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Existence of Almost Periodic Solutions to Some Singular Differential Equations

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Abstract: In this paper we make use of the well-known Drazin inverse to study and obtain the existence of almost periodic solutions to some singular systems of first- and second-order differential equations with complex coefficients in the case when the forcing term is almost periodic. In order to illustrate our abstract results, an example will be discussed at the end of the paper.

Keywords: *Drazin inverse; almost periodic; singular system of differential equation; singular system of second-order differential equation.*

Mathematics Subject Classification (2010): 34A30, 34C20.

1 Introduction

Let $\mathbb{C}^m$ be the m-dimensional complex space, which we equip with its natural Euclidean norm $|\cdot|$ and inner product $<\cdot,\cdot>$. Let $M(m,\mathbb{C})$ stand for the collection of all $m\times m$-square matrices with complex entries. If $A\in M(m,\mathbb{C})$ then its index which we will denote by $i(A)$ is the smallest nonnegative integer k such that

$$\mathrm{rank}(A^k)=\mathrm{rank}(A^{k+1}).$$

If $A\in M(m,\mathbb{C})$, then the Drazin inverse A^D of A is the matrix $X\in M(m,\mathbb{C})$ satisfying the following three properties:

$$AX=XA,\quad XAX=X,\quad XA^{k+1}=A^k,$$

where $k=i(A)$.

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If we denote the zero matrix of $\mathbb{C}^m$ by O, and if we assume that the Jordan decomposition of $A \in M(m, \mathbb{C})$ is given by

$$A = T \begin{pmatrix} C & O \\ O & N \end{pmatrix} T^{-1},$$

where $C \in M(r, \mathbb{C})$ is (nonsingular) invertible and $N \in M(n-r, \mathbb{C})$ is nilpotent of order k ($N^k = O$ and $N^{k-1} \neq O$), then A^D is given by

$$A^D = T \begin{pmatrix} C^{-1} & O \\ O & O \end{pmatrix} T^{-1}.$$

It should be mentioned that if A is nilpotent, then $A^D = O$. Similarly, if A is (nonsingular) invertible, then $A^D = A^{-1}$. Now, the special case $i(A) = 1$ is equivalent to $N = O$. In this event, A^D is called the group inverse of A and is denoted by $A^{\#}$.

The Drazin inverse is a powerful tool when it comes to studying singular systems of differential equations, singular systems of difference equations, Markov Chains, see for instance Campbell [6]. For more on the Drazin inverse and related issues we refer the reader to the landmark books of Campbell [3, 4].

In this paper we make use of the Drazin inverse to study and obtain the existence of almost periodic solutions to the singular system of differential differential equation

$$Au'(t) + Bu(t) = f(t), \quad t \in \mathbb{R}, \tag{1}$$

where A, B are (possibly singular) $m \times m$-square matrices with complex entries and $f : \mathbb{R} \mapsto \mathbb{C}^m$ is $C^{(k)}$-almost periodic with $k = i(A)$ (Theorem 3.5).

Next, we make use of Theorem 3.5 and its consequences to study and obtain the existence of almost periodic solutions to some general singular systems of second-order differential equations (Corollary 3.2).

Our work will be heavily based upon that of Campbell [3, 4] on the existence of solutions to singular systems of differential equations. In particular, we will consider two important cases. We first consider the case when $AB = BA$ and $N(A) \cap N(B) = \{0\}$. The second case which will be a consequence of the first one requires the existence of a $\lambda \in \mathbb{C}$ such that $(\lambda A + B)^{-1}$ exists.

An important assumption that we will make consists of assuming that $A^D B$ (respectively, $A_z^D B_z$) is symmetric, has a spectral decomposition, and that $\sigma(A^D B) - \{0\} \neq \emptyset$ with $\Re e\, \lambda > 0$ for all $\lambda \in \sigma(A^D B) - \{0\}$. This assumption excludes in particular the case when $A^D B$ (respectively, $A_z^D B_z$) is nilpotent.

The existence of almost periodic solutions to differential equations is one of the most attractive topics in qualitative theory of differential equations due to applications [1, 8, 10, 11]. However, to the best of the authors knowledge, the existence of almost periodic solutions to singular systems of differential equations of the form (1) remains an untreated question, which is the mean motivation of this paper.

This paper is organized as follows. Section 2 will cover almost periodic and $C^{(n)}$-almost periodic functions [10]. Section 3 discusses our main results and its consequences. Section 4 will be devoted to the case of singular systems of second-order differential equations. In Section 5, we consider an illustrative example.

2 Almost Periodic and $C^{(l)}$-Almost Periodic Functions

Most of the material of this section is taken from [1, 8, 10]. Let $C(\mathbb{R}, \mathbb{C}^m)$ stand for the collection of continuous functions from $\mathbb{R}$ into $\mathbb{C}^m$. Define $C^{(l)}(\mathbb{R}, \mathbb{C}^m)$ as the collection of functions $f : \mathbb{R} \mapsto \mathbb{C}^m$ such that $f^{(k)}$ exists and belongs to $C(\mathbb{R}, \mathbb{C}^m)$ for $k = 0, 1, 2, ..., l$. (The symbol $f^{(k)}$ being the k-derivative of f with $f^{(0)}$ corresponding to the continuity of the function f.)

Define $BC^{(l)}(\mathbb{R}, \mathbb{C}^m)$ as the collection of all functions $f \in C^{(l)}(\mathbb{R}, \mathbb{C}^m)$ such that

$$\|f\|_{(l)} := \sup_{t\in\mathbb{R}} \sum_{k=0}^{l} |f^{(k)}(t)| < \infty.$$

Clearly, $(BC^{(l)}(\mathbb{R}, \mathbb{C}^m), \|\cdot\|_{(l)})$ is a Banach space.

In this paper, the symbols $f^{(0)}$, $\|\cdot\|_{(0)}$, $C^{(0)}(\mathbb{R}, \mathbb{C}^m)$, $BC^{(0)}(\mathbb{R}, \mathbb{C}^m)$, and $AP^{(0)}(\mathbb{C}^m)$ stand respectively for f, $\|\cdot\|_\infty$, $C(\mathbb{R}, \mathbb{C}^m)$, $BC(\mathbb{R}, \mathbb{C}^m)$, and $AP(\mathbb{C}^m)$.

2.1 Almost periodic functions

Definition 2.1 [1, 8] (Bor) A function $f \in C(\mathbb{R}, \mathbb{C}^m)$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that $|f(t + \tau) - f(t)| < \varepsilon$ for all $t \in \mathbb{R}$. The collection of those functions is denoted by $AP(\mathbb{C}^m)$.

Definition 2.2 [1, 8] (Bochner) A continuous function $f : \mathbb{R} \to \mathbb{C}^m$ is said to be Bochner almost periodic if for every sequence of real numbers $(\sigma'_n)_{n\in\mathbb{N}}$ has a subsequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\{f(\sigma_n + t)\}$ converges uniformly in $t \in \mathbb{R}$.

It is well-known that Definition 2.1 and Definition 2.2 are equivalent (see Corduneanu [8]). In what follows we give another equivalent definition using trigonometric polynomials.

Basic properties of almost periodic functions include the following:

Theorem 2.1 *If $f : \mathbb{R} \to \mathbb{C}^m$ is almost periodic, then f is uniformly continuous in $t \in \mathbb{R}$. Moreover, the range $R(f) = \Big\{f(t) : t \in \mathbb{R}\Big\}$ is precompact in $\mathbb{C}^m$.*

Corollary 2.1 *Any almost periodic function is bounded on $\mathbb{R}$.*

Theorem 2.2 *If $f, g \in AP(\mathbb{C}^m)$ and $\mu \in \mathbb{C}$, then*

(i) μf and $f \pm g$ belong to $AP(\mathbb{C}^m)$.

(ii) If f, g are $\mathbb{C}$-valued, then $fg \in AP(\mathbb{C})$.

(iii) If $g \in AP(\mathbb{C})$ and $\inf_{t\in\mathbb{R}} |g(t)| = m > 0$, then $\dfrac{f}{g} \in AP(\mathbb{C}^m)$.

(iv) If $h \in L^1(\mathbb{C})$, then $(h * f)$, the convolution between h and f defined by

$$(h * f)(t) = \int_{-\infty}^{+\infty} h(s)f(t - s)\, ds$$

belongs to $AP(\mathbb{C}^m)$.

Theorem 2.3 *The space $AP(\mathbb{C}^m)$ equipped with the supnorm $\|\cdot\|_\infty$ is a Banach space.*

2.2 $C^{(n)}$-almost periodic functions

Definition 2.3 A function $f \in C^{(l)}(\mathbb{R}, \mathbb{C}^m)$ is said to be $C^{(l)}$-almost periodic if $f^{(k)} \in AP(\mathbb{C}^m)$ for $k = 0, 1, ..., l$. The collection of $C^{(l)}$-almost periodic functions is denoted by $AP^{(l)}(\mathbb{C}^m)$, which turns out to be a Banach space when equipped with the norm $\| \cdot \|_{(l)}$.

Clearly, the following inclusions hold

$$... \hookrightarrow AP^{(l+2)}(\mathbb{C}^m) \hookrightarrow AP^{(l+1)}(\mathbb{C}^m) \hookrightarrow AP^{(l)}(\mathbb{C}^m) \hookrightarrow ... \hookrightarrow AP^{(1)}(\mathbb{C}^m) \hookrightarrow AP(\mathbb{C}^m).$$

Theorem 2.4 *[10] The space $AP^{(l)}(\mathbb{C}^m)$ equipped with the norm $\left\| \cdot \right\|_{(l)}$ is a Banach space.*

Theorem 2.5 *[10] If $f \in AP^{(l)}(\mathbb{C}^m)$ and if $g \in L^1(\mathbb{C})$, then their convolution $f * g \in AP^{(l)}(\mathbb{C}^m)$.*

Proposition 2.1 *[10] If $(f_n)_{n\in\mathbb{N}} \subset AP(\mathbb{C}^m)$ converges uniformly to f on $\mathbb{R}$, then $f \in AP(\mathbb{C}^m)$.*

Theorem 2.6 *[10] If $f \in AP^{(l)}(\mathbb{C}^m)$ such that $f^{(l+1)}$ is uniformly continuous, then $f \in AP^{(l+1)}(\mathbb{C}^m)$.*

3 Existence of Almost Periodic Solutions

In this paper if $C \in M(m, \mathbb{C})$, we then denote the collection of its eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ by $\sigma(C)$.

In this section we first recall some of the results obtained by Campbell [5] on the existence of solutions to Eq. (1). We then make extensive use of those results to study the existence of almost periodic solutions to Eq. (1) in the case when $f \in AP^{(k)}(\mathbb{C}^m)$. We next use the results for Eq. (1) to study the existence of almost periodic solutions to some general singular second-order differential equations formulated through Eq. (10).

It is well-known that real symmetric matrices can be diagonalized. That is not always the case for complex symmetric matrices [9].

Definition 3.1 A symmetric matrix $C \in M(m, \mathbb{C})$ with r distinct eigenvalues λ_j is said to have a spectral decomposition if there exist matrices P_j for $j = 1, 2, ..., r$ such that

$$C = \sum_{j=1}^{r} \lambda_j P_j = \sum_{j=1, \lambda_j \neq 0}^{r} \lambda_j P_j, \tag{2}$$

where $P_i P_j = 0$ if $i \neq j$, and $P_j^2 = P_j$ for all $i, j = 1, ..., r$, and $I = \displaystyle\sum_{j=1}^{r} P_j$.

This setting requires the following assumptions: let f, $A, B \in M(m, \mathbb{C})$ satisfy the following assumptions:

(H.1) $AB = BA$.

(H.2) $N(A) \cap N(B) = \{0\}$.

(H.3) $f \in AP^{(k)}(\mathbb{C}^m)$ where $k = i(A)$.

(H.4) $\sigma^*(A^D B) := \sigma(A^D B) - \{0\} \neq \emptyset$ with $\Re e\lambda > 0$ for all $\lambda \in \sigma^*(A^D B)$.

(H.5) $A^D B$ is symmetric and has a spectral decomposition.

Remark 3.1 The case when A is non-singular will not be considered here as this is well-understood. Indeed, if A^{-1} exists, then Eq. (1) can be written as follows

$$u' + B_1 u = f_1, \tag{3}$$

where $B_1 = A^{-1}B$, and $f_1 = A^{-1}f$.

In the rest of the paper, we associate with Eq. (1), its homogeneous equation given by

$$Au' + Bu = 0. \tag{4}$$

Theorem 3.1 *[5] Under assumption* (H.1), $u = e^{-A^D Bt} AA^D \xi$ *is a solution to Eq. (4) where ξ is an arbitrary vector in $\mathbb{C}^m$.*

Proof. Indeed, $Au' = -AA^D Be^{-A^D Bt} AA^D \xi = -Be^{-A^D Bt} AA^D \xi = -Bu$. The proof is complete.

Corollary 3.1 *[5] If assumption* (H.1) *holds and if $A^D Af = f$, then*

$$u = e^{-A^D Bt} \int e^{-A^D Bt} f(t)\, dt$$

is a particular solution to Eq. (1).

Lemma 3.1 *[5] If assumptions* (H.1)–(H.2) *hold, then* $(I - AA^D)BB^D = (I - AA^D)$.

Theorem 3.2 *[5] If assumptions* (H.1)–(H.2) *hold, then* $u = e^{-A^D Bt} AA^D \xi$ *where $\xi \in \mathbb{C}^m$, is the general solution to Eq. (4).*

Theorem 3.3 *[5] Suppose* (H.1)–(H.2) *hold and let $k = Ind(A)$. If f is of class C^k and $\mathbb{C}^m$-valued, then Eq. (1) is consistent and a particular solution of it is given*

$$u = A^D e^{-A^D Bt} \int_a^t e^{A^D Bs} f(s)\, ds + (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)},$$

where $a \in \mathbb{R}$ is arbitrary.

Theorem 3.4 *[5] Suppose* (H.1)–(H.2) *hold and let $k = Ind(A)$. If f is of class C^k and $\mathbb{C}^m$-valued, then the general solution to Eq. (1) is explicitly given by*

$$u = e^{-A^D Bt} A^D A\xi + A^D e^{-A^D Bt} \int_a^t e^{A^D Bs} f(s)\, ds + (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)},$$

where ξ is arbitrary constant vector, and $a \in \mathbb{R}$ is arbitrary.

Lemma 3.2 *If $C \in M(n,\mathbb{C})$ is symmetric, has a spectral decomposition, and $\sigma^*(C) \neq \emptyset$ with $\Re e\lambda > 0$ for all $\lambda \in \sigma^*(C)$, then there exist $M > 0$ and $\omega > 0$ such that*

$$\|e^{-tC}\| \leq Me^{-\omega t}$$

for $t \geq 0$.

Proof. Using Definition 3.1, one can write $C = \sum_{j=1}^{r} \lambda_j P_j = \sum_{j=1,\ \lambda_j \neq 0}^{r} \lambda_j P_j$ and hence

$$e^{-tC} = \sum_{j=1,\ \lambda_j \in \sigma^*(C)}^{r} e^{-\lambda_j t} P_j, \quad t \geq 0.$$

Now

$$\begin{aligned} \|e^{-tC}\| &= \| \sum_{j=1,\ \lambda_j \in \sigma^*(C)}^{r} e^{-\lambda_j t} P_j \| \leq \sum_{j=1,\ \lambda_j \in \sigma^*(C)}^{r} e^{-\Re e\, \lambda_j t} \|P_j\| \\ &\leq \sum_{j=1,\ \lambda_j \in \sigma^*(C)}^{r} e^{-\omega t} \|P_j\| \leq Me^{-\omega t} \end{aligned}$$

for $t \geq 0$, where $\omega = \min\{\Re e\lambda_j : \lambda_j \neq 0, \quad j = 1, 2, ..., r\}$ and $M = \sum_{j=1}^{r} \|P_j\| < \infty$.

Lemma 3.3 *Suppose* (H.1)–(H.2) *hold. Then all the solutions to Eq. (4) on the real number line $\mathbb{R}$ are of the form*

$$w_0(t) = e^{-A^D B(t-s)} w_0(s) \ \ \textit{for all} \ \ t, s \in \mathbb{R}, \quad t \geq s. \tag{5}$$

Proof. Let w be an arbitrary solution to Eq. (4). Now from Theorem 3.2, it follows that the solution w can be written as $w(t) = e^{-A^D Bt} AA^D \xi$ where $\xi \in \mathbb{C}^n$ is an arbitrary vector. Similarly, $w(s) = e^{-A^D Bs} AA^D \xi$. Thus for $t \geq s$,

$$e^{-A^D B(t-s)} w(s) = e^{-A^D B(t-s)} e^{-A^D Bs} AA^D \xi = e^{-A^D Bt} AA^D \xi = w(t).$$

Theorem 3.5 *Under assumptions* (H.1)–(H.2)–(H.3)–(H.4)–(H.5), *Eq. (1) has a unique almost periodic solution which is explicitly given by*

$$u_0(t) = A^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s) \mathrm{d}s + (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)}(t) \tag{6}$$

for all $t \in \mathbb{R}$.

Proof. We first show that the function u_0 given by

$$u_0(t) = A^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s) \mathrm{d}s + (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)}(t), \quad t \in \mathbb{R},$$

is a solution to Eq. (1). Indeed,

$$
\begin{aligned}
Au_0'(t) &= -AA^DA^DB\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s + AA^De^{-A^DBt}e^{A^DBt}f(t)\\
&+ A(I-AA^D)\sum_{l=0}^{k-1}(-1)^l(AB^D)^lB^Df^{(l+1)}(t)\\
&= -AA^DA^DB\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s + AA^Df(t)\\
&+ (I-AA^D)\sum_{l=0}^{k-1}(-1)^l(AB^D)^{l+1}f^{(l+1)}(t)\\
&= -B(A^DAA^D)\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s + AA^Df(t)\\
&+ (I-AA^D)[\sum_{l=0}^{k-2}(-1)^l(AB^D)^{l+1}f^{(l+1)}(t) + (-1)^{k-1}(AB^D)^{k-1+1}f^{(k-1+1)}(t)]\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s + AA^Df(t)\\
&+ (I-AA^D)[\sum_{l=0}^{k-2}(-1)^l(AB^D)^{l+1}f^{(l+1)}(t) + 0]\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s + AA^Df(t)\\
&- (I-AA^D)\sum_{l=0}^{k-2}(-1)^{l+1}(AB^D)^{l+1}f^{(l+1)}(t)\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s\\
&- (I-AA^D)\sum_{j=1}^{k-1}(-1)^j(AB^D)^jf^{(j)}(t) + AA^Df(t)\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s\\
&- (I-AA^D)[\sum_{j=0}^{k-1}(-1)^j(AB^D)^jf^{(j)}(t) - f(t)] + AA^Df(t)\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s\\
&- (I-AA^D)\sum_{j=0}^{k-1}(-1)^j(AB^D)^jf^{(j)}(t) + (I-AA^D)f(t) + AA^Df(t)\\
&= -BA^D\int_{-\infty}^{t} e^{-A^DB(t-s)}f(s)\mathrm{d}s - (I-AA^D)\sum_{l=0}^{k-1}(-1)^l(AB^D)^lf^{(l)}(t) + f(t)
\end{aligned}
\tag{7}
$$

$$\begin{aligned}
&= -BA^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s)\mathrm{d}s \\
&- (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D BB^D)^l f^{(l)}(t) + f(t) \\
&= -BA^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s)\mathrm{d}s \\
&- (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l (BB^D)^l f^{(j)}(t) + f(t) \\
&= -BA^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s)\mathrm{d}s \\
&- (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l (BB^D) f^{(l)}(t) + f(t) \\
&= -B[A^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s)\mathrm{d}s \\
&- (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)}(t)] + f(t) = -Bu_0(t) + f(t),
\end{aligned}$$

and hence $u_0(t)$ is a solution to Eq. (1).

We next show that u_0 given above is bounded. Indeed, since by assumption $Re\lambda > 0$ for all $\lambda \in \sigma(A^D B) - \{0\}$, then using Lemma 3.2 it follows that there exist $M > 0$ and $\omega > 0$ such that

$$\|e^{-tA^D B}\| \leq M e^{-\omega t}, \quad t \geq 0.$$

First of all, note that

$$\left| (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)}(t) \right| \leq \left(1 + \|AA^D\|\right) \|f\|_{(k)} \sum_{l=0}^{k-1} \|AB^D\|^l \|B^D\| < \infty,$$

where $\|f\|_{(k)} = \sup_{t\in\mathbb{R}} \sum_{l=0}^{k} |f^{(l)}(t)| < \infty$ as $f \in AP^{(k)}(\mathbb{C}^m)$. Similarly,

$$\begin{aligned}
\left| A^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s) ds \right| &\leq ||A^D|| \int_{-\infty}^{t} ||e^{-A^D B(t-s)}|| \cdot |f(s)| ds \\
&\leq ||A^D|| \cdot ||f||_\infty \int_{-\infty}^{t} M e^{-\omega(t-s)} ds \\
&= M ||A^D|| \cdot ||f||_\infty \omega^{-1} < \infty
\end{aligned}$$

and hence $u_0 \in BC(\mathbb{R}, \mathbb{C}^m)$.

The next step consists of showing that the function u_0 given above is the unique (bounded) solution to Eq. (1). Indeed, suppose $u_1, u_2 \in BC(\mathbb{R}, \mathbb{C}^m)$ are two solutions to Eq. (1). Thus $w = u_1 - u_2 \in BC(\mathbb{R}, \mathbb{C}^m)$ is a solution to Eq. (4). Using Lemma 3.3 it follows that $w(t) = e^{-A^D B(t-s)} w(s)$ for $t \geq s$.

Now

$$
\begin{aligned}
|w(t)| &= |e^{-A^D B(t-s)} w(s)| \\
&\leq M e^{-\omega(t-s)} \cdot |w(s)| \\
&\leq M e^{-\omega(t-s)} \cdot \|w\|_\infty \quad \text{for all} \quad t \geq s.
\end{aligned}
$$

Now let $(s_l)_{l\in\mathbb{N}}$ be a sequence of real numbers such that $s_l \to -\infty$ as $l \to \infty$. Clearly, for any fixed $t \in \mathbb{R}$, there exists a subsequence $(s_{l_p})_{p\in\mathbb{N}}$ of $(s_l)_{l\in\mathbb{N}}$ such that $s_{l_p} < t$ for all $p \in \mathbb{N}$. In view of the above, letting $p \to \infty$ yields $w(t) = 0$ for all $t \in \mathbb{R}$. Therefore, $u_1 = u_2$.

We next show that the function u_0 given above is almost periodic. Indeed, since $f \in AP^{(k)}(\mathbb{C}^m)$ and all the operators involved in the sum

$$
\varphi(t) := (I - AA^D) \sum_{l=0}^{k-1} (-1)^l (AB^D)^l B^D f^{(l)}(t)
$$

are matrices (hence are bounded linear operators), it follows that $t \mapsto \varphi(t) \in AP^{(k)}(\mathbb{C}^n) \subset AP(\mathbb{C}^m)$.

Now since $f \in AP(\mathbb{C}^m)$, it follows that for all $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon) > 0$ contains a τ with the property

$$
|f(t+\tau) - f(t)| \leq \frac{\varepsilon\omega}{M\|A^D\|}
$$

for all $t \in \mathbb{R}$.

Now setting $\psi(t) := A^D \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s) ds$ it follows that

$$
\begin{aligned}
|\psi(t+\tau) - \psi(t)| &\leq \|A^D\| \cdot \left| \int_{-\infty}^{t+\tau} e^{-A^D B(t+\tau-s)} f(s) ds - \int_{-\infty}^{t} e^{-A^D B(t-s)} f(s) ds \right| \\
&= \|A^D\| \cdot \left| \int_{-\infty}^{t} e^{-A^D B(t-s)} \left(f(s+\tau) - f(s) \right) ds \right| \\
&\leq \|A^D\| \cdot \frac{\varepsilon\omega}{M\|A^D\|} \int_{-\infty}^{t} \|e^{-A^D B(t-s)}\| ds \\
&\leq \frac{\varepsilon\omega}{M} M \int_{-\infty}^{t} e^{-\omega(t-s)} ds \\
&= \varepsilon
\end{aligned}
$$

and hence $\psi \in AP(\mathbb{C}^m)$ which yields $u_0 = \varphi + \psi \in AP(\mathbb{C}^m)$.

We now consider the case when A and B may or may not commute. Moreover, both A and B can be taken nonsingular. In what follows, we set

$$
\rho_{A,B} = \{\lambda \in \mathbb{C} : (\lambda A + B)^{-1} \ \text{ exists}\}.
$$

If $\lambda \in \rho_{A,B}$, we also set

$$
A_\lambda = (\lambda A + B)^{-1} A, \quad B_\lambda = (\lambda A + B)^{-1} B, \quad \text{and} \quad f_\lambda = (\lambda A + B)^{-1} f.
$$

Consider

$$
A_z u' + B_z u = f_z, \quad t \in \mathbb{R}. \tag{8}
$$

Corollary 3.2 *Suppose $\rho_{A,B} \neq \emptyset$. Let $z \in \rho_{A,B}$ such that $A_z^D B_z$ is symmetric, has a spectral decomposition, $\sigma^*(A_z^D B_z) \neq \emptyset$ with $Re\lambda > 0$ for all $\lambda \in \sigma^*(A_z^D B_z)$. Moreover, we suppose that $f \in AP^{(k)}(\mathbb{C}^m)$ with $i(A_z) = k$. Then Eq. (8) has a unique almost periodic solution which is explicitly given by*

$$u_z(t) = A_z^D \int_{-\infty}^{t} e^{-A_z^D B_z (t-s)} f_z(s) \mathrm{d}s + (I - A_z A_z^D) \sum_{l=0}^{k-1} (-1)^l (A_z B_z^D)^l B_z^D f_z^{(l)}(t) \qquad (9)$$

for all $t \in \mathbb{R}$.

Therefore, Eq. (1) has a unique almost periodic solution.

Proof. Since $\rho_{A,B} \neq \emptyset$, suppose $\rho_{A,B}$ contains a $z \in \mathbb{C}$. To complete the proof we have to show that Eq. (8) has a unique almost periodic solution. For that, we have to show that assumptions (H.1)–(H.2)—(H.3) are fulfilled when A is replaced with A_z, B with B_z, and f with f_z.

Let us first show that A_z and B_z commute. This is based upon the fact $zA_z + B_z = I$, which yields $B_z = I - zA_z$.

Now

$$A_z B_z = A_z(I - zA_z) = A_z - zA_z^2 \text{ and } B_z A_z = (I - zA_z)A_z = A_z - zA_z^2.$$

We next show that $N(A_z) \cap N(B_z) = \{0\}$. First of all, note that

$$N(A_z) \cap N(B_z) = N(A) \cap N(B).$$

Now, if $u \in N(A) \cap N(B)$, then $(zA + B)u = 0$, which yields $(zA + B)^{-1}(zA + B)u = u = 0$. Therefore, $N(A) \cap N(B) = \{0\}$.

Since $f \in AP^{(k)}(\mathbb{C}^m)$, it easy follows that $f_z \in AP^{(k)}(\mathbb{C}^m)$.

To complete the proof it suffices to apply Theorem 3.5 to the case when A replaced by A_z, B with B_z, and f with f_z. Doing so yields the existence and uniqueness of an almost periodic solution to Eq. (8), which is explicitly given by

$$u_z(t) = A_z^D \int_{-\infty}^{t} e^{-A_z^D B_z (t-s)} f_z(s) \mathrm{d}s + (I - A_z A_z^D) \sum_{m=0}^{k-1} (-1)^m (A_z B_z^D)^m B_z^D f_z^{(m)}(t), \quad t \in \mathbb{R}.$$

Therefore, Eq. (1) has a unique almost periodic solution.

4 Second-Order Singular Differential Equations

In this Section we study and obtain the existence of almost periodic solutions to the singular system of second-order differential differential equations given by

$$Au''(t) + Bu'(t) + Cu(t) = f(t), \quad t \in \mathbb{R}, \qquad (10)$$

where A, B, C (possibly singular) are $m \times m$-square matrices with complex entries and $f : \mathbb{R} \mapsto \mathbb{C}^m$ is $C^{(k)}$-almost periodic with $k = i(\mathcal{A})$. For that, our strategy consists of following the work of Campbell [3, p. 161] and rewriting Eq. (10) as a first-order singular differential equation and making extensive use of the results of the previous Section to establish the existence and uniqueness of an almost periodic solution to Eq. (10). Indeed, assuming that $u : \mathbb{R} \mapsto \mathbb{C}^m$ is twice differentiable and setting

$$w := \begin{pmatrix} u \\ u' \end{pmatrix},$$

then Eq. (10) can be rewritten on $\mathbb{C}^m \times \mathbb{C}^m$ in the following form

$$\mathcal{A}w'(t) + \mathcal{B}w = \mathcal{F}(t),\ t \in \mathbb{R}, \tag{11}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{F}$ are defined by

$$\mathcal{A} = \begin{pmatrix} B & A \\ I & 0 \end{pmatrix},\quad \mathcal{B} = \begin{pmatrix} C & O \\ O & -I \end{pmatrix},\quad \text{and}\quad \mathcal{F} = \begin{pmatrix} f \\ O \end{pmatrix}.$$

Let $\rho_{\mathcal{A},\mathcal{B}} = \{\lambda \in \mathbb{C} : (\lambda\mathcal{A} + \mathcal{B})^{-1} \text{ exists}\}$. If $\lambda \in \rho_{\mathcal{A},\mathcal{B}}$, we then set

$$\mathcal{A}_\lambda = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{A},\quad \mathcal{B}_\lambda = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{B},\quad \text{and}\quad \mathcal{F}_\lambda = (\lambda\mathcal{A} + \mathcal{B})^{-1}\mathcal{F}.$$

Consider

$$\mathcal{A}_z w' + \mathcal{B}_z w = \mathcal{F}_z,\quad t \in \mathbb{R}. \tag{12}$$

Corollary 4.1 *Suppose $\rho_{\mathcal{A},\mathcal{B}} \neq \emptyset$. Let $z \in \rho_{\mathcal{A},\mathcal{B}}$ such that $\mathcal{A}_z^D\mathcal{B}_z$ is symmetric, has a spectral decomposition, and $\sigma^*(\mathcal{A}_z^D\mathcal{B}_z) \neq \emptyset$ such that $Re\lambda > 0$ for all $\lambda \in \sigma^*(\mathcal{A}_z^D\mathcal{B}_z)$. Moreover, we suppose $\mathcal{F} \in AP^{(k)}(\mathbb{C}^m \times \mathbb{C}^m)$ with $k = i(\mathcal{A})$. Then Eq. (12) has a unique almost periodic solution which is explicitly given by*

$$w_z(t) = \mathcal{A}_z^D \int_{-\infty}^t e^{-\mathcal{A}_z^D\mathcal{B}_z(t-s)}\mathcal{F}_z(s)\mathrm{d}s + (I - \mathcal{A}_z\mathcal{A}_z^D)\sum_{l=0}^{k-1}(-1)^l(\mathcal{A}_z\mathcal{B}_z^D)^l\mathcal{B}_z^D\mathcal{F}_z^{(l)}(t) \tag{13}$$

for all $t \in \mathbb{R}$.

Therefore, Eq. (10) has a unique almost periodic solution u. Moreover, since $u, u' \in AP(\mathbb{C}^m)$, it follows that $u \in AP^{(1)}(\mathbb{C}^m)$.

The proof of Corollary 4.1 follows along the same lines as that of Corollary 3.2 and hence is omitted.

5 Example

In this section we give an example to illustrate Theorem 3.5. For that, let $m = 3$ and fix $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\Re e\, \alpha > 0$, $\Re e\, \beta > 0$, and $\Re e\, \gamma > 0$.

Consider the singular system of differential equations given by

$$\begin{cases} \alpha u'(t) + \beta v'(t) + \alpha u(t) + \beta v(t) = \sin t + i \sin\sqrt{2}t, \\ \alpha v'(t) + \alpha v(t) = \cos t + i\cos \pi t, \\ \gamma w(t) = \cos t + i \sin\sqrt{3}t, \end{cases} \tag{14}$$

for all $t \in \mathbb{R}$.

Clearly the matrices $A, B \in M(3, \mathbb{C})$ and $f : \mathbb{R} \mapsto \mathbb{C}^3$ associated with the system Eq. (14) are given by

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \text{and} \quad f(t) = \begin{pmatrix} \sin t + i \sin \sqrt{2}t \\ \cos t + i \cos \pi t \\ \cos t + i \sin \sqrt{3}t \end{pmatrix}.$$

Moreover, assumptions (H.1)–(H.2), (H.4)–(H.5) hold as

$$A^D = \begin{pmatrix} \frac{1}{\alpha} & -\frac{\beta}{\alpha^2} & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A^D B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{is symmetric with} \quad \sigma^*(A^D B) = \{1\}.$$

Furthermore, $i(A) = 1$ and $f \in AP^{(1)}(\mathbb{C}^3)$. Therefore, from Theorem 3.5 the singular system of first-order differential equation

$$Az'(t) + Bz(t) = f(t), \quad t \in \mathbb{R},$$

has a unique almost periodic solution, that is,

$$z_{\alpha,\beta,\gamma}(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} \in AP(\mathbb{C}^3).$$

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