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Adaptive Synchronization for Oscillators in ϕ^6 Potentials

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Abstract: In this paper, we investigate adaptive synchronization for multi-parameter oscillators with ϕ^6 potentials. We consider the synchronization for known and unknown system parameters for the ϕ^6 Van-der Pol and Duffing oscillator based on a simple adaptive control technique; and show that a single-state adaptive feedback is sufficient to steer two identical oscillators to stable synchronization. We obtain some estimates of the unknown parameters for both systems and present numerical simulations to show the effectiveness of our approach.

Keywords: *synchronization; adaptive control; ϕ^6 oscillators.*

Mathematics Subject Classification (2010): 34C28, 34D06, 93C40, 93D21.

1 Introduction

The synchronization of chaotic oscillator is an intriguing phenomenon that has received considerable research attention during the last two decades. The increasing and enormous research activities on chaos synchronization is partly motivated by several promising real life applications; spanning areas such as secure communications, chaos generators design, chemical reactions, lasers, biological systems, information science, neural networks, etc [1–7]. For this reason, the study of chaos synchronization has grown rapidly since its discovery in 1990 by Pecora and Carroll [1]; and a wide variety of linear and nonlinear approaches have been proposed and well developed for achieving specific synchronization

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goals [1, 8–22]. These methods have been applied to many real physical model systems, such as the Nonlinear Bloch equations [31–33], lasers [34,35], Josephson junctions [36–38], various forms of the Van-der Pol and Duffing oscillators, including the ϕ^6 oscillators [23–29], to mention but a few.

The dynamics of nonlinear systems with ϕ^6 potentials [39–45], have become more fascinating because, ϕ^6 oscillators, in particular present more complex dynamics than their corresponding ϕ^4 oscillators. This property makes them better models for security of encrypted information during transmission. Thus, investigating chaos synchronization for ϕ^6 oscillators is very relevant for secure communications.

In [29], the author introduced a modified active control method for realizing the identical and non-identical synchronization of chaotic oscillators in ϕ^6 potentials. The proposed active control in [29] was specifically aimed at treating the problem of controller complexity arising in the application of the active control formalism. However, one of the two control inputs, that play the key role in driving the two oscillators to a synchronized state is still complex relative to the controlled systems, implying that the proposed approach remains questionable with regard to practical applications. Moreover, the parameters of the synchronizing systems in [29] were assumed to be known in advance, in all the cases considered. Since in practice the parameters of chaotic systems are not usually known in advance, it would be significant to investigate the synchronization for the case of unknown parameters as in [46]. In particular, for multi-parameter systems such as the ϕ^6 oscillators, estimating the unknown parameters of the systems is essential in the synchronization process.

In general, the problem of design flexibility and controller complexity has remained a crucial and long standing issue in control theory research [19, 47]. Recently, Guo [48], proposed a simple adaptive controller for the identical chaos synchronization. This technique was applied to achieve the adaptive and reduced-order synchronization for Josephson junctions and time-varying lower-order systems [38,49]; and further extended to realize the stabilization of chaotic systems [50, 51].

The goal of the present paper is to investigate the synchronization of ϕ^6 Van-der Pol and Duffing oscillators based on our proposed simple adaptive control [38, 48, 49]. We will show that for two identical ϕ^6 oscillators, a single-state adaptive feedback is sufficient to drive the oscillators to a stable synchronized state. Furthermore, we will consider the synchronization for unknown system parameters and obtain an estimate of all the unknown parameters of the systems. In the next section, we would give a brief theory of our proposed method. In Section 3, synchronization for identical ϕ^6 Van-der Pol and Duffing oscillators would be treated, both for known and unknown parameters; while in Section 4, we deal with the estimation of the unknown parameters. The paper is concluded in Section 5.

2 Theory of Adaptive-Feedback Control

2.1 Adaptive control for chaos synchronization

Let us introduce in this section, the adaptive control method [48] briefly. For a master chaotic system given as,

$$\dot{x} = f(x), \tag{1}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathrm{R}^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T : \mathrm{R}^n \to \mathrm{R}^n$ is a nonlinear vector function. Without loss of generality, let $\Omega \subset \mathrm{R}^n$ be a chaotic bounded set of system (1) which is globally attractive. For the vector function $f(x)$, we give a general assumption.

Assumption 1 $\forall x = (x_1, x_2, \cdots, x_n)^T \in \Omega$ and $y = (y_1, y_2, \cdots, y_n)^T \in \Omega$ there exists a constant $l > 0$ satisfying

$$| f_i(x) - f_i(y) | \leq l \mid x - y \mid_\infty, i = 1, 2, \cdots, n, \tag{2}$$

where $| x - y |_\infty$ is the ∞-norm of $x - y$. i.e., $| x - y |_\infty = \max_j | x_j - y_j |, j = 1, 2, \cdots, n$.

Remark 1 This condition is very loose, and in fact, holds as long as $\partial f_i/\partial x_j (i, j = 1, 2, \cdots, n)$ are bounded. Thus, the class of systems in the form of (1) and (2) includes almost all well-known finite-dimensional chaotic and hyperchaotic systems. The corresponding slave system to system (1) is as follows,

$$\dot{y} = f(y) + k_1(y - x) = f(y) + u, \tag{3}$$

where the controller $u = k_1 e = (k_1 e_1, k_1 e_2, \cdots, k_1 e_n)^T, e_i = y_i - x_i$. Unlike the usual linear feedback control, the feedback gain k_1 is duly adapted according to the following update law,

$$\dot{k}_1 = -\gamma \sum_{i=1}^{n} e_i^2, \tag{4}$$

where γ is an arbitrary positive constant. The controller $u = k_1 e$ can realize the synchronization of the master and slave chaotic systems (1) and (2).

Remark 2 The feedback gain k_1 is automatically adapted to a suitable strength k_0 depending on the initial values, which is significantly different from the well known linear feedback.

Remark 3 The controller $u = k_1 e$ can employ only one feedback term e_i for some chaotic systems. The feedback term e_i is selected such that, if $e_i = 0$ then $e_j = 0, j = 1, 2, \cdots n, j \neq i$, so that the set $E = \{(e, k_1) \in \mathrm{R}^{n+1} | e = 0, k_1 = k_0\}$. Thus, leading to the above conclusion.

2.2 Adaptive control for chaos synchronization with unknown parameters

In our previous paper [49], we obtained a novel adaptive controller for chaos synchronization with unknown parameters. This is introduced in brief herein. Consider a nonlinear dynamical system

$$\dot{x} = f(x) + g(x)p, \tag{5}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in \mathrm{R}^n$ denotes the state variables, $p = (p_1, p_2, \cdots, p_k)^T \in \mathrm{R}^k$ denotes the uncertain parameters, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))^T$ and $g(x) = [g(x)]_{n \times k}$ represent differential nonlinear vector function and matrix function respectively. The vector function $f(x)$ satisfies Assumption 1.

We consider system (5) as the master system and introduce a controlled slave system

$$\dot{y} = f(y) + g(y)p + u, \tag{6}$$

where $y = (y_1, y_2, \cdots, y_n)^T \in \mathrm{R}^n$ denotes the state variables, and $u = (u_1, u_2, \cdots, u_n)^T$ is a controller. The main goal is to design a suitable controller u to synchronize the two identical systems in spite of their uncertain parameters. We denote the synchronization error between the two systems as $e = y - x \in \mathrm{R}^n$ and subtract system (5) from system (6) and thus obtain the error dynamical system

$$\dot{e} = f(y) - f(x) + [g(y) - g(x)]p + u. \tag{7}$$

We can introduce the control function

$$u = -[g(y) - g(x)]\hat{p} + k_1 e, \tag{8}$$

where $\hat{p}$ is the estimate of p, and $k_1 e = (k_1 e_1, k_1 e_2, \cdots, k_1 e_n)^T \in \mathrm{R}^n$ is the linear feedback control with the updated gain $k_1 \in \mathrm{R}^1$. Thus, the synchronization error system is reduced to

$$\dot{e} = [f(y) - f(x)] + [g(y) - g(x)]\tilde{p} + k_1 e, \tag{9}$$

where $\tilde{p} = p - \hat{p}$ is the parameter estimation mismatch between the real value of the unknown parameter and its corresponding estimated value. Then the above discussion can be summarized in the following theorem.

Theorem 1 *If the estimations of the unknown parameters and the feedback gain contained in the adaptive controller (8) are updated by the following laws*

$$\begin{cases} \dot{\hat{p}} = [g(y) - g(x)]^T e, \\ \dot{k}_1 = -\gamma e^T e = -\gamma \displaystyle\sum_{i=1}^{n} e_i^2, \end{cases} \tag{10}$$

then, the synchronization between system (5) and (6) will be achieved.

Remark 4 The control term $k_1 e$ can include only one feedback term e_i for some chaotic systems. The feedback term e_i is selected such that if $e_i = 0$ then $e_j = 0, j = 1, 2, \cdots n, j \neq i$, therefore the set $E = \{(e, \tilde{p}, k_1) \in \mathrm{R}^{n+k+1} | e = 0, \tilde{p} = 0, k_1 = -k^*\}$, so that the conclusion in (10) is obtained.

3 Synchronization of Two Identical ϕ^6 Van-der Pol and Duffing Oscillators

3.1 Example 1. ϕ^6 Van-der Pol oscillators

The ϕ^6 Van-der Pol oscillators could be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 - \delta x_1^5 + f cos(\omega t), \end{aligned} \tag{11}$$

where $\mu, \alpha, \beta, \delta, f, \omega$ are parameters of the system (11). Let system (11) be the master system.

Case 1: The parameters $(\mu, \alpha, \beta, \delta, f, \omega)$ of the master system (11) are assumed to be known. According to Ref. [48], the slave system with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + k_1(y_1 - x_1), \\ \dot{y}_2 &= \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 - \delta y_1^5 + f cos(\omega t), \end{aligned} \tag{12}$$

where the feedback gain k_1 is adapted according to the following update law $\dot{k}_1 = -\gamma e_1^2$. The strength of the arbitrary constant γ would influence the controller performance. For instance, large γ would lead to fast stabilization of the augment system; while the feedback gain k_1 would quickly approach a suitable negative constant. On the contrary, the speed of stabilization would be very slow for small γ; so that the feedback gain k_1 would slowly approach a suitable negative constant.

Next, we give numerical simulations to verify the above theoretical result. Firstly, let the initial conditions of the master system (11) be: $x_1(0) = 1, x_2(0) = 2$, the slave system (12): $y_1(0) = -3, y_2(0) = 4$ and $\mu = 0.4, \alpha = 1.0, \beta = -0.7, \delta = 0.1, f = 9, \omega = 3.14$. With the initial value of the controller $k_1(0) = -1$, Figure 1 shows the synchronization performance. The error system of two identical ϕ^6 Van-der Pol oscillators approaches zero asymptotically as $t \to \infty$, while the feedback gain k_1 tends to a negative constant. For other sets of initial conditions and system parameters, the synchronization is still achievable. For instance, let the initial conditions be $x_1(0) = 1, x_2(0) = 2$, and $y_1(0) = -3, y_2(0) = 1$; while the system parameters are: $\mu = 0.4, \alpha = 0.46, \beta = 1.0, \delta = 0.1, f = 4.5, \omega = 0.86$. Using the same initial value of the controller $k_1(0) = -1$, as before, Figure 2 shows that the two identical Φ^6 Van-der Pol oscillators achieves asymptotic synchronization as $t \to \infty$, while the feedback gain k_1 tends to a negative constant.

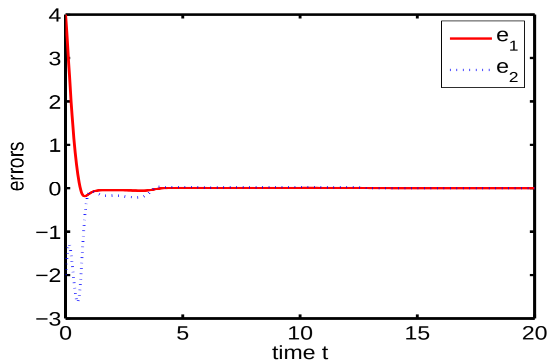

Figure 1: The error system of two identical Φ^6 Van-der Pol oscillators ($\mu = 0.4, \alpha = 1.0, \beta = -0.7, \delta = 0.1, f = 9, \omega = 3.14$) is asymptotically stable as $t \to \infty$.

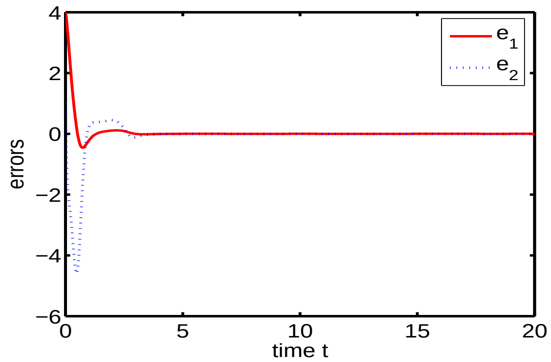

Figure 2: The error system of two identical Φ^6 Van-der Pol oscillators ($\mu = 0.4, \alpha = 0.46, \beta = 1.0, \delta = 0.1, f = 4.5, \omega = 0.86$) is asymptotically stable as $t \to \infty$.

Case 2: The parameters $(\mu, \alpha, \beta, \delta)$ of system (11) are unknown.

According to our method in Ref. [49], the slave system with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= \mu(1 - y_1^2)y_2 - \alpha y_1 - \beta y_1^3 - \delta y_1^5 + f cos(\omega t) + u_2, \end{aligned} \tag{13}$$

where the controller $u = (u_1, u_2)^T$ is defined as,

$$\begin{aligned} u_1 &= k_1(y_1 - x_1), \\ u_2 &= -(1 - y_1^2)y_2 - (1 - x_1^2)x_2)\mu_e - (x_1 - y_1)\alpha_e - (x_1^3 - y_1^3)\beta_e - (x_1^5 - y_1^5)\delta_e. \end{aligned} \tag{14}$$

In eq. (14) μ_e, α_e, β_e, and δ_e are the estimated values of the parameters μ, α, β, and δ, respectively. Again, the feedback gain k_1 is dully adapted according to the update law $\dot{k}_1 = -\gamma e_1^2$.

To verify that the controller $u = (u_1, u_2)^T$ would drive the systems to synchrony, we give some numerical simulation results, firstly, by selecting the initial conditions of the master system (11): $x_1(0) = 1, x_2(0) = 2$, the slave system (12): $y_1(0) = -3, y_2(0) = 4$, $f = 9, \omega = 3.14$. The initial values of the estimated parameters are $\mu_e = 0.6, \alpha_e = 1.2, \beta_e = -0.4$, and $\delta_e = 0.3$. Using the same initial controller gain, $k_1(0) = -1$, we illustrate in Figure 3 the synchronization behaviour of the two identical ϕ^6 Van-der Pol oscillators with unknown parameters. Clearly, asymptotically synchronization is achieved as $t \to \infty$, while the feedback gain k_1 tends to a negative constant. We may also consider other sets of initial conditions and estimating parameters. For instance, let $x_1(0) = 1$, $x_2(0) = 2$, and $y_1(0) = -3$, $y_2(0) = 4$, $f = 9, \omega = 3.14$. Similarly, let the initial values of the estimated parameters be $\mu_e = 0.5$, $\alpha_e = 0.5$, $\beta_e = 1.1$, $\delta_e = 0.12$. This case is shown in Figure 4, confirming that the synchronization is fully guaranteed for the two identical ϕ^6 Van-der Pol oscillators.

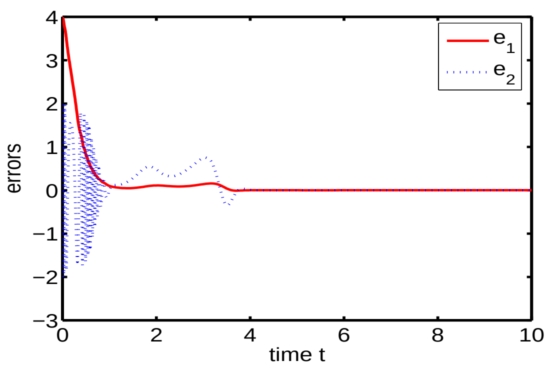

Figure 3: The error system of two identical Φ^6 Van-der Pol oscillators with unknown parameters is asymptotically stable as $t \to \infty$.

3.2 Example 2. ϕ^6 Duffing oscillators

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\lambda x_2 - \alpha x_1 - \beta x_1^3 - \delta x_1^5 + f cos(\omega t), \end{aligned} \tag{15}$$

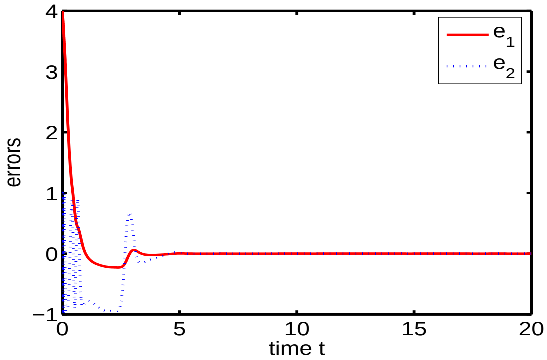

Figure 4: The error system of two identical Φ^6 Van der Pol oscillators with unknown parameters is asymptotically stable as $t \to \infty$.

where λ, α, β, δ, f, and ω are parameters of the system (15). Let system (15) be the master oscillator.

Case 1: All the parameters $(\lambda, \alpha, \beta, \delta, f, \omega)$ of system (15) are known in advance. According to our method in Ref. [48], the slave oscillator with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + k_1(y_1 - x_1), \\ \dot{y}_2 &= -\lambda y_2 - \alpha y_1 - \beta y_1^3 - \delta y_1^5 + f cos(\omega t), \end{aligned} \tag{16}$$

the feedback gain k_1 is adapted according to the following update law $\dot{k}_1 = -\gamma e_1^2$.

Again, we give numerical simulations to verify the above theoretical results. First, we select the initial states values of the master system (15) as follows: $x_1(0) = 1, x_2(0) = 2$, the slave system (16): $y_1(0) = -3, y_2(0) = 4$ and $\lambda = 0.4$, $\alpha = 1.0$, $\beta = -0.7$, $\delta = 0.1$, $f = 9$, $\omega = 3.14$. With the initial value of the adaptive controller $k_1(0) = -1$, Figure 5 shows that the two identical ϕ^6 Duffing oscillators achieve stable synchronization as $t \to \infty$, while the feedback gain k_1 tends to a negative constant. For other set of initial conditions, namely, for the master: $x_1(0) = 1, x_2(0) = 2$, the slave system: $y_1(0) = -3, y_2(0) = 1$ and $\lambda = 0.4$, $\alpha = 0.46$, $\beta = 1.0$, $\delta = 0.1$, $f = 4.5$, $\omega = 0.86$, Figure 6 shows that the synchronization is also attained with the initial adaptive controller $k_1(0) = -1$. The feedback gain k_1 also tends to a negative constant.

Case 2: The parameters $(\lambda, \alpha, \beta, \delta)$ of system (15) are unknown.

Using our method in Ref. [49], the slave system with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1, \\ \dot{y}_2 &= -\lambda y_2 - \alpha y_1 - \beta y_1^3 - \delta y_1^5 + f cos(\omega t) + u_2, \end{aligned} \tag{17}$$

and the controller $u = (u_1, u_2)^T$ is as follows,

$$\begin{aligned} u_1 &= k_1(y_1 - x_1), \\ u_2 &= -(-y_2 + x_2)\lambda_e - (x_1 - y_1)\alpha_e - (x_1^3 - y_1^3)\beta_e - (x_1^5 - y_1^5)\delta_e, \end{aligned} \tag{18}$$

where $\lambda_e, \alpha_e, \beta_e, \delta_e$ are the estimating value of the parameters $\lambda, \alpha, \beta, \delta$ respectively, and the feedback gain k_1 is adapted according to the update law $\dot{k}_1 = -\gamma e_1^2$.

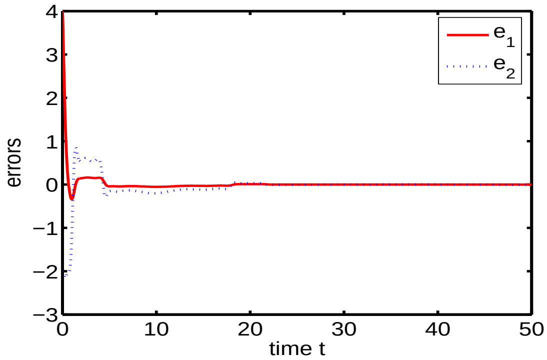

Figure 5: The error system of two identical Φ^6 Duffing oscillators ($\lambda = 0.4$, $\alpha = 1.0$, $\beta = -0.7$, $\delta = 0.1$, $f = 9$, $\omega = 3.14$) is asymptotically stable as $t \to \infty$.

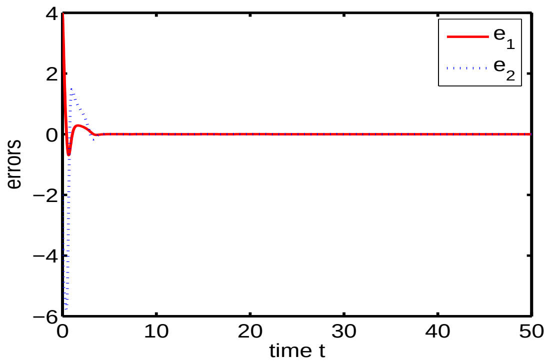

Figure 6: The error system of two identical Φ^6 Duffing oscillators ($\lambda = 0.4$, $\alpha = 0.46$, $\beta = 1.0$, $\delta = 0.1$, $f = 4.5$, $\omega = 0.86$) is asymptotically stable as $t \to \infty$.

Figure 7 shows the results of the numerical simulations. We first, selected the initial states values of the master system (17) as: $x_1(0) = 1, x_2(0) = 2$, and the slave system (18) as: $y_1(0) = -3, y_2(0) = 4$, $f = 9$, $\omega = 3.14$, the initial values of the estimating parameters are $\lambda_e = 0.6$, $\alpha_e = 1.2$, $\beta_e = -0.4$, $\delta_e = 0.3$. With the initial value of the controller being $k_1(0) = -1$, the error system of two identical ϕ^6 Duffing oscillators with unknown parameters is asymptotically stabilized as $t \to \infty$, while the feedback going k_1 tends to a negative constant; implying that synchronization is achieved. We consider also, other choice of initial conditions, namely, $x_1(0) = 1, x_2(0) = 2$ (master) and $y_1(0) = -3, y_2(0) = 4$ (slave) and the parameters values $f = 9, \omega = 3.14$, $\lambda_e = 0.5, \alpha_e = 0.5, \beta_e = 1.1$, and $\delta_e = 0.12$; and find that synchronization is still achieved as shown in Figure 8.

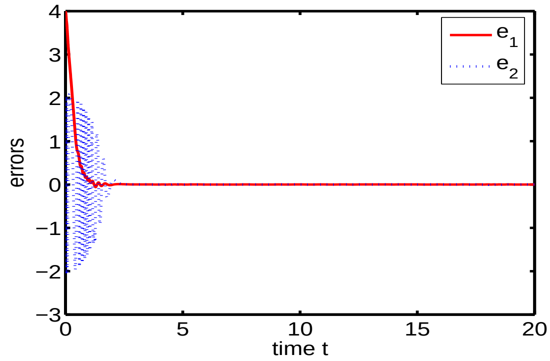

Figure 7: The error system of two identical Φ^6 Duffing oscillators with unknown parameters is asymptotically stable as $t \to \infty$.

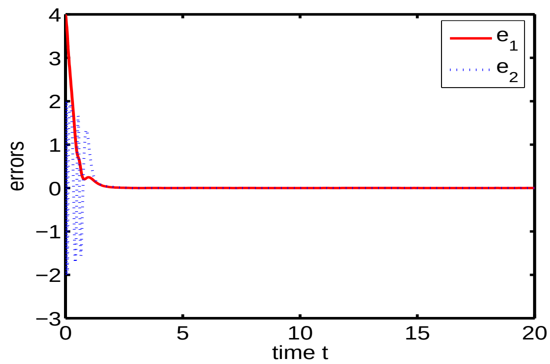

Figure 8: The error system of two identical Φ^6 Duffing oscillators with unknown parameters is asymptotically stable as $t \to \infty$.

4 Estimation of the Unknown Parameters of Van-der Pol and Duffing Oscillators

Example 1 ϕ^6 Van-der Pol oscillators:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - \alpha x_1 - \beta x_1^3 - \delta x_1^5 + f cos(\omega t), \end{aligned} \tag{19}$$

where $\mu = 0.4$, $\alpha = 1$, $\beta - 0.7$, $\delta = 0.1$, $f = 9$, and $\omega = 3.14$ are parameters of the system (19), and we let system (19) be the master system; where the parameters μ, α, β are unknown. According to Ref. [46], the slave system with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + k_1(y_1 - x_1), \\ \dot{y}_2 &= \mu_e(1 - y_1^2)y_2 - \alpha_e y_1 - \beta_e y_1^3 - \delta y_1^5 + f cos(\omega t) + k_1(y_2 - x_2), \end{aligned} \tag{20}$$

where μ_e, α_e, and β_e are the estimated parameters μ, α, and β, respectively, which are adapted according to the update law,

$$\begin{aligned} \dot{\mu}_e &= -(1-y_1^2)y_2(y_2-x_2), \\ \dot{\alpha}_e &= y_1(y_2-x_2), \\ \dot{\beta}_e &= y_1^3(y_2-x_2), \end{aligned} \tag{21}$$

and the feedback gain k_1 is adapted according to the update law $\dot{k}_1 = -(e_1^2+e_2^2)$.

In what follows, we give numerical simulations to verify the above theoretical results. Firstly, by selecting the initial state values of the master system (19): $x_1(0) = 1, x_2(0) = 2$, the slave system (8): $y_1(0) = -3, y_2(0) = 4$, $f = 9, \omega = 3.14$, the initial values of the estimating parameters are $\lambda_e = 0.6, \alpha_e = 1.2, \beta_e = -0.4$. With the initial value of the controller $k_1(0) = -1$, Figure 9 shows that the error system of two identical Φ^6 Van-der Pol oscillators with unknown parameters is asymptotically stable as $t \to \infty$, while the feedback gain k_1 tends to a negative constant. Figure 10 shows that the estimating parameters λ_e, α_e and β_e converge to its true value λ, α and β, respectively.

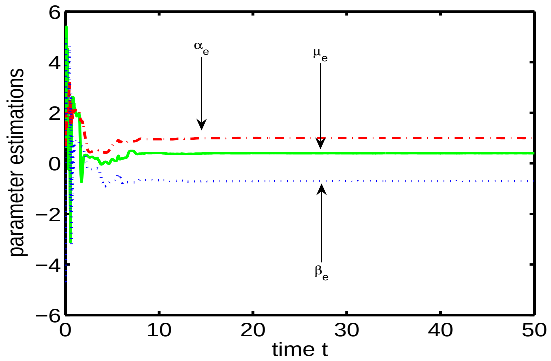

Figure 9: The estimated parameters μ_e, α_e, β_e, converge to its true value μ, α, β respectively.

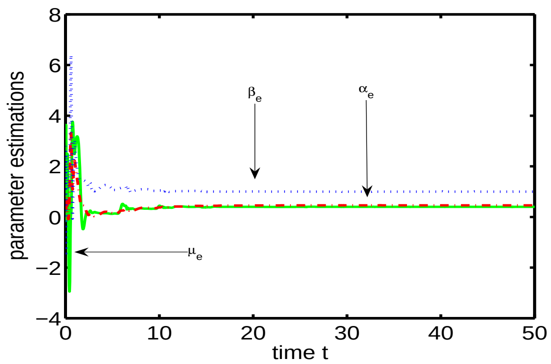

Figure 10: The estimated parameters μ_e, α_e, β_e, converge to its true value μ, α, β respectively.

Example 2 ϕ^6 Duffing oscillators:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\lambda x_2 - \alpha x_1 - \beta x_1^3 - \delta x_1^5 + f cos(\omega t), \end{aligned} \tag{22}$$

where $\lambda = 0.01$, $\alpha = 1.0$, $\beta - 0.495$, $\delta = 0.05$,f=0.78, and $\omega = 0.55$ are parameters of system (19). Let system (19) be the master system; where the parameters α, and β are unknown. Following Ref. [46], the slave system with adaptive controller is as follows,

$$\begin{aligned} \dot{y}_1 &= y_2 + k_1(y_1 - x_1), \\ \dot{y}_2 &= -\lambda y_2 - \alpha_e y_1 - \beta_e y_1^3 - \delta y_1^5 + f cos(\omega t) + k_1(y_2 - x_2), \end{aligned} \tag{23}$$

where α_e, β_e is the estimated parameters α, β respectively, which are adapted according to the update law,

$$\begin{aligned} \dot{\alpha}_e &= y_1(y_2 - x_2), \\ \dot{\beta}_e &= y_1^3(y_2 - x_2), \end{aligned} \tag{24}$$

and the feedback gain k_1 is adapted according to the update law $\dot{k}_1 = -(e_1^2 + e_2^2)$.

Numerical simulation results for the drive-response system (23) and (24), we carried out using the following initial conditions for the master system (23): $x_1(0) = 1, x_2(0) = 2$, and for the slave system (24): $y_1(0) = -3, y_2(0) = 4$. The other parameters are set as follows: $f = 9, \omega = 3.14$, while the initial values of the estimating parameters are $\alpha_e = 1.2$, and $\beta_e = -0.4$. With the initial value of the controller $k_1(0) = -1$, Figure 11 shows that the error system of two identical Φ^6 Duffing oscillators with unknown parameters is asymptotically stabilized as $t \to \infty$, while the feedback gain k_1 approaches a negative constant. Figure 12 shows also that the estimating parameters α_e, and β_e, converge to its true values α and β, respectively.

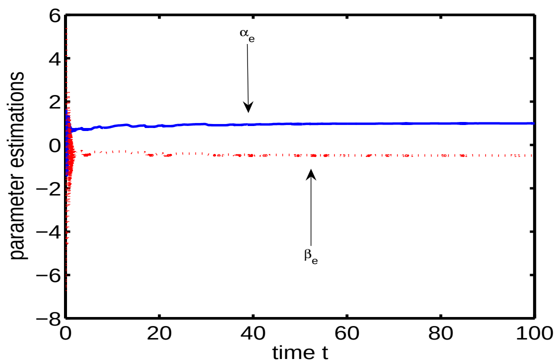

Figure 11: The estimated parameters α_e, β_e, converge to its true values α, β respectively.

5 Conclusion

In this paper, we have examined the synchronization of oscillating particles in ϕ^6 potentials based on adaptive control approach. The adaptive approach that we have employed is such that a single-state feedback is sufficient to steer the synchronization of two identical oscillators. We demonstrate this method for the ϕ^6 Van-der Pol and Duffing oscillators and realized the synchronization also for the case of unknown system parameters. For

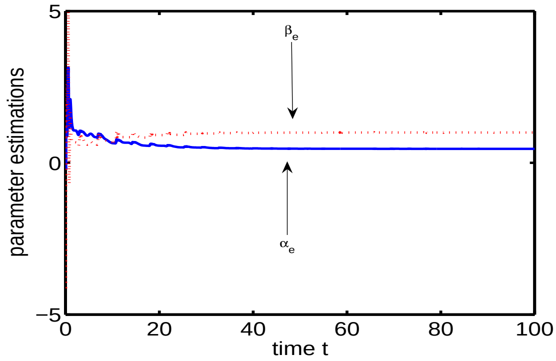

Figure 12: The estimated parameters α_e, β_e, converge to its true value α, β respectively.

the case of parameter unknown, we estimated the all the unknown parameters. We note that our proposed approach gives rise to simpler control control input, especially when the parameters are known in advance. Such adaptive control approach would have advantage in practical applications, since it could be easily realized compared to existing methods.

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