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Orthogonal Functions Approach for Model Order Reduction of LTI and LTV Systems

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Abstract: In this paper, we elaborate new methods for model-order reduction of linear time invariant (LTI) and time variant (LTV) systems by using orthogonal functions. These techniques which can be efficiently applied in SISO (single-input single-output) and MIMO (multi-input multi-output) cases are based on the projection of the system parameters and variables on an orthogonal functions basis. The useful properties of the orthogonal functions basis such as operational matrices combined with the Kronecker product permit the conversion of the system differential equations into algebraic ones allowing the determination of the reduced model parameters.

Keywords: *model-order reduction; LTI and LTV systems; orthogonal functions; operational matrices; shifted Legendre polynomials.*

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1 Introduction

In all engineering fields, an accurate modeling is necessary to have good results in control and analysis of complex systems. If the system is internally complex, the use of modern control techniques such as optimal control, μ-synthesis or robust control may lead to a controller having a comparable order as the considered system. In order to study, simulate and control those systems and to avoid time consuming in computing procedures, it is convenient and sometimes necessary to reduce their complexity, preserving the input-output behavior.

The primary problem of interest in model reduction is the efficient computation of an accurate low-order model approximating a given dynamical system. The low-order model must match the original one in some sense. However, the conditions of accuracy, speed,

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stability and low order cannot always be reached at the same time. The model-order reduction (MOR) reaches far from electrical engineering and touches various disciples of science and engineering fields such as aerospace science [1, 2], chemical processes [3], protection of civil structures, modeling of biological systems [4], power systems [47] and mechanical engineering [5].

So far, the main MOR techniques were introduced and developed for linear systems and precisely LTI systems and were lately extended to LTV systems and nonlinear systems [6, 7]. The main MOR methods fall into three classes [8]:

– Singular Value Decomposition (SVD) or Gramian-based methods including optimal Hankel MOR [9, 10], and balanced truncation realization first introduced by Moore [11] and improved during the last decades [12–14].

– Krylov subspace-based methods [15] including techniques based on Lanczos procedure [16, 17] or Arnoldi algorithm [18, 19].

– Proper orthogonal decomposition (POD) or Karhunen-Loève expansion [2, 20].

Many recent techniques give an alternative to these classical methods such as the MOR by least squares [21] and using LMIs [22]. The MOR techniques for LTI systems were later extended to modeling linear time varying (LTV) systems [7, 23, 24].

In this paper, we introduce new analytic methods for model-order reduction (MOR) of linear time invariant (LTI) and time variant (LTV) systems starting from a state space realization or a transfer function system description. Those approaches using the orthogonal functions as a tool of approximation can be applied not only for SISO systems but also for the MIMO ones. This paper is organized as follows: in Section 2, the orthogonal functions are presented with their interesting properties. The dynamical systems description by orthogonal functions is introduced in Section 3. The proposed methods for model order reduction of LTI and LTV systems using orthogonal functions are derived in Section 4. The last section is devoted to simulation examples to emphasize the effectiveness of the presented approaches.

2 Orthogonal Functions for Dynamical Systems Description

In recent decades, the approximation of time functions by orthogonal functions has been considered by many researchers to solve modeling and control problems [48]. The main characteristic of this technique is that it reduces the differential equations to algebraic ones, thus greatly simplifying the problem.

This approach originated from the use of Walsh [25] and block-pulse [26] functions was later extended to orthogonal polynomial series such as the Laguerre [27], the Chebychev [28], the Hermite [29] and the Legendre polynomials [30]. They were also used with nonlinear systems [31] and for PID control of LTI [32] and LTV systems [33]. The development in Fourier or Taylor polynomial series can give convenient results but their quick convergence is not always guaranteed or their use can be sometimes inadequate.

2.1 Orthogonal functions and properties

2.1.1 Approximation using orthogonal functions

Orthogonal functions were introduced in the field of system control because of their interesting properties as a sharp tool of approximation. Given $\Phi = \{\phi_i(t), i \in \mathbb{N}\}$ a set of functions defined over a certain interval $[a, b]$. Any function $f(t)$ absolutely integrable over $[a, b]$ can be developed as follows

$$f(t) = \sum_{i=0}^{\infty} f_i \phi_i(t), \tag{1}$$

where $f_i = \int_a^b w(t) f(t) \phi_i(t)\, dt$, for $i \in \mathbb{N}$, $w(t)$ is a positive and integrable function as the weighting function of the scalar product. For practical use, the development (1) is truncated up to an order N, thus giving the following time approximation of the function

$$f(t) \cong \sum_{i=0}^{N-1} f_i \phi_i(t) = F_N^T \Phi_N(t) \tag{2}$$

with

$$F_N = \begin{bmatrix} f_0 & f_1 & \dots & f_{N-1} \end{bmatrix}^T, \ \Phi_N(t) = \begin{bmatrix} \phi_0(t) & \phi_1(t) & \dots & \phi_{N-1}(t) \end{bmatrix}^T,$$

where $\Phi_N(t)$ is the vector of the orthogonal functions basis. The coefficients f_i and the orthogonal functions $\{\phi_i(t), i \in \mathbb{N}\}$ have the particularity to minimize the error:

$$\varepsilon = \int_a^b \left(f(t) - \sum_{i=0}^{N-1} f_i \ \phi_i(t) \right)^2 dt. \tag{3}$$

The orthogonal functions obey the orthogonality relation

$$< \phi_i(t), \phi_j(t) >= \int_a^b w(t) \phi_i(t) \phi_j(t)\, dt = \delta_{ij} \ c_i, \tag{4}$$

where δ_{ij} is the Kronecker delta. If $c_i = 1$, then the functions are not only orthogonal, but orthonormal.

2.1.2 Shifted definition interval

If the function $f(t)$ is defined over an interval $[t_0, t_f]$ and the orthogonal functions $\phi_i(t)$ over the interval $[a, b]$, we can shift the defining domain by considering the functions :

$$\forall i \in \mathbb{N}, \psi_i(t) = \phi_i\left(\frac{t-\mu}{\lambda}\right)$$

with $t \in [t_0, t_f]$, $\lambda = \dfrac{t_0 - t_f}{a - b}$ and $\mu = \dfrac{a t_f - b t_0}{a - b}$.

The functions $\psi_i(t), \forall i \in \mathbb{N}$ are also orthogonal over $[t_0, t_f]$ with the weighting function $w'(t) = w(\frac{t-\mu}{\lambda})$.

2.1.3 Matrix functions approximation

A time dependent matrix function $A(t) \in \mathbb{R}^{n \times m}$ given by $A(t) = [a_{ij}(t)]$ where $a_{ij}(t)$ are integrable over an interval $[a, b]$. The matrix $A(t)$ can be developed into orthogonal functions series with a truncation to an order N under the following relation

$$A(t) \cong \sum_{i=0}^{N-1} A_{Ni} \phi_i(t), \tag{5}$$

where $A_{Ni} \in \mathbb{R}^{n \times m}$ for $i \in \{0, 1, \dots, N-1\}$ are matrices with constant coefficients.

2.1.4 Operational matrix of integration

The operational matrix of integration is a constant coefficient function $P_N \in \mathbb{R}^{N\times N}$ verifying the integral property of the orthogonal functions basis vector $\Phi_N(t)$:

$$\underbrace{\int \cdots \int_{\alpha}^{t}}_{k \text{ times}} \Phi_N(t)\, dt^k \cong P_N^k \Phi_N(t). \tag{6}$$

Clearly, the form of P_N depends on the particular choice of the basis vector $\Phi_N(t)$.

2.1.5 Operational matrix of product

The operational vectors of product K_{ij} [35] have constant coefficients and verify the property:

$$\forall\, i, j \in \{0, 1, \cdots, N-1\},\;\; \phi_i(t)\phi_j(t) \cong K_{ij}^T \Phi_N(t). \tag{7}$$

From the relationship (7), we can readily get the operational matrix of product:

$$M_{iN} = \begin{bmatrix} K_{0i}^T \\ \vdots \\ K_{N-1,i}^T \end{bmatrix} \tag{8}$$

that allows the approximation

$$\phi_i(t)\Phi_N(t) \cong M_{iN}\Phi_N(t). \tag{9}$$

2.1.6 Legendre polynomials

The Legendre polynomials may have advantages over other orthogonal functions. This was shown by way of examples [30] where Legendre polynomials converge to the exact solution of a differential equation faster than the other types of orthogonal functions, such as, for example, Walsh functions, Hermite and Laguerre polynomials. The Legendre polynomials are defined for the time interval $x \in [-1, 1]$ and they have the following analytical form given by the Olinde-Rodrigues formula [36]:

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}. \tag{10}$$

Using the above expression for $L_n(x)$, one may readily determine the first few Legendre polynomials : $L_0(x) = 1$, $L_1(x) = x$, ...

The Legendre polynomials are also given by the recursive formula [34]:

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x). \tag{11}$$

The polynomials $L_i(x)$ form a complete set and are orthogonal [30] with

$$\int_{-1}^{1} L_i(x)L_j(x)dx = \frac{2}{2i+1}\delta_{ij}. \tag{12}$$

2.1.7 Shifted Legendre polynomials

For practical use of Legendre polynomials in the time interval $t \in [0, t_f]$, it is necessary to shift the defining domain of Legendre polynomials from the interval $[-1, 1]$ to $[0, t_f]$ through the variable transformation:

$$x = \frac{2t}{t_f} - 1, \quad 0 \le t \le t_f. \tag{13}$$

Thus, the shifted Legendre polynomials $s_i(t)$ ($\forall i \in \mathbb{N}$) for $0 \le t \le t_f$ are thus given by

$$s_{n+1}(t) = \frac{2n+1}{n+1}\,\frac{2t-t_f}{t_f}\,s_n(t) - \frac{n}{n+1}s_{n-1}(t) \tag{14}$$

with $s_0(t) = 1$ and $s_1(t) = \dfrac{2t}{t_f} - 1$.

It is apparent that polynomials $s_n(t)$ also constitute a complete set and are orthogonal [37] with

$$\int_0^{t_f} s_i(t)s_j(t)\,dt = \frac{t_f}{2i+1}\delta_{ij}. \tag{15}$$

Any time function $f(t)$ that is absolutely integrable on the time interval $[0, t_f]$ may be expanded into shifted Legendre series as follows

$$f(t) = \sum_{i=0}^{\infty} f_i s_i(t), \tag{16}$$

where [38]

$$f_i = \frac{2i+1}{t_f}\int_0^{t_f} f(t)s_i(t)\,dt. \tag{17}$$

If equation (16) is truncated up to its first N terms, then it may be written as

$$f(t) \cong \sum_{i=0}^{N-1} f_i s_i(t) = F_N^T S_N(t) \tag{18}$$

with $F_N = \begin{bmatrix} f_0 & f_1 & \dots & f_{N-1} \end{bmatrix}^T$ and $S_N(t) = \begin{bmatrix} s_0(t) & s_1(t) & \dots & s_{N-1}(t) \end{bmatrix}^T$. The shifted Legendre polynomials and coefficients f_i, $(i = 0, 1, \dots, N-1)$ have the particularity to minimize the integral squared-error:

$$\varepsilon = \int_0^{t_f}\left(f(t) - \sum_{i=0}^{N-1} f_i s_i(t)\right)^2 dt. \tag{19}$$

2.1.8 Operational matrix of integration

Since the shifted Legendre polynomials $s_i(t)$, $(i = 0, 1, \dots)$ satisfy [34] the differential equation:

$$s_i(t) = \frac{t_f}{2(2i+1)}\left[\frac{ds_{i+1}}{dt} - \frac{ds_{i-1}}{dt}\right] \tag{20}$$

and $s_i(0) = (-1)^i$, it can be easily shown that the integrals of $s_i(t)$, $(i = 0, 1, \dots)$ are given by

$$\int_0^t s_i(\tau)d\tau = \begin{cases} \frac{t_f}{2}\left[s_1(t) - s_0(t)\right] & \text{, for } i = 0, \\ \frac{t_f}{2(2i+1)}\left[s_{i+1}(t) - s_{i-1}(t)\right] & \text{, for } i > 0. \end{cases} \tag{21}$$

From equation (21) we can obtain the integral of truncated shifted Legendre vector as follows

$$\int_0^t S_N(\tau)d\tau \cong P_N S_N(t), \tag{22}$$

where P_N is the constant operational matrix of integration given in [39] and [40].

3 Dynamical systems description using orthogonal functions

3.1 LTI systems described by a transfer function

Given a linear time invariant system described by a transfer function:

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + ... + b_m s^m}{a_0 + a_1 s + ... + a_n s^n} \tag{23}$$

with $m \leq n$, or a linear differential equation in time domain with constant coefficients, the input $u(t)$ and the output $y(t)$:

$$a_0 y(t) + a_1 y'(t) \cdots + a_n y^{(n)}(t) = b_0 u(t) + b_1 u'(t) \cdots + b_m u^{(m)}(t). \tag{24}$$

Upon integration of both sides of equation (24) n times, we have:

$$\begin{aligned} &a_0 \underbrace{\int \cdots \int_0^t}_{n \text{ times}} y(\tau)d\tau^n + a_1 \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} y(\tau)d\tau^{n-1} + \cdots + a_n y(t) = \\ &b_0 \underbrace{\int \cdots \int_0^t}_{n \text{ times}} u(\tau)d\tau^n + b_1 \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} u(\tau)d\tau^{n-1} + \cdots + b_m \underbrace{\int \cdots \int_0^t}_{n-m \text{ times}} u(\tau)d\tau^{n-m}. \end{aligned} \tag{25}$$

The projection of the input $u(t)$ and the output $y(t)$ on an orthogonal functions basis with truncated developments to an order N over a time interval $[0, t_f]$ yields:

$$y(t) \cong Y_N \Phi_N(t), \tag{26}$$

$$u(t) \cong U_N \Phi_N(t), \tag{27}$$

where Y_N and U_N are constant coefficient vectors.

By introducing the projections (26) and (27) in equation (25) and considering the case where the initial conditions are equal to zero, we obtain the relation

$$\begin{aligned} &a_0 Y_N \underbrace{\int \cdots \int_0^t}_{n \text{ times}} \Phi_N(\tau)d\tau^n + a_1 Y_N \underbrace{\int \cdots \int_0^t}_{n-1 \text{ times}} \Phi_N(\tau)d\tau^{n-1} + \cdots + a_n y(t) = \\ &b_0 U_N \underbrace{\int \cdots \int_0^t}_{n \text{ times}} \Phi_N(\tau)d\tau^n + \cdots + b_m U_N \underbrace{\int \cdots \int_0^t}_{n-m \text{ times}} \Phi_N(\tau)d\tau^{n-m}. \end{aligned} \tag{28}$$

In the case where the initial conditions are different from zero, they can be projected on the orthogonal basis and then integrated in the equation (28). By using the operational matrix of integration and the property (6), the equation (28) yields:

$$\begin{aligned} Y_N \left(a_0 P_N^n + a_1 P_N^{n-1} \cdots + a_n I_N\right) \Phi_N(t) = \\ U_N \left(b_0 P_N^n + b_1 P_N^{n-1} + \cdots + b_m P_N^{n-m}\right) \Phi_N(t). \end{aligned} \tag{29}$$

This equality is available for all time $t \in [0, t_f]$ then the simplification by $\Phi_N(t)$ in the equality (29) leads to the following description of the considered system:

$$Y_N \,\mathbb{M} = U_N \,\mathbb{T} \quad \text{or} \quad Y_N = U_N \,\mathbb{T}\,\mathbb{M}^{-1} \tag{30}$$

with

$$\begin{aligned} \mathbb{M} &= a_0 P_N^n + a_1 P_N^{n-1} + ... + a_n I_N, \\ \mathbb{T} &= b_0 P_N^n + b_1 P_N^{n-1} + ... + b_m P_N^{n-m}. \end{aligned} \tag{31}$$

3.2 LTI systems described by a state representation

Consider a linear time invariant (LTI) MIMO system given by the following state realization:

$$\begin{cases} \dot{X}(t) = A\,X(t) + B\,U(t), & X(0) = 0, \\ Y(t) = C\,X(t), & t \in [0, t_f], \end{cases} \tag{32}$$

with the state vector $X(t) \in \mathbb{R}^n$, the inputs vector $U(t) \in \mathbb{R}^m$ and the output one $Y(t) \in \mathbb{R}^p$. The matrices A, B, and C have respectively the dimensions $n \times n$, $n \times m$ and $p \times n$. The integration of the state equation (32) with zero initial conditions gives:

$$X(t) = A \int_0^t X(\tau) d\tau + B \int_0^t U(\tau) d\tau. \tag{33}$$

The projection of the state vector $X(t)$, the input U and the output Y, on an orthogonal basis functions $\{\varphi_i(t), i \in \{0, 1, \ldots, N-1\}\}$ with a truncated development to an order N over the interval $[0, t_f]$ leads to:

$$X(t) \cong X_N \Phi_N(t), \tag{34}$$

$$U(t) \cong U_N \Phi_N(t), \tag{35}$$

$$Y(t) \cong Y_N \Phi_N(t), \tag{36}$$

where the matrices X_N and U_N are constant coefficients matrices. With the developments (34) and 35), the integrated state equation (33) can be written under the following form

$$X_N \Phi_N(t) = A \int_0^t X_N \Phi_N(\tau) d\tau + B \int_0^t U_N \Phi_N(\tau) d\tau. \tag{37}$$

The use of the operational matrix of integration that approximates the integration of the orthogonal basis vector $\Phi_N(t)$:

$$\int_0^t \Phi_N(\tau)\, d\tau \cong P_N \;\Phi_N(t) \tag{38}$$

leads to the relation:

$$X_N\ \Phi_N(t) = A\ X_N\ P_N \Phi_N(t) + B\ U_N\ P_N\ \Phi_N(t). \tag{39}$$

Simplifying by the orthogonal functions basis vector $\Phi_N(t)$ yields:

$$X_N - A\ X_N\ P_N = B\ U_N\ P_N. \tag{40}$$

For rearranging the equation (40), we use the Vec operator [41] that reshapes a matrix by stacking its columns into a long vector. This vector denoted by $Vec(A)$ is associated with a matrix A and has the following property:

$$Vec(E\ F\ G) = \left(G^T \otimes E\right) Vec(F), \tag{41}$$

where E, F and G are matrices having appropriate dimensions and $\otimes$ is the Kronecker product.

Mathematically, let $R = [r_{ij}] \in \mathbb{R}^{m\times n}$ and $W = [w_{ij}] \in \mathbb{R}^{p\times q}$, the Kronecker product of R and W, denoted by $R \otimes W \in \mathbb{R}^{mp\times nq}$ is defined by [41]:

$$R \otimes W = \begin{bmatrix} r_{11}W & r_{12}W & \dots & r_{1n}W \\ r_{21}W & r_{21}W & \dots & r_{2n}W \\ \vdots & \vdots & \vdots & \vdots \\ r_{m1}W & r_{m1}W & \dots & r_{mn}W \end{bmatrix}. \tag{42}$$

By applying the property (41) to the equation (40), we get the following algebraic relation

$$Vec(X_N) = \left[I_{n\times N} - \left(P_N^T \otimes A\right)\right]^{-1} \left(P_N^T \otimes B\right) Vec(U_N) \tag{43}$$

and in the same way, the output relation in (32) can be written as:

$$Vec(Y_N) = (I_N \otimes C)\, Vec(X_N). \tag{44}$$

3.3 LTV systems described by a state representation

In this section, we consider the linear time varying (LTV) systems described by the following state space realization

$$\begin{cases} \dot{X}(t) = A(t)\ X(t) + B(t)\ U(t), \\ Y(t) = C(t)\ X(t), \end{cases} \tag{45}$$

with $A(t)$, $B(t)$ and $C(t)$ varying in time t with respective dimensions $n \times n$, $n \times m$ and $p \times n$. The expressions of matrices $A(t)$, $B(t)$ and $C(t)$ are supposed to be known with:

$$A(t) = \begin{bmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \dots & \dots & \dots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad C(t) = \begin{bmatrix} c_1(t) & \dots & c_n(t) \end{bmatrix}.$$

Notice that this state description can be derived from an input-output LTV differential equation. A technique of LTV systems identification was proposed in [42]. The integration of the state equation gives:

$$X(t) = \int_0^t A(\tau)X(\tau)d\tau + \int_0^t B(\tau)U(\tau)d\tau. \tag{46}$$

By exploiting the matrix functions approximation (5), the variable in time parameters of the system can be projected into th orthogonal basis and then written under the form:

$$A(t) \cong \sum_{i=0}^{N-1} A_{Ni}\ \varphi_i(t), \tag{47}$$

$$B(t) \cong \sum_{i=0}^{N-1} B_{Ni}\ \varphi_i(t), \tag{48}$$

$$C(t) \cong \sum_{i=0}^{N-1} C_{Ni}\ \varphi_i(t), \tag{49}$$

where A_{Ni}, B_{Ni} and C_{Ni} are constant coefficients matrices having respectively the same dimensions as $A(t)$, $B(t)$ and $C(t)$.

With the same projections (34), (35) and (36) of the state vector, the input vector and the output vector, the equation (46) becomes:

$$X_N\,\Phi_N(t) = \int_0^t \sum_{i=0}^{N-1} A_{iN}\phi_i(\tau)X_N\Phi_N(\tau)d\tau + \int_0^t \sum_{i=0}^{N-1} B_{iN}\phi_i(\tau)U_N\Phi_N(\tau)d\tau. \tag{50}$$

The orthogonal functions $\phi_i(t)$ are scalar functions, so:

$$X_N\,\Phi_N(t) = \int_0^t \sum_{i=0}^{N-1} A_{iN}X_N\phi_i(\tau)\Phi_N(\tau)d\tau + \int_0^t \sum_{i=0}^{N-1} B_{iN}U_N\phi_i(\tau)\Phi_N.(\tau)d\tau \tag{51}$$

By using the operational matrix of product [35] and the property (9), one has:

$$X_N\,\Phi_N(t) = \int_0^{t_f} \sum_{i=0}^{N-1} A_{iN}X_N M_{iN}\Phi_N(t)dt + \int_0^{t_f} \sum_{i=0}^{N-1} B_{iN}U_N M_{iN}\Phi_N(t)dt \tag{52}$$

and with the operational matrix of integration [39, 40], it comes out:

$$X_N\,\Phi_N(t) = \sum_{i=0}^{N-1} A_{iN}X_N M_{iN}P_N\Phi_N(t) + \sum_{i=0}^{N-1} B_{iN}U_N M_{iN}P_N\Phi_N(t). \tag{53}$$

We can simplify by the orthogonal function basis vector and eliminate the time depending parameters in the equation (53). The application of the property of the Vec operator (41) yields:

$$\begin{aligned}\left[I_{n\times N} - \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes A_{iN}\right)\right] Vec\,(X_N) = \\ \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes B_{iN}\right) Vec\,(U_N)\end{aligned} \tag{54}$$

or

$$Vec\,(X_N) = \mathbb{G}^{-1}\,\mathbb{H}\,Vec\,(U_N) \tag{55}$$

with the constant matrices

$$\mathbb{G} = I_{n\times N} - \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes A_{iN}\right),\ \ \mathbb{H} = \sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes B_{iN}.$$

On the other hand, we have

$$Y(t) \cong Y_N\, \Phi_N(t) = \sum_{i=0}^{N-1} C_{iN}\, \phi_i(t)\, X_N \Phi_N(t) \cong \sum_{i=0}^{N-1} C_{iN}\, X_N\, M_{iN} \Phi_N(t). \qquad (56)$$

The application of the Vec operator yields:

$$Vec(Y_N) = \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN} \right) Vec\left(X_N\right). \qquad (57)$$

4 Model-order reduction (MOR) using orthogonal functions

4.1 MOR with a transfer function representation

Consider a linear time invariant system described by the transfer function (23). The order of reduction can be chosen by the Hankel singular values. The reduced-order model have an order k and the following transfer function:

$$\frac{Y_r(p)}{U(p)} = \frac{d_0 + d_1 p + ... + d_l p^l}{c_0 + c_1 p + ... + c_{q-1} p^{q-1} + p^q} \qquad (58)$$

with $l \leq q < n$. The input-output differential equation of the reduced order system will be written as:

$$c_0 y_r(t) + c_1 y_r'(t) \cdots + c_{q-1} y_r^{(q-1)}(t) + y_r^{(q)}(t) = d_0 u(t) + d_1 u'(t) \cdots + d_l u^{(l)}(t), \qquad (59)$$

where $u(t)$ is the input and $y_r(t)$ is the output of the reduced order system.

The description of the reduced order system by orthogonal functions will have an analogue form to (30), given by the following relation:

$$Y_{rN} = U_N \mathbb{T}_r\, \mathbb{M}_r^{-1}, \qquad (60)$$

where $\mathbb{M}_r(c_0, \ldots, c_{q-1}) = c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N + I_N$ and $\mathbb{T}_r(d_0, \ldots, d_l) = d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l}$ are matrices depending on the parameters of the reduced order system and P_N the operational matrix of integration depending of the chosen orthogonal functions basis.

The reduced-order system is computed such that it has a similar input-output dynamical behavior to the original system for all inputs $u(t)$. When projected into the orthogonal functions basis, this condition yields:

$$Y_N\, \Phi_N(t) = Y_{rN}\, \Phi_N(t) \quad \Leftrightarrow \quad Y_N = Y_{rN}. \qquad (61)$$

The developments (60) and (30) lead to:

$$U_N \mathbb{T} \mathbb{M}^{-1} = U_N \mathbb{T}_r \mathbb{M}_r^{-1}, \qquad (62)$$

where $\mathbb{T}$ and $\mathbb{M}$ are constant matrices depending on the known parameters of the original system and the operational matrix of integration P_N given by (31).

The relation (62) must be verified for any input $u(t)$ (i.e. to any U_N). Therefore, we can formulate the equality (62) as:

$$\mathbb{T}\mathbb{M}^{-1} \left(c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N + I_N \right) = d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l} \qquad (63)$$

or

$$\left(d_0 P_N^q + d_1 P_N^{q-1} + ... + d_l P_N^{q-l}\right) - \mathbb{TM}^{-1}\left(c_0 P_N^q + c_1 P_N^{q-1} + ... + c_{q-1} P_N\right) = \mathbb{TM}^{-1}. \quad (64)$$

Let Θ be the vector of reduced order system parameters:

$$\Theta^T = \begin{bmatrix} d_0 & d_1 & \cdots & d_l & c_0 & \cdots & c_{q-1} \end{bmatrix}, \quad (65)$$

$\mathbb{A} = \begin{bmatrix} A_l & A_r \end{bmatrix}$ with

$$A_l = \begin{bmatrix} Vec(P_N^q) & Vec(P_N^{q-1}) \cdots & Vec(P_N^{q-l}) \end{bmatrix},$$

$$A_r = \begin{bmatrix} Vec(-\mathbb{TM}^{-1} P_N^q) & \cdots & Vec(-\mathbb{TM}^{-1} P_N) \end{bmatrix},$$

and

$$\mathbb{B} = Vec(\mathbb{TM}^{-1}).$$

Then the equation (64) can be written as

$$\mathbb{A}\,\Theta = \mathbb{B}. \quad (66)$$

The vector of the unknown parameters Θ are derived by means of least square resolution

$$\Theta = (\mathbb{A}^T\,\mathbb{A})^{-1}\mathbb{A}^T\,\mathbb{B}. \quad (67)$$

Remark 4.1 Extension to the MIMO LTI system case.
For MIMO LTI system described by a transfer matrix

$$H(s) = \begin{bmatrix} H_{11}(s) & \cdots & H_{1p}(s) \\ \cdots & \cdots & \cdots \\ H_{k1}(s) & \cdots & H_{kp}(s) \end{bmatrix} \quad (68)$$

the order reduction of $H(s)$ can be led by considering the order reduction of each partial transfer function $H_{ij}(s)$ between the i-input and j-output. Note that the reduced order choice of each transmittance $H_{ij}(s)$ can be made using the Hankel singular values technique [44].

4.2 Model order reduction with a state space LTI realization

Consider a linear time invariant (LTI) system described by the state realization (32). We are searching for a reduced order system having an order $r < n$ and the following realization

$$\begin{cases} \dot{X}_r(t) = A_r\,X_r(t) + B_r\,U(t), \\ Y_r(t) = C_r\,X_r(t). \end{cases} \quad (69)$$

Using the orthogonal functions (43) for the reduced-order model description, one obtains:

$$Vec(X_{Nr}) = \left[I_{r\times N} - \left(P_N^T \otimes A_r\right)\right]^{-1}\left(P_N^T \otimes B_r\right) Vec(U_N). \quad (70)$$

The reduced system is computed such that it has the same dynamical output as the original system for any input $U(t)$. This condition is equivalent to

$$Y = Y_r \quad \text{or} \quad C\,X = C_r\,X_r. \tag{71}$$

By projecting the relation (71) in the orthogonal functions basis, one has:

$$CX_N\Phi_N(t) = C_rX_{Nr}\Phi_N(t). \tag{72}$$

The simplification by the vector of orthogonal functions $\Phi_N(t)$ and the application of the Vec operator yields:

$$(I_N \otimes C)Vec(X_N) = (I_N \otimes C_r)Vec(X_{Nr}). \tag{73}$$

With combination by substitution of the equations (43), (70) and (73), we obtain the relation

$$\begin{array}{c}(I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B)Vec(U_N) = \\ (I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r)Vec(U_N).\end{array} \tag{74}$$

The relation (74) must be verified to get a convenient reduced system for any input U (i.e. for any matrix $U_N\ \Phi_N(t)$). Therefore, it gives the following equation which must be verified by the parameters of the reduced system:

$$(I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B) = (I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r). \tag{75}$$

The parameters of the reduced system with the realization (A_r, B_r, C_r) derived [45] by minimizing the norm ξ of the difference between both parts of the equation (75). This unconstrained minimization can be led by using the functions of the optimization tools or genetic algorithms. Then, the reduced model determination is brought back to the following optimization problem: derive A_r, B_r and C_r such that they minimize:

$$\xi = \left\| \begin{array}{c} (I_N \otimes C)(I_{n\times N} - P_N^T \otimes A)^{-1}(P_N^T \otimes B) \\ -(I_N \otimes C_r)(I_{r\times N} - P_N^T \otimes A_r)^{-1}(P_N^T \otimes B_r) \end{array} \right\|. \tag{76}$$

4.3 Model order reduction of LTV systems

In this section, we consider the order model reduction of the LTV systems defined by the realization (45). The reduced order system is taken equal to r and the state space description of the reduced system is the following:

$$\left\{ \begin{array}{l} \dot{\tilde{X}}(t) = \tilde{A}(t)\,\tilde{X}(t) + \tilde{B}(t)\,U(t), \\ \tilde{Y}(t) = \tilde{C}(t)\,\tilde{X}(t), \end{array} \right. \tag{77}$$

with $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$ varying in time with respective dimensions $r \times r$, $r \times m$ and $p \times r$. The description of the original system (45) using an orthogonal functions basis $\Phi_N(t)$ is given by the relations (55) and (57).

In the same manner, the variable in time parameters of the reduced LTV system will be defined by their projections on the orthogonal functions basis truncated to an order N:

$$\tilde{A}(t) \cong \sum_{i=0}^{N-1} \tilde{A}_{Ni}\,\varphi_i(t), \quad \tilde{B}(t) \cong \sum_{i=0}^{N-1} \tilde{B}_{Ni}\,\varphi_i(t), \quad \tilde{C}(t) \cong \sum_{i=0}^{N-1} \tilde{C}_{Ni}\,\varphi_i(t), \tag{78}$$

where $\tilde{A}_{Ni}$, $\tilde{B}_{Ni}$ and $\tilde{C}_{Ni}$ are constant with the same dimensions as $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$. Then, the description of the reduced-order model using the orthogonal functions basis can be written as:

$$\begin{array}{c}\left[I_{r\times N} - \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{A}_{iN}\right)\right] Vec\left(\tilde{X}_N\right) = \\ \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{B}_{iN}\right) Vec\left(U_N\right)\end{array} \tag{79}$$

and

$$Vec(\tilde{Y}_N) = \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes \tilde{C}_{iN}\right) Vec\left(\tilde{X}_N\right). \tag{80}$$

The equalization between the original system and the reduced system outputs can be expressed by the following relation : $\tilde{Y}_N = Y_N$ which can be written as

$$\left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN}\right) Vec\left(X_N\right) = \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes \tilde{C}_{iN}\right) Vec\left(\tilde{X}_N\right). \tag{81}$$

The substitution in (81) of $Vec(X_N)$ and $Vec(\tilde{X}_N)$ by their expressions (55) and (79) yields the following equality:

$$\begin{array}{l}\left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN}\right) \mathbb{G}^{-1}\mathbb{H}\, Vec\left(U_N\right) = \\ \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes \tilde{C}_{iN}\right) Q \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{B}_{iN}\right) Vec\left(U_N\right),\end{array} \tag{82}$$

where $Q = \left[I_{r\times N} - \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{A}_{iN}\right)\right]^{-1}$. This relation must be verified for any input $U(t)$. Then, one obtains:

$$\begin{array}{l}\left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN}\right) \mathbb{G}^{-1}\mathbb{H} = \\ \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes \tilde{C}_{iN}\right) \left[I_{r\times N} - \sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{A}_{iN}\right]^{-1} \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{B}_{iN}\right)\end{array} \tag{83}$$

with the constant matrices

$$\mathbb{G} = I_{n\times N} - \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes A_{iN}\right),\ \mathbb{H} = \sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes B_{iN}.$$

The parameters of the reduced order system $\tilde{A}_{iN}$, $\tilde{B}_{iN}$ and $\tilde{C}_{iN}$ can be derived by the minimization of the norm of the difference between both sides of the equation (83). Thus, the problem of the reduced model determination can be formulated as follows: determine $\tilde{A}_{iN}$, $\tilde{B}_{iN}$ and $\tilde{C}_{iN}$ in order to minimize

$$\xi = \left\| \begin{array}{c}\left(\sum_{i=0}^{N-1} M_{iN}^T \otimes C_{iN}\right) \mathbb{G}^{-1}\mathbb{H} \\ - \left(\sum_{i=0}^{N-1} M_{iN}^T \otimes \tilde{C}_{iN}\right) Q \left(\sum_{i=0}^{N-1} (M_{iN}P_N)^T \otimes \tilde{B}_{iN}\right)\end{array} \right\|. \tag{84}$$

Remark 4.2 Stability of the reduced model.
The reduced-order model is determined such that the error between its output vector and that of the original full-order system is minimized regardless the input signals. When the time horizon of approximation is sufficiently large to take in consideration the transient response and the steady state, then one may conclude that if the original system is stable, the reduced order one will be also stable.

Remark 4.3 Number of the orthogonal basis functions.
The accuracy and validity of the reduced model depend on the number of the orthogonal basis functions. The higher is the number of the basis functions, the more accurate is the obtained reduced order model. However, the size of the matrices interfering in the computation of the reduced order parameters and the calculus time cost increase with respect to the orthogonal basis dimension. Thus, the number of the basis functions is generally chosen such that it satisfies a compromise between the accuracy of the searched model and the computational constraints.

5 Simulation study

In order to illustrate the availability of the developed approaches for system order reduction, we consider in this section different examples of high order systems that we will reduce using a set of shifted Legendre polynomials with order $N = 16$ as an orthogonal functions basis.

5.1 LTI SISO system example

We consider the LTI system studied in [46] and given by the following transfer function

$$G(s) = \frac{s^4+35s^3+291s^2+1093s+1700}{s^9+9s^8+66s^7+294s^6+1029s^5+2541s^4+4689s^3+5856s^2+4620s+1700}. \tag{85}$$

The order reduction of this system has been led by both approaches developed in paragraph 4.1 using the transfer function representation and paragraph 4.2 using the state space description. The reduced order is taken $r = 3$.

The first approach yields the following reduced transfer function:

$$H_r(p) = \frac{0.3298\, s^2 - 1.713\, s + 3.232}{s^3 + 3.05\, s^2 + 4.992\, s + 3.232}$$

and by the second approach technique, we obtain the reduced state space description (32) with

$$A_r = \begin{bmatrix} 0.01032 & 1.349 & 8.391 \\ 0.08643 & -0.1717 & 3.45 \\ -0.5394 & 0.07006 & -2.741 \end{bmatrix}, \quad B_r = \begin{bmatrix} 3.508 \\ 3.375 \\ -1.235 \end{bmatrix},$$

$$C_r = \begin{bmatrix} 2.338 & 1.138 & 9.559 \end{bmatrix}.$$

Figure 1 shows the step responses of the original system (85) and the reduced systems (by transfer function and by state space methods). It appears from these simulations that the behavior of the reduced models obtained by the developed methods in this paper is very close to that of the original system which shows the availability of the proposed techniques. This property can also be verified with different inputs applied to the reduced order model.

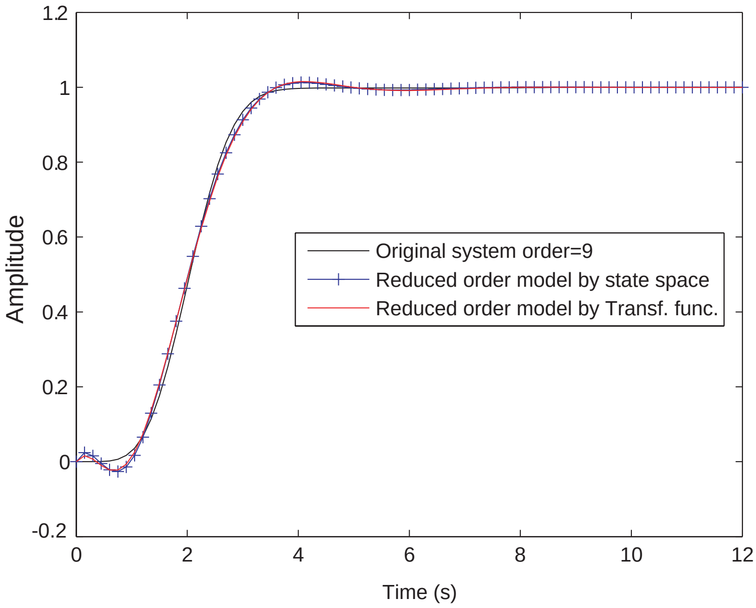

Figure 1: Step responses of the original system order $n = 9$ and reduced order models $r = 3$ obtained by the proposed techniques : starting from a state space realization and from a transfer function.

5.2 LTI MIMO system example: CD player

The proposed technique using orthogonal functions has been applied to the model of a CD player. This example is widely treated in many papers concerning MOR [49]. The considered model of CD player describes the dynamics between the lens actuator and the radial arm position as shown in Figure 2 and it is obtained using finite element approximation. Detailed description can be found in [50].

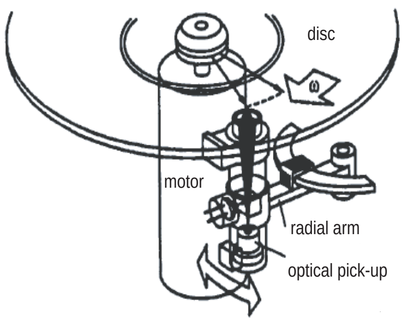

Figure 2: **CD player model. Source [50].**

The full-order model of the CD player is LTI MIMO having 120 states, 2 inputs and

2 outputs [21]. Gugercin and Antoulas proposed a reduced model having an order $r = 12$ by considering the system as LTI SISO. Chu and al. [51] reduced the LTI MIMO model to an order $r = 12$. The proposed technique has been applied to the full-order MIMO model and the reduction order is chosen to $r = 10$. The simulation of the error between the step responses of the original model and the reduced order ones obtained by the following techniques:
– The proposed technique using a shifted Legendre polynomials basis truncated to an order 10 on the time domain $[0, 50s]$,
– Balanced Hankel based (HMR) model reduction via square root method,
– Stochastic model truncation via Schur method (BST),
shows that the proposed reduction technique using orthogonal functions gives a minimal error converging to zero and the behavior of the obtained reduced model is close to the original 120-states full-model for any control input.

5.3 LTV system example

We consider now the LTV SISO system described by the state space realization (45) with

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 + e^{-t} & -2 + \frac{1}{8t} & -3 - 0.7\cos(-0.01t) & -2 + 0.5\cos(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 + 0.15\, cos(1.2t - 0.5) \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The variable in time system parameters are projected on the orthogonal functions basis. The obtained matrices $A_{iN} \in \mathbb{R}^{4\times4}, i \in \{0, 1, \ldots, 15\}$ and vectors $B_{iN} \in \mathbb{R}^{4\times1}, i \in \{0, 1, \ldots, 15\}$ resulting from this truncated development to the order 16, are used for computing the reduced order LTV model as shown in Section 4.3. The reduced order is chosen equal 2 ($r = 2$).

Figure 3 represents the time plot of the variable in time parameters of the reduced order model. Figure 4 shows that the step response of the reduced-order model (order $r = 2$) is close to the original system with order $n = 4$.

6 Conclusion

In this paper, new approaches have been introduced for the model order reduction of LTI and LTV systems using orthogonal functions as a tool of approximation. The proposed techniques can be applied to the order simplification of models defined either by an input-output relation or a state representation. Indeed, the projection of the input, the output and system variables on an orthogonal functions basis and the use of the operational matrices of integration and product have permitted the conversion of the system model from differential equations to algebraic ones. The minimization of the difference between the algebraic original system description and the algebraic reduced model have allowed the determination of the reduced order parameters.

Notice that the proposed order reduction techniques constitute an important contribution in the field of dynamical model simplification. These techniques come to reinforce

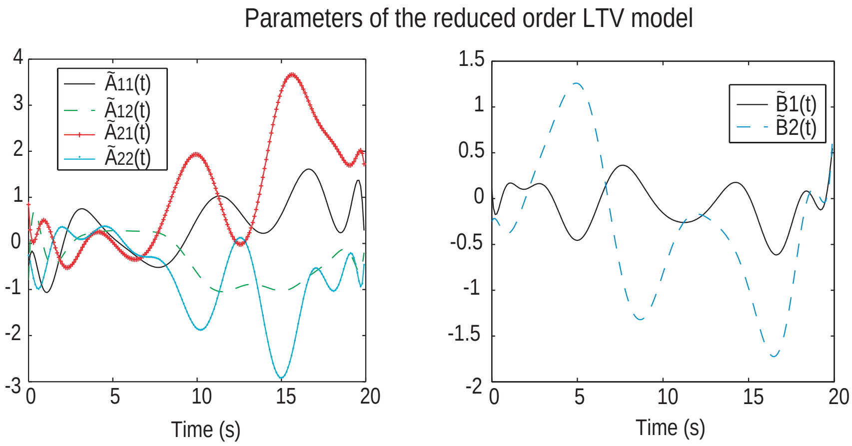

Figure 3: Time plots of the reduced order LTV system parameters obtained by the proposed technique.

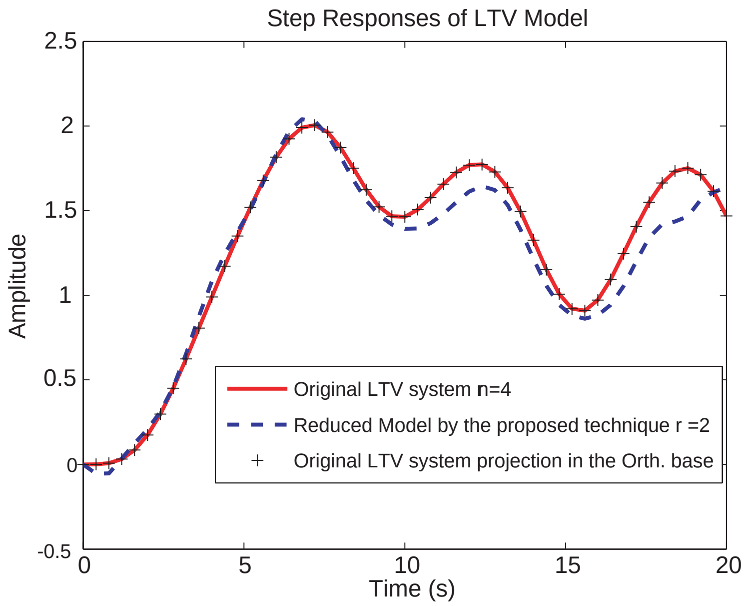

Figure 4: Step responses of the original LTV system, the projected system into Legendre shifted polynomials and the reduced order LTV model.

the existing approaches, especially in the case of LTV systems where only few methods with limited efficiency are published on the order reduction subject.

Finally, let us point out that the presented results in this paper are concerned with linear systems in both cases : time invariant and time variant parameters. However, it seems that they can be extended to some classes of nonlinear systems as bilinear systems.

This subject will be considered in our further works.

References

[1] Gugercin, S., Antoulas, A.C. and Bedrossian, N. Approximation of the International Space Station 1r and 12a models. In: *Proc. the IEEE 40th Conf. on Decision and Control.* Orlando, Florida, Dec. 2001, **2** pp. 1515–1516.

[2] Hall, K.C., Thomas, J.P. and Dowell, E.H. Reduced-order modeling of unsteady small-disturbance flows using a frequency-domain proper orthogonal decomposition technique. *The 37th Aerospace Sciences Meeting and Exhibit.* AIAA, No. 99–0655, 1999.

[3] Baldea, M. and Daoutidis, P. Model reduction and control of reactor-heat exchanger networks. *J. Process Cont.* **16** (2006) 265–274.

[4] Korenberg, M.J. and Hunter, I.W. The identification of nonlinear biological systems: Volterra Kernel approaches. *Annals of Biomedical Engineering* **24** (1996) 250–268.

[5] Lall, S. and Marsden, J.E. A subspace approach to balanced truncation for model reduction of nonlinear systems. *Int. J. Robust and Nonlinear Control* **12** (2002) 519–535.

[6] Sandberg, H. A case study in model reduction of linear time-varying systems. *Automatica* **42** (2006) 467–472.

[7] Sandberg, H. and Rantzer, A. Balanced truncation of linear time varying systems. *IEEE Trans. Aut. Cont.* **49** (2) (2004) 217–229.

[8] Antoulas, A.C., Sorensen, D.C. and Gugercin, S. A survey of model reduction methods for large-scale systems, Contemporary Mathematics. *AMS Publications* (280) (2001) 193–219.

[9] Glover, K. All optimal Hankel norm approximations of linear multivariable systems and their L^{∞} error bounds. *Int. J. of Control.* **39** (6) (1984) 1115–1193.

[10] Kung, S.Y. and Lin, D.W. Optimal Hankel-norm model reductions: multivariable systems. *IEEE Trans. Aut. Contr.* **1** (1981) 832–852.

[11] Moore, B.C. Principal component analysis in linear systems: controllability, observability and model reduction. *IEEE Trans. Automat. Contr.* 26 **1** (1981) 17–32.

[12] Pernebo, L. and Silverman, L.M. Model reduction via balanced state space representation. *IEEE Trans. Automat. Contr.* **27** (1982) 382–387.

[13] Phillips, J., Daniel,L. and Silveira, L.M. Guaranteed passive balancing transformations for model order reduction. *The 39th Design Automation Conference* (2002) 52–57.

[14] Sorensen, D.C. and Antoulas, A.C. The Sylvester equation and approximate balanced reduction. *Linear Algebra and its Applications* (2002) 671–700.

[15] Grimme, E.J. Krylov projection methods for model reduction. *Ph.D. Thesis.* ECE Dep., U. of Illinois, Urbana-Champaign, (1997).

[16] Gallivan, K., Grimme, E. and Van Dooren, P. Asymptotic waveform evaluation via a Lanczos method. *Applied Mathematics Letters* **7** (5) (1994) 75–80.

[17] Feldmann, P. and Freund, R.W. Efficient linear circuit analysis by Padé approximation via the Lanczos process. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* **14** (5) (1995) 639–649.

[18] Balk, I. On a Passivity of the Arnoldi based model order reduction for full-wave electromagnetic modeling *IEEE Transactions on Advanced Packaging* **24** (3) (2001) 304–308.

[19] Willcox, K., Peraire, J. and White, J. An Arnoldi approach for generation of reduced-order models for turbomachinery. *Computers and Fluids* **31** (3) (2002) 369–389.

[20] Willcox, K. and Peraire, J. Balanced model reduction via the proper orthogonal decomposition. *AIAA Journal* **40** (11) (2002) 2323–2330.

[21] Gugercin, S. and Antoulas, A.C. Model reduction of large scale systems by least squares. *Linear algebra and its applications* **415** (2006) 290–321.

[22] Ebihara, Y. and Hagiwara, T. On H_∞ model reduction using LMIs. *IEEE Trans. Aut. Cont.* **49** (7) (2004) 1187–1191.

[23] Shokoohi,S., Silverman, L.M. and Van Dooren, P.M. Linear time-variable systems: balancing and model reduction *IEEE Trans. Automat. Contr.* **28** (8) (1983) 810–822.

[24] Farhood, M., Beck, C.L. and Dullerud, G.E. Model reduction of periodic systems: A lifting approach. *Automatica.* (41) (2005) 1085–1090.

[25] Chen, C. and Hsiao, H. Time-domain synthesis via Walsh functions. *IEEE* **122** (1975) 565 – 570.

[26] Shih, L.S., Yeung, C.K. and McInis, B.G. Solution of state-space equations via Block-pulse functions. it Int. J. Contr. **28** (1978) 383–392.

[27] King, R.E. and Paraskevopoulos, P.N. Parametric identification of discrete-time SISO systems. *Int. J. Contr.* **30** (1979) 1023–1029.

[28] Paraskevopoulos, P.N. Chebychev series approach to system identification, analysis and optimal control. *J. Frankin. Inst.* **316** (1983) 135–157.

[29] Paraskevopoulos, P.N. and Kekkeris, G.Th. Hermite series approach to system identification, analysis and optimal control. In: *Proc. Meas. Contr. Conf.* Vol. **2**, Athens Greece, 1983, 146–149.

[30] Paraskevopoulos, P.N. Legendre series approach to identification and analysis of linear systems. *Trans. IEEE. Automat. Contr.* **AC-30** (6) (1985) 585–589.

[31] Benhadj Braiek, N. Application des fonctions de Walsh and des fonctions modulatrices à la modèlisation des systèmes continus non linéaires. *Ph.D. Thesis.* Université des Sciences and Techniques de Lille, Flandres Artois (1990).

[32] Ayadi, B. and Benhadj Braiek, N. MIMO PID Controllers synthesis using orthogonal functions. In: *IFAC 16th World Congress.* International Federation of Automatic Control, Prague, Czech Republic, July 2005.

[33] Ayadi, B. and Benhadj Braiek, N. Commande PID des systèmes linéaires à temps variant par utilisation des fonctions orthogonales. In: *CIFA, Conférence Internationale Francophone d'Automatique.* Bordeaux, France, 2006.

[34] Gradshteyn, I.S. and Ryzhik, I.M. Tables of Integrals, Series and Products, Academic Press, New York, 1979.

[35] Rotella, F. and Dauphin-Tanguy, G. Non-linear systems identification and optimal control. *Int. J. Control* **48** (2) (1988) 525–544.

[36] Bell, M.M. *Special functions for scientists and engineers* Van Nostrand, Princeton, NJ, 1968.

[37] Hwang, C. and Chen, M.Y. Analysis and parameter identification of bilinear systems via shifted Legendre polynomials. *Int. J. Contr.* **44** (2) (1986) 351–362.

[38] Hwang, C. and Guo, T.Y. Transfer function matrix identification in MIMO systems via shifted Legendre polynomials. *Int. J. Contr.* **39** (4) (1984) 807–814.

[39] Hwang, C. and Shih, Y.P. Parameter identification of a class of time-varying systems via orthogonal shifted Legendre polynomials. *J. Frankin. Inst.* **318** (1) (1984) 56–69.

[40] Chang, R.Y. and Wang, M.L. Parameter identification via shifted Legendre polynomials. *Int. J. Syst. Sci.* **13** (10) (1982) 1125–1135.

[41] Brewer, J.W. Kronecker products and matrix calculus in systems theory. *IEEE Trans. Circ. and Syst.* **CAS-25** (9) (1978) 772–781.

[42] Benhadj Braiek, N. and Rotella, F. Identification of non-stationary continuous systems using modulating functions. *J. Franklin Inst.* **327** (5) (1990) 831–840.

[43] N. Benhadj Braiek, F. Rotella, LOGID: a non linear system identification software in modeling and simulation of systems, P. Breedveld and al., J. C. Baltzer AG, Scientific Publishing Co., (1989) 211–217.

[44] Safonov,M.G., Chiang, R.Y. and Limebeer, J.N. Optimal Hankel model reduction for non-minimal systems. *IEEE Trans. Aut. Contr.* **34** (4) (1990) 496–502.

[45] Ayadi, B. and Benhadj Braiek, N. Réduction des systèmes LTI par utilisation des fonctions orthogonales. In: *JDMACS.* Reims, France, July 2007.

[46] Mukherjee, S., Satakshi, S. and Mittal, R.C. Model order reduction using response-matching technique. *J. Franklin Inst.* **342** (5) (2005) 503–519.

[47] Elloumi, S. and Benhadj Braiek, N. A decentralized stabilisation approach of a class of nonlinear polynomial interconnected systems application. *Nonlinear Dynamics and Systems Theory* **12** (2) (2012) 57–70.

[48] Sayem, H., Benhadj Braiek, N. and Hammouri, H. Trajectory planning and tracking of bilinear systems using orthogonal functions. *Nonlinear Dynamics and Systems Theory* **10** (3) (2010) 295–304.

[49] Chahlaoui, Y. and Dooren, P. V. *A collection of benchmark examples for model reduction of linear time invariant dynamical systems.* Technical Report 2002, 2, SLICOT, Feb. 2002.

[50] Steinbuch, M., Van Gross, P.J.M., Schootsra, G., Wortelboer, P.M.R. and Bosgra, O.H. Mu-synthesis of a compact disc player. *Int. J. Robust Nonlinear Control* **8** (1998) 169–189.

[51] Chu, C. C. et al. Model order reduction for MIMO systems using global Krilov subspace methods. *Math. Comput. Simul.*, 2007.