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A Compactness Condition for Solutions of Nonlocal Boundary Value Problems of Orders $n = 3, 4$ & 5 †

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Abstract: For the ordinary differential equation, $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, of order $n = 3, 4$, or 5, it is shown that the existence of unique solutions of certain 4-point nonlocal boundary value problems implies a compactness condition on uniformly bounded sequences of solutions.

Keywords: *boundary value problem; nonlocal; continuous dependence; compactness condition.*

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1 Introduction

In a recent paper, for $n \geq 3$ and $1 \leq k \leq n-1$, Henderson [6] studied solutions of the ordinary differential equation,

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \quad a < x < b, \tag{1}$$

satisfying the $(k+2)$-point nonlocal boundary conditions,

$$\begin{aligned} &y^{(i-1)}(x_j) = y_{ij},\ 1 \leq i \leq m_j,\ 1 \leq j \leq k, \\ &y(x_{k+1}) - y(x_{k+2}) = y_n, \end{aligned} \tag{2}$$

for positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n-1$, points $a < x_1 < x_2 < \cdots < x_k < x_{k+1} < x_{k+2} < b$, real values $y_{ij}, 1 \leq i \leq m_j, 1 \leq j \leq k$, and $y_n \in \mathbb{R}$. In particular, sufficient conditions were given under which the existence of solutions for 4-point nonlocal boundary value problems for (1), (2), (that is, when $k = 2$), led to the existence of unique solutions of $(k+2)$-point nonlocal boundary value problems for (1), (2), for all $1 \leq k \leq n-1$.

Fundamental to that paper's main result was the following list of assumptions on solutions of (1).

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(A) $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

(B) Solutions of initial value problems for (1) are unique and extend to (a, b).

(C) Boundary value problems (1), (2), for $k = 2$, have solutions on (a, b).

(D) Boundary value problems (1), (2), for $k = n - 2$, have at most one solution.

(E) If $\{y_\nu(x)\}$ is a sequence of solutions of (1) which is uniformly bounded on a non-degenerate compact subinterval $[c, d] \subset (a, b)$, then there is a subsequence $\{y_{\nu_j}(x)\}$ such that $\{y^{(i)}_{\nu_j}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $i = 0, \ldots, n - 1$.

Under these assumptions, and in conjunction with a uniqueness implies existence result by Eloe and Henderson [3], the following existence result was the main result of paper [6].

Theorem 1.1 *Assume that with respect to* (1)*, conditions* (A)–(E) *are satisfied. Then, for each* $1 \le k \le n - 1$*, solutions of* (1)*,* (2) *exist and are unique on* (a, b)*.*

One question that arises, and which is the motivation for this paper, is whether conditions (A) – (D) imply the so-called "Compactness Condition" (E) on sequences of solutions of (1). The study of hypotheses sufficient to imply (E) has a long history, especially in the context of boundary value problems for (1) satisfying ℓ-point *conjugate* boundary conditions, for $2 \le \ell \le n$, of the form,

$$y^{(i-1)}(t_j) = r_{ij},\ 1 \le i \le p_j,\ 1 \le j \le \ell, \tag{3}$$

where $p_1, \ldots, p_\ell$ are positive integers such that $p_1 + \cdots + p_\ell = n$, $a < t_1 < \cdots < t_\ell < b$, and $r_{ij} \in \mathbb{R}, 1 \le i \le p_j, 1 \le j \le \ell$.

In the conjugate boundary value problem context, a principal question of the 1960's through the mid-1980's involved whether conditions (A) and (B) and uniqueness of solutions of n-point conjugate boundary value problems (1), (3) implied the Compactness Condition (E). This was answered in the affirmative for equation (1), when $n = 2$ and 3, by Jackson [10] and Jackson and Schrader [13]. Other extensive inroads were made in addressing the question for (1) of arbitrary order n in the papers [1, 5, 7–9, 11, 12, 14–17]. In 1985, in an unpublished paper, Schrader [18] announced that the conjecture had been verified. Later, Agarwal [2] gave a detailed presentation of the history and resolution of the conjecture for conjugate boundary value problems.

Much in the spirit of the work done regarding (E) with respect to solutions of conjugate boundary value problems, we show in this paper that when (1) is of any of the orders, $n = 3,\ 4$, or 5, then existence of unique solutions of (1), (2), for $k = 2$, and conditions (A) and (B) imply the Compactness Condition (E).

Each of these cases for n will depend on continuous dependence of solutions of (1), (2) on boundary conditions. We will refer to the following continuous dependence theorem [3], whose proof relies on a standard application of the Brouwer theorem on invariance of domain [19].

Theorem 1.2 *Assume that with respect to* (1), (2)*, conditions* (A) *and* (B) *are satisfied. Assume that, for* $k = 2$ *and any positive integers* m_1 *and* m_2 *such that* $m_1 + m_2 = n - 1$*, solutions of the corresponding nonlocal boundary value problem* (1), (2) *are unique, when they exist. Given a solution* $y(x)$ *of* (1)*, an interval* $[c, d]$*, points* $c < x_1 <$

$x_2 < x_3 < x_4 < d$ and an $\epsilon > 0$, there exists $\delta(\epsilon, [c,d]) > 0$ such that, if $|x_i - \xi_i| < \delta$, $i = 1,2,3,4$, and $c < \xi_1 < \xi_2 < \xi_3 < \xi_4 < d$, and if $|y^{(i-1)}(x_j) - z_{ij}| < \delta$, $1 \le i \le m_j$, $j = 1,2$ and $|y(x_3) - y(x_4) - z_n| < \delta$, then there exists a solution $z(x)$ of (1) satisfying $z^{(i-1)}(\xi_j) = z_{ij}$, $1 \le i \le m_j$, $j = 1,2$, $z(\xi_3) - z(\xi_4) = z_n$, and $|y^{(i)}(x) - z^{(i)}(x)| < \epsilon$ on $[c,d]$, $0 \le i \le n-1$.

2 The Compactness Condition: n =3, 4, 5

In this section, we show that, for $n = 3, 4$, or 5, conditions (A) and (B) and the existence of unique solutions of (1), (2), for $k = 2$, imply the Compactness Condition (E).

Theorem 2.1 *For $n = 3$, 4, or 5, assume that with respect to* (1)*, conditions* (A) *and* (B) *hold, and in addition, that there exist unique solutions of* (1)*,* (2)*, for $k = 2$. Then condition* (E) *also holds.*

Proof. We will address the case of each n independently.
(a) $n = 3$. In this case, we are assuming that, for each pair of positive integers m_1 and m_2 such that $m_1 + m_2 = n - 1 = 2$ (that is, $m_1 = m_2 = 1$), there exist unique solutions of (1), (2); that is, there exists a unique solution of (1) satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y(x_3) - y(x_4) = y_3,$$

where $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3 \in \mathbb{R}$. From Rolle's theorem, solutions of 3-point conjugate boundary value problems (1), (3) are unique, when they exist. As a consequence of the Jackson and Schrader [13] result for third order conjugate boundary value problems, or as a result of the more general result by Schrader [18] which was detailed in the Introduction, it follows that the Compactness Condition (E) is satisfied.

(b) $n = 4$. In this case, we are assuming that, for each pair of positive integers m_1 and m_2 such that $m_1 + m_2 = n - 1 = 3$, there are unique solutions of (1), (2); that is, for any $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3, y_4 \in \mathbb{R}$, there exists a unique solution of (1) satisfying

$$y(x_1) = y_1,\ y'(x_1) = y_2,\ y(x_2) = y_3,\ y(x_3) - y(x_4) = y_4,$$

and there exists a unique solution of (1) satisfying

$$y(x_1) = y_1,\ y(x_2) = y_2,\ y'(x_2) = y_3,\ y(x_3) - y(x_4) = y_4.$$

We now assume there are $a < c < d < b$, a number $M > 0$, and a sequence $\{y_\nu\}$ of solutions of (1) such that, for each $\nu \ge 1$,

$$|y_\nu(x)| \le M, \quad c \le x \le d.$$

Next, let the points $c < \eta_1 < x_2 < x_3 < x_4 < d$ be given. Then, for each $\nu \ge 1$, there exists $\xi_\nu \in (c, \eta_1)$ such that

$$|y'_\nu(\xi_\nu)| \le \frac{2M}{\eta_1 - c}.$$

This leads to the five bounded sequences of real numbers,

$$\{\xi_\nu\} \subset (c, \eta_1), \quad \{y_\nu(\xi_\nu)\} \subset [-M, M], \quad \{y'_\nu(\xi_\nu)\} \subset \left[\frac{-2M}{\eta_1 - c}, \frac{2M}{\eta_1 - c}\right],$$
$$\{y_\nu(x_2)\} \subset [-M, M], \text{ and } \{y_\nu(x_3) - y_\nu(x_4)\} \subset [-2M, 2M].$$

Hence, there exist a subsequence $\{\nu_j\} \subset \{\nu\}$, a point $x_1 \in [c, \eta_1]$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}$ such that

$$\xi_{\nu_j} \to x_1, \quad y_{\nu_j}(\xi_{\nu_j}) \to \gamma_1, \quad y'_{\nu_j}(\xi_{\nu_j}) \to \gamma_2,$$
$$y_{\nu_j}(x_2) \to \gamma_3, \text{ and } \quad \{y_{\nu_j}(x_3) - y_{\nu_j}(x_4)\} \to \gamma_4.$$

Now, let $y(x)$ be the solution of (1), (2), for $k = 2$, satisfying

$$y(x_1) = \gamma_1, \ y'(x_1) = \gamma_2, \ y(x_2) = \gamma_3, \text{ and } y(x_3) - y(x_4) = \gamma_4.$$

It follows from Theorem 1.2 that

$$\lim y^{(i)}_{\nu_j}(x) = y^{(i)}(x) \text{ uniformly on } [c, d],$$

for each $i = 0, 1, 2, 3$. It follows in turn, from (A) and (B) and the Kamke Convergence Theorem [4, page 14, Theorem 3.2], that these convergences are uniform on each compact subinterval of (a, b).

(c) $n = 5$. This time, we assume that, for each pair of positive integers m_1 and m_2 such that $m_1 + m_2 = n - 1 = 4$, there are unique solutions of (1), (2); that is, for any $a < x_1 < x_2 < x_3 < x_4 < b$ and $y_1, y_2, y_3, y_4, y_5 \in \mathbb{R}$, there exists a unique solution of (1) satisfying

$$y(x_1) = y_1, \ y'(x_1) = y_2, \ y''(x_1) = y_3, \ y(x_2) = y_4, \ y(x_3) - y(x_4) = y_5,$$

there exists a unique solution of (1) satisfying

$$y(x_1) = y_1, \ y'(x_1) = y_2, \ y(x_2) = y_3, \ y'(x_2) = y_4, \ y(x_3) - y(x_4) = y_5,$$

and there exists a unique solution of (1) satisfying

$$y(x_1) = y_1, \ y(x_2) = y_2, \ y'(x_2) = y_3, \ y''(x_2) = y_4, \ y(x_3) - y(x_4) = y_5.$$

Again, we assume there are $a < c < d < b$, a number $M > 0$, and a sequence $\{y_\nu\}$ of solutions of (1) such that, for each $\nu \geq 1$,

$$|y_\nu(x)| \leq M, \quad c \leq x \leq d.$$

Let the points $c < \eta_1 < \eta_2 < \eta_3 < x_3 < x_4 < d$ be given. Then, for each $\nu \geq 1$, there exist $\xi_\nu \in (c, \eta_1)$ and $\sigma_\nu \in (\eta_2, \eta_3)$ such that

$$|y'_\nu(\xi_\nu)| \leq \frac{2M}{\eta_1 - c} \text{ and } |y'_\nu(\sigma_\nu)| \leq \frac{2M}{\eta_3 - \eta_2}.$$

Then we have the seven bounded sequences of real numbers,

$$\{\xi_\nu\} \subset (c, \eta_1), \ \{\sigma_\nu\} \subset (\eta_2, \eta_3), \ \{y_\nu(\xi_\nu)\} \subset [-M, M], \ \{y'_\nu(\xi_\nu)\} \subset \left[\frac{-2M}{\eta_1 - c}, \frac{2M}{\eta_1 - c}\right],$$
$$\{y_\nu(\sigma_\nu)\} \subset [-M, M], \ \{y'_\nu(\sigma_\nu)\} \subset \left[\frac{-2M}{\eta_3 - \eta_2}, \frac{2M}{\eta_3 - \eta_2}\right], \ \& \ \{y_\nu(x_3) - y_\nu(x_4)\} \subset [-2M, 2M].$$

As in the previous case, there exist a subsequence $\{\nu_j\} \subset \{\nu\}$, points $x_1 \in [c, \eta_1]$ and $x_2 \in [\eta_2, \eta_3]$, and $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \mathbb{R}$ such that,

$$\xi_{\nu_j} \to x_1,\ \sigma_{\nu_j} \to x_2,\ y_{\nu_j}(\xi_{\nu_j}) \to \gamma_1,\ y'_{\nu_j}(\xi_{\nu_j}) \to \gamma_2,$$
$$y_{\nu_j}(\sigma_{\nu_j}) \to \gamma_3,\ y'_{\nu_j}(\sigma_{\nu_j}) \to \gamma_4, \text{ and } \ \{y_{\nu_j}(x_3) - y_{\nu_j}(x_4)\} \to \gamma_5.$$

Let $y(x)$ be the solution of (1), (2), for $k = 2$, satisfying

$$y(x_1) = \gamma_1,\ y'(x_1) = \gamma_2,\ y(x_2) = \gamma_3,\ y'(x_2) = \gamma_4, \text{ and } y(x_3) - y(x_4) = \gamma_5.$$

As above, it follows from Theorem 1.2 that

$$\lim y^{(i)}_{\nu_j}(x) = y^{(i)}(x) \text{ uniformly on } [c, d],$$

for each $i = 0, 1, 2, 3, 4$, and from (A) and (B) and the Kamke Convergence Theorem [4, page 14, Theorem 3.2], these convergences are uniform on each compact subinterval of (a, b). □

We remark that in [6], it was proved that condition (D) implies uniqueness of solutions of (1), (2), when solutions exist, for $1 \le k \le n - 2$. As a consequence of that and by Theorem 2.1, we can give a stronger result than Theorem 1.2, for $n = 3, 4, 5$.

Theorem 2.2 *For $n = 3$, 4, or 5, assume that with respect to* (1)*, conditions* (A)–(D) *are satisfied. Then, for each $1 \le k \le n-1$, solutions of* (1)*,* (2) *exist and are unique on* (a, b).

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