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A computational Method for Solving a System of Volterra Integro-differential Equations

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Abstract: In this paper we present a reliable algorithm for solving a system of Volterra integro-differential equations using Taylor series expansion method and computer algebra. This method converts a system of Volterra integro-differential equations to a system of linear algebraic equations. Some illustrative examples have been presented to illustrate the implementation of the algorithm and efficiency of the method.

Keywords: *system of Volterra integro-differential equations; Taylor-series expansion method; ordinary differential equations; system of linear algebraic equations.*

Mathematics Subject Classification (2010): 45J05, 34K28.

1 Introduction

A number of problems in chemistry, physics and engineering are modeled in terms of system of Volterra integro-differential equations. Various methods have been developed to prove existence and uniqueness of solutions to integro-differential equations [3].

In this paper, we use a modified Taylor-series expansion method for solving system of Volterra integro-differential equations. This method was first presented by Kanwal and Liu et. al. [1] for solving integral equations and in [2, 6] for solving Fredholm integral equations of second kind. Daftardar-Gejji et. al. have used this method for solving system of ordinary differential equations [4]. Maleknejad et. al. have applied this method for solving Volterra integral equations and system of Volterra integral equations

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of second kind [5,7]. Yalcinbas and Sezer [8] have studied the following type of nonlinear Fredholm-Volterra integral equations

$$y(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)[y(t)]^p\,dt + \lambda_2 \int_a^b k_2(x,t)y(t)\,dt, \qquad (1)$$

and the high-order linear and nonlinear Volterra-Fredholm integro-differential equations have been considered in [9,10]

$$\sum_{j=0}^{m} P_j(x)\,y^{(j)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)y(t)\,dt + \lambda_2 \int_a^b k_2(x,t)y(t)\,dt, \qquad (2)$$

$$\sum_{j=0}^{m} P_j(x)\,y^{(j)}(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)[y(t)]^p\,dt + \lambda_2 \int_a^b k_2(x,t)[y(t)]^q\,dt. \qquad (3)$$

In this paper, the basic ideas of the previous work [6] are developed and applied to the high-order system of Volterra integro-differential equation of the form

$$\sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijs}(x)\,y_i^{(j)}(x) = f_s(x) + \sum_{i=1}^{n}\int_a^x k_{is}(x,t)y_i(t)dt, \qquad s = 1,2,\cdots,n, \qquad (4)$$

where $a_{ijs}(x)$, $f_s(x)\,(s = 1,2,\cdots,n)$ and $k_{is}(x,t)$ are known functions which are lth derivatable on interval $a \leq x,t \leq b$.

We assume that (4) has a unique solution. Suppose the solution of (4), can be expressed in the form:

$$y_i(x) = \sum_{r=0}^{N} \frac{1}{r!} y_i^{(r)}(\xi)\,(x-\xi)^r,\ a \leq x,\xi \leq b, \qquad (5)$$

which is a Taylor polynomial of degree N, where $N \geq \{n_{ijs}, n_s\}$, and $y^{(s)}(\xi)\,(s = 0,1,\cdots,N)$ are the coefficients to be determined.

2 Analysis of Method

First, we rewrite (4) in the following form

$$D(x) = I(x), \qquad (6)$$

where $D(x) = [D_1(x), D_2(x), ..., D_n(x)]^T, \quad I(x) = [I_1(x), I_2(x), ..., I_n(x)]^T,$

$$D_s(x) = \sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijs}(x)y_i^{(j)}(x), \qquad I_s(x) = f_s(x) + \sum_{i=1}^{n} V_{is}(x), \qquad s = 1,2,\cdots,n, \quad (7)$$

with

$$V_{is}(x) = \int_a^x k_{is}(x,t)y_i(t)dt. \qquad (8)$$

Then $D(x)$ is called the differential part and $I(x)$ the integral part of (4). Differentiating Eq. (6) N times with respect to x, we get

$$D_s^l(x) = I_s^l(x), \quad l = 0,1,\cdots,N,\ s = 1,2,\cdots,n. \qquad (9)$$

In the following part, we will analyse the expressions $D_s^l(x)$ and $I_s^l(x)$. It is easy to see that

$$\begin{aligned} D_s^{(l)}(x) &= \left[\sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijs}(x)\, y_i^{(j)}(x)\right]^{(l)} \\ &= \left[\sum_{j=0}^{m} a_{1js}(x)\, y_1^{(j)}(x)\right]^{(l)} + \cdots + \left[\sum_{j=0}^{m} a_{njs}(x)\, y_n^{(j)}(x)\right]^{(l)}, \\ & \quad l = 0, 1, \cdots, N,\ s = 1, 2, \cdots, n \end{aligned} \tag{10}$$

Using Leibnitz's rule (dealing with differentiation of products of functions), simplifying and then substituting $x = \xi$ into the resulting relation, we can get

$$D_s^{(l)}(x) = \sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{p=0}^{l} \binom{l}{p} a_{ijs}^{(l-p)}(x)\, y_i^{(p+j)}(x), \quad l = 1, 2, \cdots, N,\ s = 0, 1, \cdots, n. \tag{11}$$

The system (11) can be written in the matrix form as:

$$\mathbf{D} = \mathbf{W}\mathbf{Y}, \tag{12}$$

where $\mathbf{Y} = \left[y_1^{(0)}, y_1^{(1)}, \cdots, y_1^{(N)}, y_2^{(0)}, \cdots, y_2^{(N)}, \cdots, y_n^{(0)}, \cdots, y_n^{(N)}\right]^T$. Note that

$$\mathbf{W} = [W_{is}] \qquad i, s = 1, 2, \cdots, n, \tag{13}$$

is a matrix, where each W_{is} is again a matrix:

$$w_{is}^{lp} = \sum_{q=0}^{m} \binom{l}{p-m+q} a_{im-qs}^{(l-p+m-q)}(\xi), \quad l, p = 0, 1, \cdots, N. \tag{14}$$

Note: For $r < 0$, $a_{ijs}^{(r)} = 0$ and for $j < 0$ and $j > i$, $\binom{i}{j} = 0$, where i, j and r are integers. On the other hand, for the integral part $I_s^l(x)$, it is easy to know that

$$I_s^{(l)}(x) = f_s^{(l)}(x) + \sum_{i=1}^{n} V_{is}^{(l)}(x), \quad l = 0, 1, \cdots, N, \tag{15}$$

where

$$\begin{aligned} V_{is}^{(l)}(x) &= \frac{\partial^l}{\partial x^l}\int_a^x k_{is}(x,t) y_i(t) dt \\ &= \sum_{j=0}^{l-1}\left[h_{is}^j(x) y_i(x)\right]^{l-j-1} + \int_a^x \frac{\partial^l k_{is}(x,t)}{\partial x^l} y_i(t) dt \\ &= \sum_{r=0}^{l-1}\sum_{j=0}^{l-r-1} \binom{l-j-1}{r} \left(h_{is}^j(x)\right)^{(l-r-j-1)} y_i^{(r)}(x) + \int_a^x \frac{\partial^l k_{is}(x,t)}{\partial x^l} y_i(t) dt, \end{aligned} \tag{16}$$

with

$$h_{is}^{j}(x)=\frac{\partial^{j}k_{is}(x,t)}{\partial x^{j}}|_{t=x}\,.$$

By Using Leibnitz's rule and substituting (5) in (16), we can get

$$\begin{aligned} I_s^{(l)}(\xi) \quad &= f_s^{(l)}(\xi)+\sum_{r=0}^{l-1}\sum_{j=0}^{l-r-1}\binom{l-j-1}{r}\left(h_{is}^{j}(\xi)\right)^{(l-r-j-1)}y_i^{(r)}(\xi) \\ &\quad +\int_a^{\xi}\frac{\partial^{l}k_{is}(x,t)}{\partial x^{l}}|_{x=\xi}\left[\sum_{r=0}^{\infty}\frac{1}{r!}y_i^{(r)}(\xi)\,(t-\xi)^{r}\right]dt \qquad (17) \\ &= f_s^{(l)}(x)+\sum_{r=0}^{l-1}(H_{is}^{lr}+T_{is}^{lr})y_i^{(r)}(\xi)+\sum_{r=l}^{\infty}T_{is}^{lr}y_i^{(r)}(\xi). \end{aligned}$$

For the case of computing in practice, the approximate form of system (17) can be put in as follows:

$$I_s^{(l)}(\xi)=f_s^{(l)}(\xi)+\sum_{r=0}^{l-1}(H_{is}^{lr}+T_{is}^{lr})y_i^{(r)}(\xi)+\sum_{r=l}^{N}T_{is}^{lr}y_i^{(r)}(\xi), \qquad (18)$$

where for $l=1,2,\cdots,N;\;\; r=0,1,\cdots,l-1\;\;(l>r)$

$$H_{is}^{lr}=\sum_{j=0}^{l-r-1}\binom{l-j-1}{r}\left(h_{is}^{j}(\xi)\right)^{(l-r-j-1)} \qquad (19)$$

and for $l\le r,\quad H_{is}^{lr}=0.$ and

$$T_{is}^{lr}=\frac{1}{r!}\int_a^{\xi}\frac{\partial^{l}k_{is}(x,t)}{\partial x^{l}}|_{x=\xi}\,(t-\xi)^{r}dt,\quad l,r=0,1,\cdots,N. \qquad (20)$$

This system can be put in the matrix form as

$$\mathbf{I}=\mathbf{F}+\mathbf{TY}. \qquad (21)$$

(21) combined with (13)

$$\mathbf{WY}=\mathbf{F}+\mathbf{TY}\quad \textbf{or}\quad (\mathbf{W}-\mathbf{T})\mathbf{Y}=\mathbf{F}, \qquad (22)$$

where

$$\mathbf{W}=[w_{is}^{lr}]=\begin{bmatrix} w_{11}^{00} & w_{11}^{01} & \cdots & w_{11}^{0N} & \cdots & w_{1n}^{00} & w_{1n}^{01} & \cdots & w_{1n}^{0N} \\ w_{11}^{10} & w_{11}^{11} & \cdots & w_{11}^{1N} & \cdots & w_{1n}^{00} & w_{1n}^{01} & \cdots & w_{1n}^{0N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{11}^{N0} & w_{11}^{N1} & \cdots & w_{11}^{NN} & \cdots & w_{1n}^{N0} & w_{1n}^{N1} & \cdots & w_{1n}^{NN} \\ \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ w_{n1}^{00} & w_{n1}^{01} & \cdots & w_{n1}^{0N} & \cdots & w_{nn}^{00} & w_{nn}^{01} & \cdots & w_{nn}^{0N} \\ w_{n1}^{10} & w_{n1}^{11} & \cdots & w_{n1}^{1N} & \cdots & w_{nn}^{00} & w_{nn}^{01} & \cdots & w_{nn}^{0N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ w_{n1}^{N0} & w_{n1}^{N1} & \cdots & w_{n1}^{NN} & \cdots & w_{nn}^{N0} & w_{nn}^{N1} & \cdots & w_{nn}^{NN} \end{bmatrix},$$

$$\mathbf{F} = \left[f_1^{(0)},\, f_1^{(1)}, \cdots, f_1^{(N)}, f_2^{(0)}, \cdots, f_2^{(N)}, \cdots, f_n^{(0)}, \cdots, f_n^{(N)}\right]^T, \tag{23}$$

$$\mathbf{T} = [H_{is}^{lr} + T_{is}^{lr}] = \begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} & \cdots & \mathbf{A_{1n}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} & \cdots & \mathbf{A_{2n}} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{A_{n1}} & \mathbf{A_{n2}} & \cdots & \mathbf{A_{nn}} \end{bmatrix},$$

$$\mathbf{A_{is}} = \begin{bmatrix} T_{is}^{00} & T_{is}^{01} & T_{is}^{02} & \cdots & T_{is}^{0N} \\ H_{is}^{10} + T_{is}^{10} & T_{is}^{11} & T_{is}^{12} & \cdots & T_{is}^{1N} \\ H_{is}^{20} + T_{is}^{20} & H_{is}^{21} + T_{is}^{21} & T_{is}^{22} & \cdots & T_{is}^{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{is}^{N0} + T_{is}^{N0} & H_{is}^{N1} + T_{is}^{N1} & H_{is}^{N2} + T_{is}^{N2} & \cdots & T_{is}^{NN} \end{bmatrix},$$

$$\begin{aligned} \mathbf{Y} &= \left[y_1^{(0)}(\xi),\, y_1^{(1)}(\xi), \cdots, y_1^{(N)}(\xi), y_2^{(0)}(\xi), \cdots, y_2^{(N)}(\xi), \cdots, y_n^{(0)}(\xi), \cdots, y_n^{(N)}(\xi)\right]^T \\ &\equiv [y_{10},\, y_{11}, \cdots, y_{1N}, y_{20}, \cdots, y_{2N}, \cdots, y_{n0}, \cdots, y_{nN}]^T, \end{aligned} \tag{24}$$

which is to be solved.

Substituting (23) in (22), we can convert (22) into an algebraic equations with variables $y_{10}, y_{11}, \cdots, y_{1N}, y_{20}, \cdots, y_{2N}, \cdots, y_{n0}, \cdots, y_{nN}$, we can determine the variables y_{ir} (i=1,2,...,n; r=0,1,...,N), i.e., the unknown Taylor coefficients $y_i^{(r)}(\xi)$ (i=1,2,...,n; r = 0,1,...,N), thus we can get the Taylor polynomial solution of the system (4) as follows:

$$y_i(x) = \sum_{r=0}^{N} \frac{1}{r!} y_i^{(r)}(\xi)\,(x-\xi)^r. \tag{25}$$

Example 2.1 Consider the following Volterra system of integro-differential equations:

$$\begin{cases} y_1 - y_1' + 2y_1'' - 4y_2 - \frac{3}{2}y_2' + \frac{5}{4}y_2'' - 2y_3' - \frac{1}{3}y_3'' = f_1(x) + \int_0^x (t-x)\,y_1(t)dt \\ \qquad + \int_0^x 5\,y_2(t)dt + \int_0^x (5x+\frac{1}{2})\,y_3(t)dt, \\ -\frac{1}{4}y_1'' + y_2 + y_2' - 2y_2'' + \frac{1}{6}y_3'' = f_2(x) + \int_0^x (x+1)\,y_1(t)dt + \int_0^x t^2\,y_2(t)dt \\ \qquad + \int_0^x y_3(t)dt, \\ \frac{2}{3}y_1' + \frac{1}{6}y_1'' - \frac{1}{2}y_2 + y_2'' + \frac{3}{4}y_3 = f_3(x) + \int_0^x (3x-t)\,y_2(t)dt + \int_0^x x\,y_3(t)dt, \end{cases} \tag{26}$$

where $f_1(x) = -x^6 - \frac{5}{4}x^5 - \frac{1}{8}x^4 - 3x^3 + \frac{1}{2}x^2$, $f_2(x) = -\frac{9}{10}x^5 - x^4$, $f_3(x) = -\frac{1}{5}x^6 - \frac{1}{4}x^5 + \frac{7}{12}x^3 - \frac{23}{6}$ and the matrix form of the above system for $N = 4$ and $\xi = 0$ is as follows:

$$WY - KY = F,$$

where

$$W = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & -4 & -\frac{3}{2} & \frac{5}{4} & 0 & 0 & 0 & -2 & \frac{-1}{3} & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 & -4 & -\frac{3}{2} & \frac{5}{4} & 0 & 0 & 0 & -2 & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 & 0 & -4 & -\frac{3}{2} & \frac{5}{4} & 0 & 0 & 0 & -2 & \frac{-1}{3} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -4 & -\frac{3}{2} & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{6} & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{3}{4} \end{bmatrix},$$

$$F = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -18 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -24 \\ -\frac{23}{6} \\ 0 \\ 0 \\ \frac{7}{2} \\ 0 \end{bmatrix}, \; K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 10 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 15 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 20 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}.$$

Then we can get the following algebraic equations:

$$\begin{cases} -\frac{1}{4}y_{12} + y_{20} + y_{21} - 2y_{22} + \frac{1}{6}y_{32} = 0, \\ -\frac{1}{4}y_{14} + y_{22} + y_{23} - 2y_{24} + \frac{1}{6}y_{34} - 2y_{10} - y_{11} - y_{31} = 0, \\ \frac{2}{3}y_{12} + \frac{1}{6}y_{13} - \frac{1}{2}y_{21} + y_{23} + \frac{3}{4}y_{31} = 0, \\ -\frac{1}{4}y_{13} + y_{21} + y_{22} - 2y_{23} + \frac{1}{6}y_{33} - y_{10} - y_{30} = 0, \\ -\frac{1}{2}y_{24} + \frac{3}{4}y_{34} - 9y_{22} - 4y_{32} = 0, \\ y_{23} + y_{24} - 3y_{11} - y_{12} - 2y_{20} - y_{32} = 0, \\ y_{13} - y_{14} - 4y_{23} - \frac{3}{2}y_{24} - 2y_{34} + y_{11} - 5y_{22} - 15y_{31} - \frac{1}{2}y_{32} = -18, \\ \frac{2}{3}y_{13} + \frac{1}{6}y_{14} - \frac{1}{2}y_{22} + y_{24} + \frac{3}{4}y_{32} - 5y_{20} - 2y_{30} = 0, \\ y_{24} - 4y_{12} - y_{13} - 6y_{21} - y_{33} = -24, \\ y_{10} - y_{11} + 2y_{12} - 4y_{20} - \frac{3}{2}y_{21} + \frac{5}{4}y_{22} - 2y_{31} - \frac{1}{3}y_{32} = 0, \\ \frac{2}{3}y_{11} + \frac{1}{6}y_{12} - \frac{1}{2}y_{20} + y_{22} + \frac{3}{4}y_{30} = -\frac{23}{6}, \\ \frac{2}{3}y_{14} - \frac{1}{2}y_{23} + \frac{3}{4}y_{33} - 7y_{21} - 3y_{31} = \frac{7}{2}, \\ y_{14} - 4y_{24} + y_{12} - 5y_{23} - 20y_{32} - \frac{1}{2}y_{33} = -3, \\ y_{11} - y_{12} + 2y_{13} - 4y_{21} - \frac{3}{2}y_{22} + \frac{5}{4}y_{23} - 2y_{32} - \frac{1}{3}y_{33} - 5y_{20} - \frac{1}{2}y_{30} = 0, \\ y_{12} - y_{13} + 2y_{14} - 4y_{22} - \frac{3}{2}y_{23} + \frac{5}{4}y_{24} - 2y_{33} - \frac{1}{3}y_{34} + y_{10} - 5y_{21} - 10y_{30} - \frac{1}{2}y_{31} = 1. \end{cases}$$

The solution of this system of algebraic equations is as follows:

$$\begin{aligned} Y &= \{y_{12}=0, y_{23}=0, y_{30}=-4, y_{22}=2, y_{11}=-2, y_{20}=3, y_{31}=-2, y_{10}=5, \\ &\quad y_{34}=24, y_{14}=0, y_{33}=6, y_{21}=1, y_{32}=0, y_{13}=12, y_{24}=0\}. \end{aligned}$$

Then in view of (25) we can obtain the solution of (26) as

$$\begin{aligned} y_1(x) &= 2x^3-2x+5, \\ y_2(x) &= x^2+x+3, \\ y_3(x) &= x^4+x^3-2x-4. \end{aligned}$$

which is exact solution.

Example 2.2 Consider the following Volterra system of integro-differential equations:

$$\begin{cases} y_1+y_1^{'}=f_1(x)+\int_0^x(sin(x-t)-1)\,y_1(t)dt+\int_0^x(1-tcos(x))\,y_2(t)dt, \\ -y_1+y_2=f_2(x)+\int_0^x y_1(t)dt+\int_0^x(x-t)\,y_2(t)dt, \end{cases}$$

where $f_1(x)$ and $f_2(x)$ are chosen such that the exact solution is $f_1(x)=cos(x)$ and $f_2(x)=sin(x)$. Numerical results for N=12 and $\xi=0$ are given in Table 1.

x	$y_1(x)$		$y_2(x)$	
	Exact	Approximate	Exact	Approximate
0.0	1.0	1.002264127	0.0	-0.002264127328
0.1	0.9950041653	0.9970320650	0.09983341665	0.09759183946
0.2	0.9800665778	0.9818454811	0.1986693308	0.1965212957
0.3	0.9553364891	0.9568714640	0.2955202067	0.2936886078
0.4	0.9210609940	0.9224004551	0.3894183423	0.3885026325
0.5	0.8775825619	0.8788683493	0.4794255386	0.4806769283
0.6	0.8253356149	0.8268785893	0.5646424734	0.5702225351
0.7	0.7648421873	0.7672136234	0.6442176872	0.6572875444
0.8	0.6967067093	0.7008073298	0.7173560909	0.7415468153
0.9	0.6216099683	0.6285990889	0.7833269096	0.8203856200
1.0	0.5403023059	0.5510745228	0.8414709848	0.8842821006

Table 1: Numerical results for N=12 and $\xi=0$.

3 Conclusion

In this paper, we use modified Taylor-series expansion method for solving a system of Volterra integro-differential equations. By using the theories and methods of mathematical analysis and computer algebra, we convert the system of integro-differential equations into a system of linear algebraic equations and then we obtain the solution of the system of integro-differential equations. The Taylor polynomial method proposed in this investigation is simple and effective for solving various system of integro-differential equations and can provide an accuracy approximate solution or exact solution.
Maple has been used for computations in this paper.

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