Nonlinear Dynamics and Systems Theory, **13** (2) (2013) 135–145

A Separation Principle of a Class of Time-Varying Dynamical Systems

H. Damak, I. Ellouze and M.A. Hammami *

* *University of Sfax, Faculty of sciences of Sfax, Departement of Mathematics Route Soukra Km 3.5, BP 1171- 3000 Sfax, Tunisie.*

Received: June 28, 2012; Revised: March 20, 2013

Abstract: This paper studies the separation principle for a class of nonlinear time-varying dynamical systems whose dynamics are in general bounded in time. The resultant observer-based state feedback control guarantees practical stability of the state oscillation given that the system is both uniformly controllable and observable. Our separation principle relies on stability results for cascades systems.

Keywords: *nonlinear differential equations; stabilization; Lyapunov functions; practical stability; Riccatti equation.*

Mathematics Subject Classification (2010): 34H15, 37B25.

1 Introduction

The stability problem of nonlinear time-varying systems has attracted the attention of several authors and has produced many important results [8], [11], [12], [13] and [14] and the references therein. The problem of state trajectory control for nonlinear systems by output feedback has received much attention. For systems with non-periodically time-varying parameters, an output feedback control design is proposed in [4] for linear time-varying systems based on the gradient algorithm. In [5], a new design is proposed for the state feedback control of multivariable linear time-varying systems. The new design is based on inversion state transformation and a forward differential Riccati equation.

The condition that we impose on the globally stabilizing state feedback control law is that it does not vanish asymptotically for large values. Then, we will give a separation principle based on analysis results for cascaded systems, as done for instance in [1] , [2], [3], [6], [7], [9] and [10]. However, in contrast to [11] we stress that our results will be formulated for time-varying systems and hence are applicable to tracking problems. Moreover as mentioned above, in [15] the author imposes the more restrictive assumption ISS. Our cascades criteria lead to milder conditions.

The main contribution of this paper is the separation principle of nonlinear systems by a linear output feedback under a generalized conditions. A practical stability approach is obtained. Furthermore, we give an example to show the applicability of our result

* Corresponding author: mailto:MohamedAli.Hammami@fss.rnu.tn

2 General Definitions

We consider the system

$$\dot{x}(t) = F(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ is the state. The function $F : [0,+\infty[\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x.

We now introduce the notions of uniform boundedness and uniform ultimate boundedness of a trajectory of (1) (see [8]).

Definition 2.1 The system (1) is uniformly bounded if for all $R_1 > 0$, there exists a $R_2 = R_2(R_1) > 0$, such that for all $t_0 \geq 0$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R_2, \quad \forall t \geq t_0.$$

Definition 2.2 The system (1) is uniformly ultimately bounded if there exists $R > 0$, such that for all $R_1 > 0$, there exists a $T = T(R_1)$, such that for all $t_0 \geq 0$

$$\|x_0\| \leq R_1 \Rightarrow \|x(t)\| \leq R, \quad \forall t \geq t_0 + T.$$

Let $r \geq 0$ and $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$. First, we give the definition of uniform stability and uniform attractivity of B_r.

Definition 2.3 (Uniform stability of B_r) (i) B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that for all $t_0 \geq 0$

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0.$$

(ii) B_r is globally uniformly stable if it is uniformly stable and the solutions of system (1) are globally uniformly bounded.

Definition 2.4 (Uniform attractivity of B_r) B_r is globally uniformly attractive, if for all $\varepsilon > r$ and $c > 0$, there exists $T(\varepsilon, c) > 0$, such that for all $t_0 \geq 0$

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, c), \quad \|x_0\| < c.$$

Definition 2.5 The system (1) is globally uniformly practically asymptotically stable if there exists $r \geq 0$, such that B_r is globally uniformly stable and globally uniformly attractive.

Definition 2.6 B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k \geq 0$, such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$

$$\|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)) + r.$$

The system (1) is globally practically uniformly exponentially stable if there exists $r > 0$, such that B_r is globally uniformly exponentially stable.

3 Basic Results

We consider now the following dynamical system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \\ y(t) = C(t)x(t), \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $y(t) \in \mathbb{R}^p$ is the system output, $u(t) \in \mathbb{R}^m$ is the control input and $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$ are matrices whose elements are bounded continuous or piecewise continuous functions of time. The function $f(t, x)$ is continuous, locally Lipschitz in x and there exists a non negative constant f_0, such that

$$\|f(t, 0)\| \leq f_0, \ \forall t \geq 0.$$

The corresponding nominal system is described by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases} \tag{3}$$

3.1 Stabilization

We prove in this subsection the stabilization of system (2) by a state feedback control candidate. It is assumed that the system (3) is uniformly controllable (see [5]).

Definition 3.1 The pair $(A(t), B(t))$ is uniformly controllable if there exist Δ and another constant α depending on Δ, such that the controllability grammian $I(t - \Delta, t)$ satisfies

$$I(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau)B(\tau)B^T(\tau)\psi^T(t-\Delta, \tau)d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$ and is defined by

$$\frac{\partial \psi(t, t_0)}{\partial t} = A(t)\psi(t, t_0), \quad \psi(t, t) = I,$$

$$\psi(t, t_0)\psi(t_0, s) = \psi(t, s)$$

and

$$\psi(t_0, t) = \psi^{-1}(t, t_0).$$

We find from [5] the state feedback gain $K(t)$, such that the control input

$$u(t) = K(t)x(t) \tag{4}$$

with

$$K(t) = R_1^{-1}(t)\overline{B}^T(t)P(t),$$

where $P(t)$ is the solution of the forward differential Riccati equation

$$\dot{P}(t) = -\overline{A}^T(t)P(t) - P(t)\overline{A}(t) + R_1(t) - P(t)\overline{B}(t)R_2^{-1}(t)\overline{B}^T(t)P(t), \quad P(0) = P_0 > 0, \tag{5}$$

in which

$$\overline{A}(t) = -T(x)A(t)T^{-1}(x), \quad \overline{B}(t) = T(x)B(t),$$

with

$$T(x) = I - 2\frac{x(t)x^T(t)}{x^T(t)x(t)},$$

$R_1(t) > 0$, $R_2(t) > 0$ and $R_1(t), R_2(t), R_1^{-1}(t), R_2^{-1}(t)$ are all uniformly bounded.

Proposition 3.1 *(see [6]) Consider the system (3) and the state feedback control (4) and (5), if the system (3) is uniformly controllable, the closed-loop system is globally exponentially stable.*

Notice that, the system (3) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally exponentially stable, then from [6] we have for all positive definite symmetric matrix $Q_1(t)$,

$$Q_1(t) \geq c_1 I > 0, \quad \forall t \geq 0,$$

there exists a positive definite symmetric matrix $P_1(t)$,

$$0 < c_2 I < P_1(t) < c_3 I, \quad \forall t \geq 0,$$

which satisfies

$$A_K^T(t)P_1(t) + P_1(t)A_K(t) + \dot{P}_1(t) = -Q_1(t), \quad \text{where } A_K(t) = A(t) + B(t)K(t). \tag{6}$$

Now, we prove the global practical uniform stabilizability of (2). We shall suppose the following.
$(\mathcal{A}_1)$ Assume that

$$\|f(t,x) - f(t,y)\| \leq \gamma(t)\|x - y\| + \delta(t) + \varepsilon, \quad \forall t \geq 0, \ \forall \ x, y \in \mathbb{R}^n, \tag{7}$$

where $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are continuous non-negative functions with

$$\int_0^{+\infty} \gamma(s)\, ds \leq M_\gamma < +\infty$$

and

$$\int_0^{+\infty} \delta^2(s)\, ds \leq M_\delta < +\infty.$$

Theorem 3.1 *Under assumption $(\mathcal{A}_1)$, the system (3) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable.*

Proof. Let us consider the Lyapunov function $V(t, x(t)) = x^T(t)P_1(t)x(t)$. The derivative of V along the trajectories of system (2) is given by

$$\dot{V}(t, x(t)) \leq -\left(\frac{c_1}{c_3} - \frac{2c_3\gamma(t)}{c_2}\right) V(t, x(t)) + 2\frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)\sqrt{V(t, x(t))}.$$

Use the following change $v(t) = \sqrt{V(t, x(t))}$. Then, $v(t)$ satisfies the following estimation

$$v(t) \leq v(t_0)e^{-\int_{t_0}^t \alpha(s)\, ds} + \frac{c_3}{\sqrt{c_2}}\left(\int_{t_0}^t (\delta(s) + f_0 + \varepsilon)e^{\int_{t_0}^s \alpha(\tau)\, d\tau} ds\right) e^{-\int_{t_0}^t \alpha(s)\, ds}$$

with

$$\alpha(t) = \frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}\cdot$$

A simple computation shows that,

$$\left(\int_{t_0}^{t} (\delta(s) + f_0 + \varepsilon) e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds\right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds} \leq \left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0+\varepsilon)\frac{c_3}{c_1}\right) e^{\dfrac{c_3 M_\gamma}{c_2}}.$$

Thus, we obtain

$$v(t) \leq v(t_0) e^{\dfrac{c_3 M_\gamma}{c_2}} e^{-\dfrac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{\sqrt{c_2}}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0+\varepsilon)\frac{c_3}{c_1}\right) e^{\dfrac{c_3 M_\gamma}{c_2}}.$$

It follows that

$$\|x(t)\| \leq \sqrt{\frac{c_3}{c_2}} e^{\dfrac{c_3 M_\gamma}{c_2}} \|x_0\| e^{-\dfrac{c_1}{2c_3}(t-t_0)} + \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0+\varepsilon)\frac{c_3}{c_1}\right) e^{\dfrac{c_3 M_\gamma}{c_2}}.$$

This implies the global uniform exponential stability of B_κ with

$$\kappa = \frac{c_3}{c_2}\left(\sqrt{\frac{c_3 M_\delta}{c_1}} + 2(f_0+\varepsilon)\frac{c_3}{c_1}\right) e^{\dfrac{c_3 M_\gamma}{c_2}}.$$

Hence, the system (2) in closed-loop with the linear feedback $u(t) = K(t)x(t)$ is globally practically uniformly exponentially stable. □

3.2 Conception of the observer

For the concept of observer, we aim at simplifying the design of this system by exploiting the linear form of the nominal system. The system (3) is assumed to be uniformly observable (see [5]).

Definition 3.2 The pair $(A(t), C(t))$ is uniformly observable if there exist Δ and another constant α depending on Δ, such that the observability grammian $J(t-\Delta, t)$ satisfies

$$J(t-\Delta, t) = \int_{t-\Delta}^{t} \psi(t-\Delta, \tau) C(\tau) C^T(\tau) \psi^T(t-\Delta, \tau) d\tau \geq \alpha I > 0,$$

in which $\psi(t, \tau)$ is the state transition matrix $A(t)$.

Definition 3.3 (Practical exponential observer) A practical exponential observer for (2) is a dynamical system which has the following form

$$\dot{\hat{x}}(t) = F(t, \hat{x}(t), u(t)) - L(t)(C(t)\hat{x}(t) - y(t)), \tag{8}$$

where $L(t)$ is the gain matrix and the error equation with $e(t) = \hat{x}(t) - x(t)$, is given by

$$\dot{e}(t) = F(t, \hat{x}(t), u(t)) - F(t, x(t), u(t)) - L(t)C(t)e(t) \tag{9}$$

a Luenberger observer which is expected to produce an estimation of the state in the sense of global practical exponential stability. It means that, the system (9) is globally practically uniformly exponentially stable and the following estimation holds:

$$\|e(t)\| \leq \lambda_1 \|e(t_0)\| \; e^{-\lambda_2(t-t_0)} + r, \quad \forall\, t \geq t_0,$$

with $\lambda_1,\ \lambda_2,\ r > 0$.

To design an observer, we shall consider the system

$$\dot{\hat{x}} = A(t)\hat{x}(t) + B(t)u(t) + f(t, \hat{x}(t)) - L(t)(C(t)\hat{x}(t) - y(t)), \tag{10}$$

where $\hat{x}(t)$ is the state estimate of $x(t)$ and $L(t) \in \mathbb{R}^{n\times p}$ is the observer feedback gain to be determined so that $\hat{x}(t)$ tends to $x(t)$ exponentially. One such design is the well known Kalman filter design ([3]), in which the observer feedback gain $L(t)$ is chosen as

$$L(t) = Q(t)C^T(t)V_2^{-1}(t), \tag{11}$$

where $Q(t)$ satisfies a forward differential Riccati equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t), \quad Q(0) = Q_0 > 0, \tag{12}$$

in which $V_1(t) > 0$, $V_2(t) > 0$ and $V_1(t)$, $V_2(t)$, $V_1^{-1}(t)$, $V_2^{-1}(t)$ are all uniformly bounded. The error equation is given by

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A(t) - L(t)C(t))e(t) + f(t, \hat{x}(t)) - f(t, x(t)). \tag{13}$$

Proposition 3.2 *(see [9]) Consider the system (3) and the observer (11) and (12). If $(A(t), C(t))$ is uniformly observable, the closed-loop system is globally exponentially stable.*

Notice that, if the system (3) in closed-loop with the observer (11) and (12) is globally uniformly exponentially stable, then for all positive definite symmetric matrix $Q_2(t)$,

$$Q_2(t) \geq b_1 I > 0, \quad \forall t \geq 0,$$

there exists a positive definite symmetric matrix $P_2(t)$,

$$0 < b_2 I < P_2(t) < b_3 I, \quad \forall t \geq 0,$$

which satisfies

$$A_L^T(t)P_2(t) + P_2(t)A_L(t) + \dot{P}_2(t) = -Q_2(t), \quad \text{where } A_L(t) = A(t) - L(t)C(t). \tag{14}$$

Theorem 3.2 *Under assumption ($\mathcal{A}_1$), the system* (10) *is a practical exponential observer for the system (2).*

Proof. Let us consider the Lyapunov function $Y(t, e(t)) = e^T(t)P_2(t)e(t)$. The derivative of Y along the trajectories of system (13) is given by

$$\dot{Y}(t, e(t)) \leq -\left(\frac{b_1}{b_3} - \frac{2b_3}{b_2}\gamma(t)\right) Y(t, e(t)) + 2\frac{b_3}{\sqrt{b_2}}(\delta(t) + \varepsilon)\sqrt{Y(t, e(t))}.$$

Use the following change $y(t) = \sqrt{Y(t, e(t))}$. Then, $y(t)$ satisfies the following estimation

$$y(t) \leq y(t_0)e^{-\int_{t_0}^{t} \beta(s)\,ds} + \frac{b_3}{\sqrt{b_2}} \left(\int_{t_0}^{t} (\delta(s) + \varepsilon) e^{\int_{t_0}^{s} \beta(\tau)\,d\tau} ds \right) e^{-\int_{t_0}^{t} \beta(s)\,ds}$$

with

$$\beta(t) = \frac{b_1}{2b_3} - \frac{b_3 \gamma(t)}{b_2}\cdot$$

A simple computation shows that,

$$\left(\int_{t_0}^{t} (\delta(s) + \varepsilon) e^{\int_{t_0}^{s} \beta(\tau)\,d\tau} ds \right) e^{-\int_{t_0}^{t} \beta(s)\,ds} \leq \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

Thus, we obtain

$$y(t) \leq y(t_0) e^{\frac{b_3 M_\gamma}{b_2}} e^{-\frac{b_1}{2b_3}(t - t_0)} + \frac{b_3}{\sqrt{b_2}} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

Hence,

$$\|e(t)\| \leq \sqrt{\frac{b_3}{b_2}} e^{\frac{b_3 M_\gamma}{b_2}} \|e(t_0)\| e^{-\frac{b_1}{2b_3}(t - t_0)} + \frac{b_3}{b_2} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

This implies the global uniform exponential stability of B_η with

$$\eta = \frac{b_3}{b_2} \left(\sqrt{\frac{b_3 M_\delta}{b_1}} + 2\varepsilon \frac{b_3}{b_1} \right) e^{\frac{b_3 M_\gamma}{b_2}}.$$

We deduce that, the system (13) is globally practically exponentially stable. Hence, the system (10) is a practical exponential observer for the system (2). □

3.3 Separation principle

Now, we obtain a separation principle for (2). We consider the system (2) controlled by the linear feedback control $u(t) = K(t)\hat{x}(t)$ and estimated with the observer (10).

Theorem 3.3 *Under assumption ($\mathcal{A}_1$), the system*

$$\begin{cases} \dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + f(t, \hat{x}(t)) - L(t)C(t)e(t), \\ \dot{e}(t) = (A(t) - L(t)C(t))e(t) + f(t, \hat{x}(t)) - f(t, x(t)), \end{cases} \tag{15}$$

is globally practically uniformly exponentially stable.

Proof. In order to study the stabilization problem via an observer, we consider the system

$$\dot{\hat{x}}(t) = \psi(t, \hat{x}(t)) - L(t)C(t)e(t), \tag{16}$$

where

$$\psi(t, \hat{x}(t)) = (A(t) + B(t)K(t))\hat{x}(t) + f(t, \hat{x}(t)).$$

Let us consider the Lyapunov function $v(t, \hat{x}(t)) = \sqrt{\hat{x}^T(t)P_1(t)\hat{x}(t)}$, which satisfies

$$\sqrt{c_2}\|\hat{x}(t)\| \leq v(t, \hat{x}(t)) \leq \sqrt{c_3}\|\hat{x}(t)\|,$$

$$\frac{\partial v}{\partial t}(t, \hat{x}(t)) + \frac{\partial v}{\partial \hat{x}(t)}\psi(t, \hat{x}(t)) \leq \ -\alpha(t)v(t, \hat{x}(t)) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)$$

and

$$\left\| \frac{\partial v}{\partial \hat{x}}(t, \hat{x}(t)) \right\| \leq \ \frac{c_3}{\sqrt{c_2}},$$

where

$$\alpha(t) = \frac{c_1}{2c_3} - \frac{c_3\gamma(t)}{c_2}\cdot$$

The derivative of v along the trajectories of system (16) is given by

$$\begin{aligned}\dot{v}(t, \hat{x}(t)) \leq{}& -\alpha(t)v(t, \hat{x}(t)) + \frac{c_3}{\sqrt{c_2}}(\delta(t) + f_0 + \varepsilon)\\ &+ \frac{c_3}{\sqrt{c_2}}\|L(t)C(t)\| \left(\sqrt{\frac{b_3}{b_2}}\ e^{\dfrac{b_3M_\gamma}{b_2}} \|e(t_0)\| e^{-\dfrac{b_1}{2b_3}(t - t_0)} \right.\\ &\left. + \frac{b_3}{b_2}\left(\sqrt{\frac{b_3M_\delta}{b_1}} + 2\varepsilon\frac{b_3}{b_1}\right)\ e^{\dfrac{b_3M_\gamma}{b_2}} \right).\end{aligned}$$

Since $L(t)C(t)$ is bounded for all $t \geq t_0$, then there exists $R_1 > 0$, such that

$$\|L(t)C(t)\| \leq R_1, \quad \forall t \geq t_0 \geq 0.$$

Then

$$\dot{v}(t, \hat{x}(t)) \leq -\alpha(t)v(t, \hat{x}(t)) + \lambda\|e(t_0)\|e^{-\dfrac{b_1}{2b_3}(t - t_0)} + \frac{c_3}{\sqrt{c_2}}\delta(t) + R$$

with

$$\lambda = \frac{c_3}{\sqrt{c_2}}R_1\sqrt{\frac{b_3}{b_2}}\ e^{\dfrac{b_3M_\gamma}{b_2}}$$

and

$$R = \frac{c_3}{\sqrt{c_2}}(f_0 + \varepsilon) + \frac{b_3c_3}{b_2\sqrt{c_2}}R_1\left(\sqrt{\frac{b_3M_\delta}{b_1}} + 2\varepsilon\frac{b_3}{b_1}\right)\ e^{\dfrac{b_3M_\gamma}{b_2}}.$$

Using the following change

$$y(t) = v(t)e^{\displaystyle\int_{t_0}^{t}\alpha(s)\,ds},$$

we obtain

$$y(t) \leq y(t_0) + \frac{c_3}{\sqrt{c_2}} \int_{t_0}^{t} \delta(s) e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds + \lambda \|e(t_0)\| \int_{t_0}^{t} e^{-\displaystyle\frac{b_1}{2b_3}(s - t_0)} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds$$
$$+ R \int_{t_0}^{t} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds.$$

Then

$$v(t) \leq v(t_0) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds} + \frac{c_3}{\sqrt{c_2}} \left(\int_{t_0}^{t} \delta(s) e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}$$
$$+ \lambda \|e(t_0)\| \left(\int_{t_0}^{t} e^{-\displaystyle\frac{b_1}{2b_3}(s - t_0)} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}$$
$$+ R \left(\int_{t_0}^{t} e^{\displaystyle\int_{t_0}^{s} \alpha(\tau)\, d\tau} ds \right) e^{-\displaystyle\int_{t_0}^{t} \alpha(s)\, ds}.$$

A simple computation shows that

$$v(t) \leq v(t_0) e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\displaystyle\frac{c_1}{2c_3}(t - t_0)} + \frac{c_3}{\sqrt{c_2}} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}}$$
$$+ \lambda \|e(t_0)\| \frac{2b_3 c_3}{c_1 b_3 - b_1 c_3}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\displaystyle\frac{b_1}{2b_3}(t - t_0)} + 2\frac{R c_3}{c_1}.$$

Let

$$\theta = \min\left(\frac{c_1}{2c_3},\ \frac{b_1}{2b_3} \right).$$

Then

$$v(t) \leq \sqrt{c_3} \|\hat{x}_0\| e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} e^{-\theta(t - t_0)} + \frac{2\lambda b_3 c_3}{c_1 b_3 - b_1 c_3}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} \|e(t_0)\|\, e^{-\theta(t - t_0)}$$
$$+ \frac{c_3}{\sqrt{c_2}} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} + 2\frac{R c_3}{c_1}.$$

Let

$$k = \max\left(\sqrt{c_3},\ \frac{2\lambda b_3 c_3}{c_1 b_3 - b_1 c_3} \right).$$

Hence,

$$\|\hat{x}(t)\| \leq \frac{k}{\sqrt{c_2}} e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} (\|\hat{x}_0\| + \|e(t_0)\|)\, e^{-\theta(t - t_0)} + \frac{c_3}{c_2} \sqrt{\frac{c_3 M_\delta}{c_1}}\, e^{\displaystyle\frac{c_3 M_\gamma}{c_2}} + 2\frac{R c_3}{c_1 \sqrt{c_2}}.$$

Then, the cascade system (15) is globally practically uniformly exponentially stable. □

Example 3.1 Consider the system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t, x(t)), \\ y(t) = C(t)x(t) \end{cases} \tag{17}$$

with $x(t) = (x_1(t), x_2(t))^T$,

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ e^{-2t} \end{pmatrix},$$

$$C(t) = \begin{pmatrix} 1 & e^{-2t} \end{pmatrix}$$

and

$$f(t, x(t)) = e^{-kt}x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad k > 0.$$

The proposed control (4) is then applied to the system with the following design parameters $P(0) = I$, $R_1(t) = I$, $R_2(t) = I$ in (5). The matrix $P(t)$ is calculated by solving the Ricatti equation (5). The function $f(t, x(t))$ is continuous and satisfies assumption $(\mathcal{A}_1)$ because

$$\int_0^{+\infty} e^{-kt} = \frac{1}{k}, \quad k > 0.$$

We conclude that the system (2) can be globally practically uniformly exponentially stable. The observer feedback gain $L(t)$ is chosen as (11) by solving the Riccati equation (12). We conclude that the system (10) is a practical exponential observer for the system (17). Thus, Theorem 3.3 is satisfied. We conclude that, the system (15) is globally uniformly practically exponentially stable.

4 Conclusion

This paper presents a separation principle for a class of nonlinear controls systems. It is shown that the system can be globally exponentially stabilizable by means of an estimated state feedback control given by an observer design.

Acknowledgment

The authors thank the referees for their careful reading of the manuscript and insightful comments.

References

[1] Arcak, M. and Kokotovic, P. Observer-based control of systems with slope-restricted nonlinearities. *IEEE Transactions on Automatic Control* (2001) 1146-1150.

[2] Abouelsoud, A.A., Abdelfattah, H., El Metwally, M. and Nasr, M. State Feedback Controller of Robinson Nuclear Plant with States and Control Constraints. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 1-17.

[3] Benmansour, K., Tlemcani, A., Djemai, M. and De Leon, J. A New Interconnected Observer Design in Power Converter: Theory and Experimentation. *Nonlinear Dynamics and Systems Theory* **10** (3) (2010) 211-224

[4] Chen, M. Control of linear time-varying systems by the gradient algorithm. *IEEE Transactions on Automatic Control* (1997) 4549–4553.

[5] Chen, M.S. and Kao, C.Y. Control of linear time-varying systems using forward Riccati equation. *Journal of Dynamics systems, Measurement and Control* **119** (1997) 536–540.

[6] Ellouze, I. and Hammami, M.A. A separation principle of time-varying dynamical system: A practical approach. *Mathematical Modelling and Analysis* **12** (2) (2007) 1–16.

[7] Kalman, R.E. and Bucy, R.S. New results in linear filtering and prediction theory. *ASME Journal of Basic Engineering* (1961) 95–108.

[8] Khalil, H.K. *Nonlinear Systems*. Prentice-Hall, New York, 2002.

[9] Loria, A. and Panteley, E. A separation principle for Euler-Langrange systems. In: *New Directions in Nonlinear Observer Design* (H. Nijmeijer and T.I. Fossen, Eds). London: Springer Verlag, 1999, 229–247.

[10] Loria, A. and de Leon Morales, J. On persistently exciting observers and a non-linear separation principle: application to the stabilization of a generator. *International Journal of Control* **76** (6) (2003) 607–617.

[11] Martynyuk, A.A. Stability in the models of real world phenomena. *Nonlinear Dynamics and Systems Theory* **11** (1) (2011) 7–52 .

[12] Misir, A. and Ogrekci, S. Reducibility and Stability Results for Dynamic Systems on Time Scales. *Advances in Dynamical Systems and Applications* **5** (2) (2010) 191–203.

[13] Nilakantan, N. and Raghavendra, V. Global Stability Given Local Stability Via Curvature of Some Nonautonomous Differential Equations. *Nonlinear Dynamics and Systems Theory* **12** (1) (2012) 105-109 .

[14] Rugh, W.J. *Linear System Theory*. Prentice-Hall, Englewood Cliffs, 1993.

[15] Tsinias, J. Sontag's input to state stability condition and global stabilization using state detection. *Systems and Control Letters* **20** (1993) 219–226.