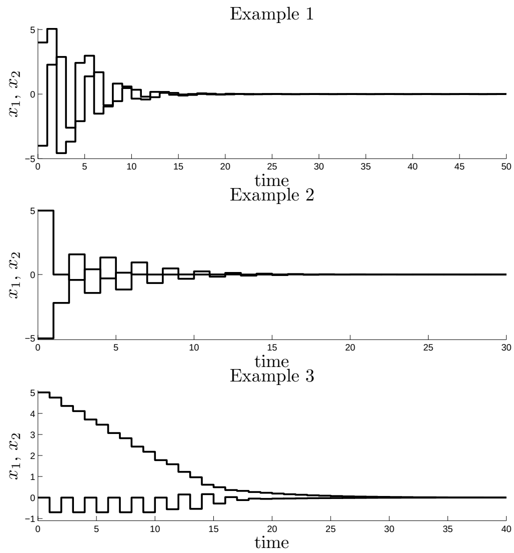

Figure 1: Simulation results – State trajectories.

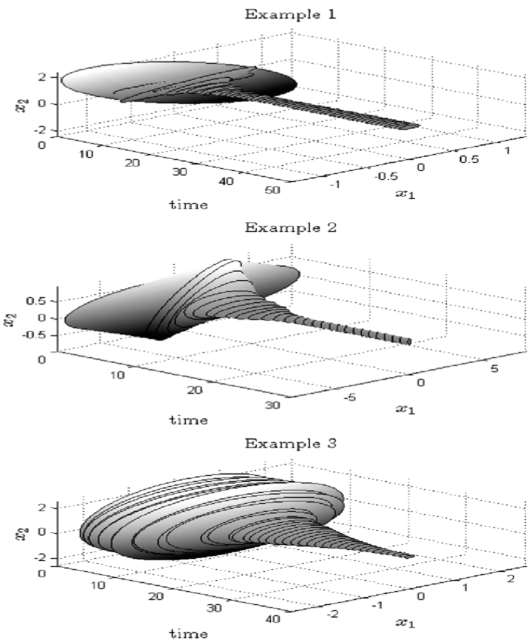

Figure 2: Simulation results – Evolution in time of the sub-level sets of V.

To design a stabilizing controller for this system we made use of (15). The following constants were chosen: $\gamma = 0.01$, $\Gamma = 100$, $\rho = 0.8$. To optimize convergence and improve feasibility, we added the cost $J_1 + J$ to (15). A state trajectory plot obtained for $x(0) = [5\ 0]^\top$, which lies on the boundary of $\mathbb{X}$, is given in Figure 1. Notice that input and state constraints are fulfilled at all times. In [15], *a non-monotone* CLF with a fixed parameter set was employed to stabilize the system for a similar initial condition.

6 Conclusions

This paper has provided results on existence and preliminary results on synthesis of parameterized-(control) Lyapunov functions for discrete-time nonlinear systems that are possibly subject to constraints. A p-LF was defined by assigning a finite set of parameters to a standard LF, which can take different values for each state. It was demonstrated that the so-obtained p-LFs offer non-conservative stability analysis conditions, even when Lyapunov functions with a particular structure, such as quadratic forms, are considered. Furthermore, a method for synthesizing p-CLFs for discrete-time nonlinear systems was proposed. It was shown that this method can be implemented by solving on-line a single low-complexity semi-definite program. Deriving *a priori* verifiable conditions under which the developed synthesis method yields a recursively feasible optimization problem makes the object of future research.

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