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Travelling Wave Solutions of Nonlocal Models for Media with Oscillating Inclusions

V.A. Danylenko and S.I. Skurativskyi *

Division of Geodynamics of Explosion, Subbotin Institute of geophysics, National Academy of Sciences of Ukraine, 63 G Bohdan Khmelnytsky str., 01054, Kyiv-54, Ukraine

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Abstract: Continual model of a complex medium with oscillating inclusions is considered. Travelling wave (TW) solutions to the source system are shown to satisfy a four-dimensional dynamical system. Qualitative study of the factorized system enables to show the existence of homoclinic and heteroclinic contours in vicinities of fixed points. Existence of the homoclinic loops results in the complex global behavior of phase trajectories, including the bifurcations of tori, that are investigated numerically.

Keywords: *travelling wave solutions; homoclinic curve; invariant tori; nonlinear normal modes.*

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1 Introduction

Experimental investigations of deformations of geomedia in the wide range of loading velocities, carried out in the last decades, testify that geomedia possess two basic features, namely, a discrete structure and oscillating motion of the discrete elements [1, 2].

Oscillating modes can be incorporated into the continual model by means of adding extra volumetric forces, causing the movements of the elements of the structure. In the papers [3, 4] a linear mathematical model for structured media taking into account the oscillations of structural elements has been suggested. In the simplest form the equations of motion can be written as follows:

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - m\rho\frac{\partial^2 w}{\partial t^2}, \qquad \frac{\partial^2 w}{\partial t^2} + \omega^2 (w-u) = 0, \tag{1}$$

* Corresponding author: mailto:skurserg@rambler.ru

where ρ is the density, σ is the stress, $u\,(x,t)$, $w\,(x,t)$ are the displacements of the bulk medium and typical oscillator with natural frequency ω, $m\rho$ is the density of oscillating inclusions.

But it is well known that the real geomaterials manifest a strong nonlinear effects when being subjected to high-intense impulse loading. In the situation when the medium is far from equilibrium, various relaxing processes within the elements of structure take place and the linear model becomes completely incorrect.

Thus, generally speaking, one should take into account both physical nonlinearity and nonlocal effects. This can be done by incorporating into the modelling system the following equation of state [5, 6]:

$$\sigma = E_1\varepsilon + E_3\varepsilon^3 + \theta\left(\sigma_{xx} - \sigma_x\frac{\varepsilon_x}{\varepsilon+1} - \eta\left[\varepsilon_{xx} - \frac{(\varepsilon_x)^2}{\varepsilon+1}\right]\right). \tag{2}$$

Equations (1), (2) form a closed system, which will be studied below. In our previous work [7], preliminary investigations of the system with $\theta = 0$ were carried out, revealing, in particular, the existence of periodic and soliton-like (especially important in nonlinear physics and engineering applications [8]) TW solutions.

The aim of the present paper is to study a set of TW solutions to (1)-(2) in the general case and to investigate an influence of spatial nonlocality on the structure of wave regimes.

2 Qualitative Analysis of the Dynamical System Describing Autowave Solutions

We restrict our consideration to the set of TW solutions, having the form

$$u = U(s), \quad w = W(s), \quad s = x - Dt. \tag{3}$$

Here the parameter D stands for the constant velocity of the wave front. Substituting (3) into the equations (1), (2), we obtain the dynamical system

$$D^2U'' = F' - mD^2W'', \tag{4}$$

$$W'' + \Omega^2\,(W - U) = 0, \tag{5}$$

$$F = e_1U' + e_3\,(U')^3 + \theta\left(F'' - F'\frac{U''}{U'+1} - \eta\left[U''' - \frac{(U'')^2}{U'+1}\right]\right), \tag{6}$$

where $\Omega = \omega D^{-1}$.

Integrating once equation (4), we get

$$F = D^2\,(U' + mW'). \tag{7}$$

Excluding the function F with the help of formula (7), we obtain the following system:

$$\begin{gathered} W'' + \Omega^2\,(W - U) = 0, \\ D^2\,(U' + mW') = e_1U' + e_3\,(U')^3 + \theta\left(F'' - F'\frac{U''}{U'+1} - \eta\left[U''' - \frac{(U'')^2}{U'+1}\right]\right). \end{gathered} \tag{8}$$

It is easily seen, that this system can be rewritten as four-dimensional dynamical system (8):

$$
\begin{gathered}
Z' = Y, \quad Y' = -\Omega^2 (Z - R), \quad R' = X, \\
X' = \frac{1}{\theta (D^2 - \eta)} \Big(-e_1 R - e_3 R^3 + \frac{X^2 \{D^2 - \eta\} \theta + D^2 \theta m X Y}{R + 1} + \\
+ \theta m D^2 \Omega^2 \{Z - R\} + D^2 \{R + mZ\} \Big)
\end{gathered}
\tag{9}
$$

Analysis shows that system (9) has three fixed (or stationary) points: the point M_1 coinciding with the origin, and the pair of the points $M_{2,3}$, given by the formulae

$$
X = Y = 0, \qquad Z_2 = R_2 = \pm \sqrt{\frac{D^2 (1 + m) - e_1}{e_3}} \equiv \pm G.
$$

It is easy to get convinced, that the Jacobi matrix of the system has the form:

$$
J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\Omega^2 & 0 & \Omega^2 & 0 \\ 0 & 0 & 0 & 1 \\ K_1 & 0 & K_2 & 0 \end{pmatrix},
$$

where $K_1 = \dfrac{mD^2 (1 + \Omega^2 \theta)}{(D^2 - \eta)\theta}$, and $K_2 = \dfrac{D^2 - e_1 - m\omega^2 \theta}{(D^2 - \eta)\theta}$ at the point M_1, and $K_2 = \dfrac{2e_1 - D^2 (2 + 3m + m\Omega^2 \theta)}{(D^2 - \eta)\theta}$ at the points $M_{2,3}$. The eigenvalues of the matrix J satisfy the biquadratic equation

$$
\lambda^4 + \lambda^2 (\Omega^2 - K_2) - (K_1 + K_2)\Omega^2 = 0.
$$

It is then obvious that $\lambda^2 = \frac{1}{2}\left(K_2 - \Omega^2 \pm \sqrt{(\Omega^2 + K_2)^2 + 4\Omega^2 K_1} \right)$. Depending on the values of λ the fixed points of the dynamical system are centers, saddles, or degenerate ones.

Some analytical results concerning the behavior of solutions in some vicinities of the fixed points can be obtained on the basis of the local asymptotic analysis. Let us consider the dynamical system (9) in the vicinity of the points $M_{2,3}$. For convenience, we replace the origin at the point M_i, $i = 1, 2$, making the change of variables $Z = x_1 + G$, $Y = y_1$, $R = x_2 + G$, $X = y_2$:

$$
\begin{aligned}
&x_1' = y_1, \quad y_1' = -\Omega^2 (x_1 - x_2), \quad x_2' = y_2, \\
&y_2' = K_1 x_1 + K_2 x_2 - \frac{3Ge_3}{(D^2 - \eta)\theta} x_2^2 - \frac{e_3}{(D^2 - \eta)\theta} x_2^3 + \frac{y_2 (-\eta y_2 + D^2 m y_1 + D^2 y_2)}{(D^2 - \eta)(1 + x_2 + G)}
\end{aligned}
\tag{10}
$$

To analyze the dynamics in a vicinity of the critical point of system (10), we introduce a formal parameter ε. Using the scaling transformation $x_i = \varepsilon x_i$, $y_i = \varepsilon y_i$ and the expansion in series $\dfrac{1}{1 + \varepsilon x_2 + G} = \sum\limits_{j=0} \dfrac{(-1)^j \varepsilon^j x_2^j}{(1 + G)^{j+1}}$ we can rewrite our system up to $O(\varepsilon^3)$ in the following form:

$$\begin{aligned} &x_1' = y_1, \quad y_1' = -\Omega^2\left(x_1 - x_2\right), \quad x_2' = y_2, \\ &y_2' = K_1 x_1 + K_2 x_2 + \varepsilon\left(H_1 x_2^2 + H_2 y_1 y_2 + H_3 y_2^2\right) + \varepsilon^2\left(L_1 x_2^3 + L_2 x_2 y_1 y_2 + L_3 x_2 y_2^2\right), \end{aligned} \tag{11}$$

where $H_1 = \dfrac{-3e_3 G}{(D^2-\eta)\theta}$, $H_2 = \dfrac{D^2 m}{(1+G)(D^2-\eta)}$, $H_3 = \dfrac{1}{1+G}$, $L_1 = \dfrac{-e_3}{(D^2-\eta)\theta}$, $L_2 = -\dfrac{D^2 m}{(1+G)^2(D^2-\eta)}$, $L_3 = -\dfrac{1}{(1+G)^2}$.

The expansion of the dynamical system (9) in vicinity of the stationary point M_1 can be written in the same form but with different coefficients $K_i,\ i = 1, 2$ and $H_i,\ L_i,\ i = 1, 2, 3$.

Now let us remind, that any linear system of coupled oscillators can be presented in an uncoupled form by means of passing to the normal modes (see e.g. [9]). This procedure is connected with the separation of general system dynamics onto the simpler motions described by systems with single degree of freedom, and expresses the principle of superposition for linear systems. For nonlinear systems analogs of the superposition principle can also be stated in many cases. For weakly non-linear systems like (11) the superposition principle can be established on the basis of the method of nonlinear normal modes [10,11]. In accordance with [12], we assume that it is possible to split the degrees of freedoms into the "master" coordinates $x_1 = u$, $y_1 = v$ and the "slave" coordinates x_2, y_2 functionally, dependent on the "master" ones: $x_2 = X_2(u,v)$, $y_2 = Y_2(u,v)$. Such relations just express the nonlinear principle of superposition. On the other hand, the nonlinear normal modes technique could be regarded as the next step of a local asymptotic analysis, following the qualitative analysis of the linearized system.

If we assume that the master system has the form

$$\begin{aligned} x_1' &= y_1, \qquad y_1' = f_1(x_i, y_i), \\ x_2' &= y_2, \qquad y_2' = f_2(x_i, y_i), \end{aligned}$$

then X_2 and Y_2 satisfy the equations

$$\begin{aligned} Y_2 &= \frac{\partial X_2}{\partial u} v + \frac{\partial X_2}{\partial v} f_1(u,v,X_2,Y_2), \\ f_2(u,v,X_2,Y_2) &= \frac{\partial Y_2}{\partial u} v + \frac{\partial Y_2}{\partial v} f_1(u,v,X_2,Y_2). \end{aligned} \tag{12}$$

Now we are going to find the solution of (12) in the form of the following series expansions:

$$\begin{aligned} X_2 &= a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2 + a_6 u^3 + a_7 u^2 v + a_8 uv^2 + a_9 v^3 + ..., \\ Y_2 &= b_1 u + b_2 v + b_3 u^2 + b_4 uv + b_5 v^2 + b_6 u^3 + b_7 u^2 v + b_8 uv^2 + b_9 v^3 + ... \end{aligned} \tag{13}$$

Inserting (13) into (12) and equating to zero the coefficients of the same monomials $u^i v^j$, we get a set of algebraic equations with respect to the parameters a_i, b_i. The first four coefficients obtained in this way are as follows:

for mode I

$$\begin{aligned} a_1 &= \frac{1}{2\Omega^2}(K_2 + \Omega^2 - \sqrt{(K_2+\Omega^2)^2 + 4K_1\Omega^2}), a_2 = 0, \\ b_1 &= 0, b_2 = \frac{1}{2\Omega^2}(K_2 + \Omega^2 - \sqrt{(K_2+\Omega^2)^2 + 4K_1\Omega^2}). \end{aligned} \tag{14}$$

for mode II

$$\begin{aligned} a_1 &= \frac{1}{2\Omega^2}(K_2 + \Omega^2 + \sqrt{(K_2+\Omega^2)^2 + 4K_1\Omega^2}), a_2 = 0, \\ b_1 &= 0, b_2 = \frac{1}{2\Omega^2}(K_2 + \Omega^2 + \sqrt{(K_2+\Omega^2)^2 + 4K_1\Omega^2}). \end{aligned} \tag{15}$$

Note that the sets of the parameters (14) and (15) correspond to the case when the linearly coupled system breaks up into a pair of uncoupled equations describing linear oscillations.

Using (14), we can express the coefficients of the quadratic monomials in the following form:

for mode I and II

$$\begin{gathered} a_3 = -\frac{a_1\left(2\,H_2\,\Omega^4\,(a_1-1)^2 + \left(H_1\,\left(K_2+\Omega^2\,(2-3\,a_1)\right) + 2\,H_3\,\Omega^4\,(a_1-1)^2\right)\,a_1\right)}{\left(K_2+\Omega^2\,(4-5\,a_1)\right)\,\left(K_2-\Omega^2\,a_1\right)}, \\ a_4 = 0, \\ a_5 = -\frac{a_1\,\left(H_2\,\left(K_2+\Omega^2\,(2-3\,a_1)\right) + \left(2\,H_1 + H_3\,\left(K_2+\Omega^2\,(2-3\,a_1)\right)\right)\,a_1\right)}{\left(K_2+\Omega^2\,(4-5\,a_1)\right)\,\left(K_2-\Omega^2\,a_1\right)}, \\ b_3 = 0, \\ b_4 = \frac{-2\,a_1\,\left(H_2\,\Omega^2\,(a_1-1) + \left(H_1 + H_3\,\Omega^2\,(a_1-1)\right)\,a_1\right)}{K_2+\Omega^2\,(4-5\,a_1)}, \quad b_5 = 0. \end{gathered} \tag{16}$$

Remark. One can easily see, that the coefficients defined by (16) become infinite, when the corresponding denominators nullify. This occurs if $K_2 - \Omega^2 a_1 = 0$, $K_2 - \Omega^2\,(5\,a_1 - 4) = 0$, $K_2 - \Omega^2\,(10\,a_1 - 9) = 0$, and so on. In these cases the corresponding resonances take place, namely $1:1$, $1:2$, $1:3$, ..., and the coupled system cannot be presented as a pair of uncoupled ones.

In the third order approximation we get:

for mode I and II

$$a_7 = a_9 = 0, b_6 = b_8 = 0, \tag{17}$$

while the rest ones are nonzero. We don't present them because they are very cumbersome.

Since $u' = v, \quad v' = f_1(u, v, X_2, Y_2)$, then taking into account the parameters values corresponding to the first mode, we get the following planar system (instead of the fourth order one):

$$u' = v, v' = \mu_1 u + \mu_2 u^2 + \mu_3 v^2 + \mu_4 u^3 + \mu_5 u v^2, \tag{18}$$

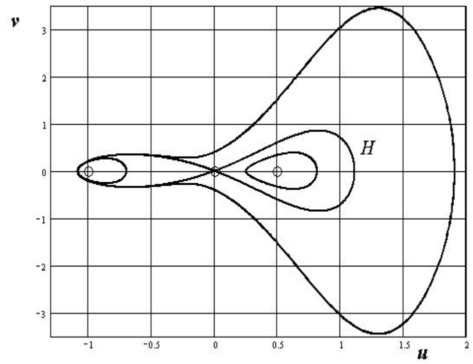
where $\mu_1 = \Omega^2(a_1 - 1)$, $\mu_2 = \Omega^2 a_3$, $\mu_3 = \Omega^2 a_5$, $\mu_4 = \Omega^2 a_6$, $\mu_5 = \Omega^2 a_8$. Note that the value $\sqrt{\mu_1}$ coincides with the pair of eigenvalues of the matrix J.

Nonlinear system (18) proves to be completely integrable. Indeed, dividing the second equation by the first one, we obtain the following equation:

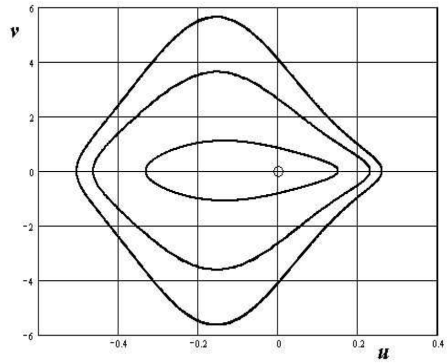
$$\frac{1}{2}\frac{d\rho}{du} = \mu_1 u + \mu_2 u^2 + \mu_3 \rho + \mu_4 u^3 + \mu_5 u \rho, \tag{19}$$

where $\rho = v^2$. The general solution of (19) can be presented in the form

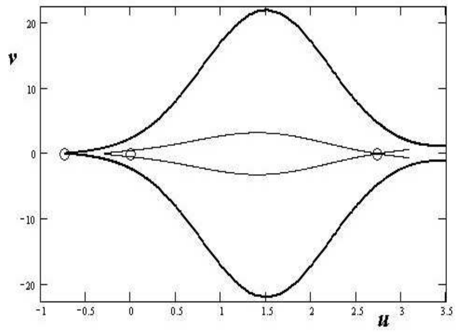
$$\begin{aligned} v^2 = 2\int_{u_0}^{u} \left(\mu_1\tau + \mu_2\tau^2 + \mu_4\tau^3\right)\exp\left[(u-\tau)(2\mu_3 + \mu_5(u+\tau))\right] d\tau + \\ + v_0^2 \exp\left[(u-u_0)(2\mu_3 + \mu_5(u+u_0))\right], \end{aligned} \tag{20}$$



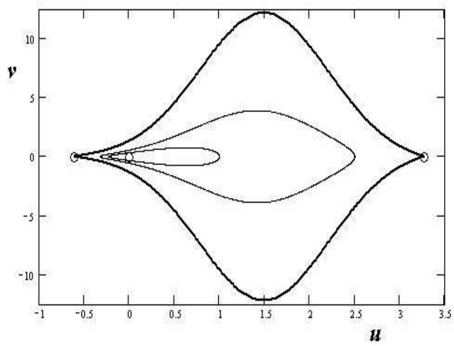
a) $\mu_1 > 0,\, \Delta > 0$



b) $\mu_1 < 0,\, \Delta < 0$



c) $\mu_1 < 0,\, \Delta > 0$



d) $\mu_1 < 0,\, \Delta > 0,\, \mu_2 = -1.3274589$

Figure 1: The phase portraits of dynamical system (19) at the different values of the parameters μ_i.

where $(u_0,\, v_0)$ stand for the initial data. Hence, the general solution of system (18) has the form $s = \int v^{-1} du$. To analyze the behavior of the solution obtained, it is desired to perform the qualitative integration [13] of the planar system (18).

The fixed points of system (18) have the coordinates

$$v = 0, u_1 = 0, u_2 = \frac{-\mu_2 - \sqrt{\mu_2^2 - 4\mu_1\mu_4}}{2\mu_4}, u_3 = \frac{-\mu_2 + \sqrt{\mu_2^2 - 4\mu_1\mu_4}}{2\mu_4}.$$

The fixed points $(u_{2,3}; 0)$ exist if $\Delta \equiv \mu_2^2 - 4\mu_1\mu_4 \geq 0$. The type of the fixed points is defined by the eigenvalues λ of the linearized matrix

$$M = \begin{pmatrix} 0 & 1 \\ \mu_1 + 2\mu_2 u_i + 3u_i^2 & 0 \end{pmatrix}.$$

For the fixed point $(0; 0)$ $\lambda^2 = \mu_1$ then if $\mu_1 < 0$ the fixed point is a center, if $\mu_1 > 0$ then it is a saddle. For another fixed points, if $\lambda^2 = \mu_1 + 2\mu_2 u_i + 3u_i^2 < 0$ then the fixed points are centers otherwise they are saddles. Let us consider the typical phase portraits of dynamical system (18).

We can distinguish the following cases

- $\mu_1 > 0$. The phase plane has a saddle $(0;0)$ if $\Delta < 0$; the phase plane has a saddle $(0;0)$ and a pair $(u_{2,3};0)$ of centers, if $\Delta > 0$ (Figure 1a).
- $\mu_1 < 0$. In the case when $\Delta < 0$, there is a center $(0;0)$ at the origin, Figure 1b. In the case $\Delta > 0$ the center is accompanied by the pair of saddles $(u_{2,3};0)$. Separatrices of one of the saddles form a homoclinic loop, whereas the separatrices of another one do not intersect, and surround the homoclinic loop (Figure 1c).

The last case is more complicated and interesting. Indeed, small changes of the parameters (e.g. μ_2) may cause a global qualitative changes of the phase portrait. The saddle separatrices under certain conditions can interconnect, forming a heteroclinic loop.

Using the exact solution (20), one can estimate the conditions of the heteroclinic loop creation. Suppose that a trajectory connecting the fixed points $(u_2,\, 0)$ and $(u_3,\, 0)$ exists. Then the coordinates of the fixed points must satisfy relation (20), where $u_0 = u_2$, $v_0 = 0$, $u = u_3$, $v = 0$. As a result, the following relation is derived

$$\int_{u_3}^{u_2} \left(\mu_1\tau + \mu_2\tau^2 + \mu_4\tau^3\right) \exp\left[(u_2 - \tau)(2\mu_3 + \mu_5(u_2 + \tau))\right] d\tau = 0.$$

It poses certain restrictions on the parameters of the dynamical system, the value of some parameter can be calculated precisely. Following this way, we succeeded in constructing the figure 1d, corresponding to $\mu_1 = -1$, $\mu_3 = 3$, $\mu_4 = 0.5$, and $\mu_5 = -2$.

3 Application of Local Analysis to the Dynamical System

Let us apply the results presented above to the investigation of the local dynamics of the system (9) in the vicinity of the fixed points. For the parameters values $D = 0.9$, $\omega = 1$, $m = 0.8$, $e_1 = 1$, $e_3 = 0.7$, $\eta = 0.105$, $\theta = 0.7$ the linearization matrix J taken at the fixed point $(Z_1;0;R_1;0)$ has the eigenvalues $(\pm 1.767i;\pm 0.606)$. At the fixed point $(Z_2;0;R_2;0)$ the eigenvalues of J are the following: $(\pm 2.256i;\pm 0.671i)$. In the vicinity of each fixed point the system (9) splits into a pair of separated planar dynamical systems,both written in the form (18), but differing by the values of the parameters μ_i.

Thus, for fixed point $(Z_2;0;R_2;0)$, the mode I is described by the dynamical system (18) with $\mu_i = \{-5.0882, -17.8235, -4.1475, -128.7057, -26.4049\}$. The corresponding phase plane of the system is depicted in Figure 1b.

The parameters $\mu_i = \{-0.4504, -0.4457, 0.3738, 0.4805, -2.4475\}$ relate to the mode II. Then dynamical system (18) has three fixed points $(0;0)$, $(-0.6097;0)$, $(1.5373;0)$ and the phase plane is presented in Figure 2.

The analysis of the system (18) in the vicinity of the fixed point $(Z_1;0;R_1;0)$ is carried out in the same way. The parameter values $\mu_i = \{-3.1214, -4.7475, -1.6104,$ $-16.4902, -5.3289\}$ correspond to the mode I. Corresponding phase portrait is shown in Figure 1b.

For the mode II we have the following values of the parameters $\mu_i = \{0.367067, -0.06678, 0.9554, -0.8903, 0.0763\}$. The phase plane of the system (18) contains the fixed points with the coordinates $(0;0)$, $(0.6057;0)$, $(-0.6807;0)$. Its phase portrait is illustrated by Figure 1a.

It is well known, that the presence of the homoclinic loops in the phase space of the multidimensional dynamical system can lead to the very complex dynamical behaviour [14]. In the case under consideration the homoclinic loops observed in the phase portrait of system (18) can rupture in the next approximations, causing the presence of

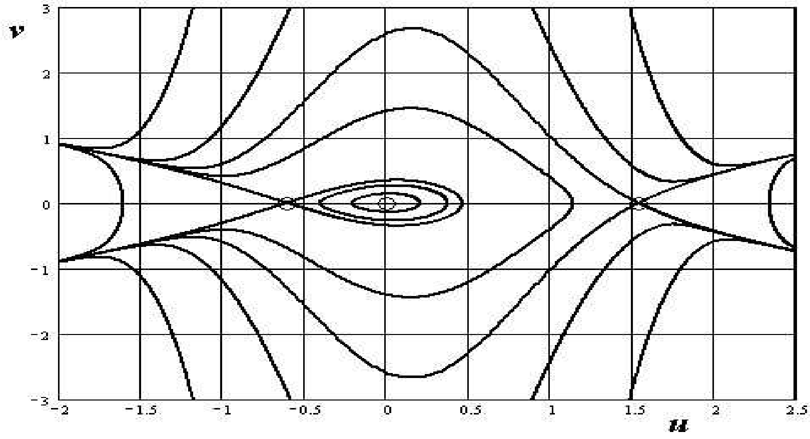
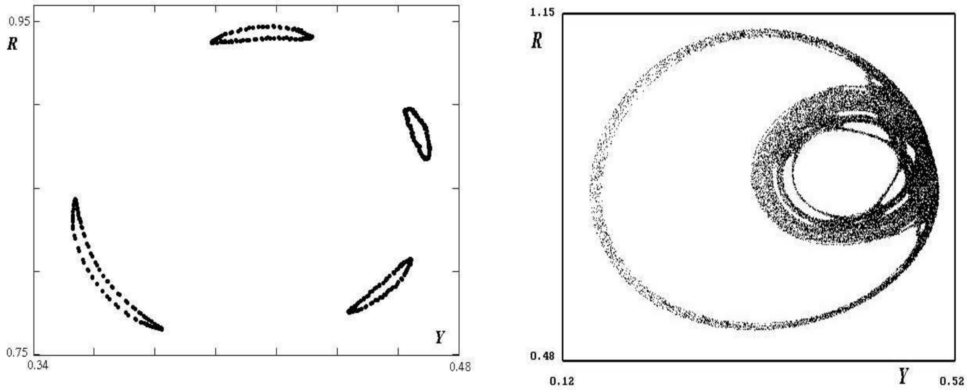


Figure 2: Phase plane of dynamical system (18).

complicated dynamics, coexisting with the homoclinic loops. In this case the complex trajectories can be observed in the phase space of a dynamical system.

In order to check the existence of complicated regimes, we integrated the dynamical system (9) numerically. In numerical experiments all the parameters but one were fixed. The θ played the rule of the bifurcation parameter. We started from the value θ of the order 0.01. Starting from the initial data $\left(10^{-6};0;0;0\right)$, we obtained the trajectories oscillating closely to the saddle separatices of the stationary point placed at the origin. Note, that for small θ a qualitative behavior of separatrices can be obtained by means of the asymptotic analysis.



a b

Figure 3: The Poincare sections of the tori existing in the phase space of dynamical system (9) at a) $\theta = 0.7$, b) $\theta = 0.72$.

Now let us consider the case when $\theta \sim O(1)$. It is evident that different initial data for dynamical system (9) lead to surfaces with different structure. We considered the

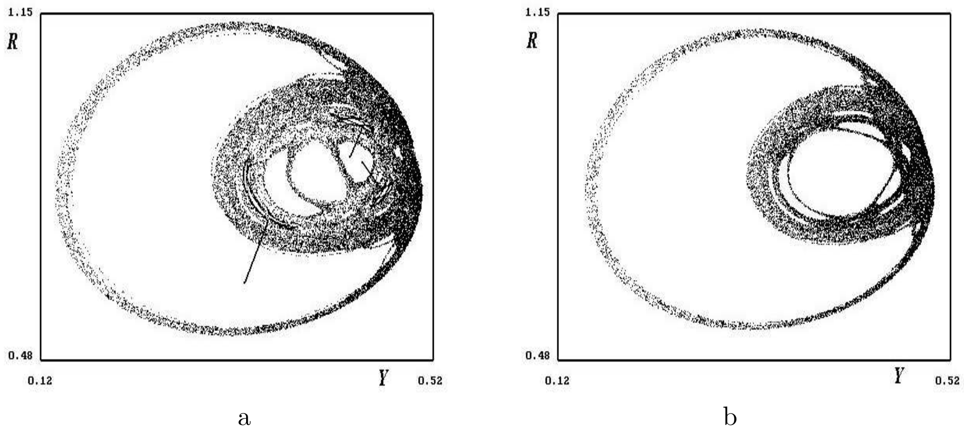


Figure 4: The Poincare sections of the tori existing in the phase space of dynamical system (9) at a) $\theta = 0.74$, b) $\theta = 0.7517$.

most interesting of them only. Integrating the dynamical system (9) with the initial data $(0.6; 0.3; 0.8; 0.4)$ and $\theta = 0.7$, one can observe the torus. For its visualization, we used the Poincare section technique. Let the surface $Z = 0.8$ be the target hyperplane. The locus of the intersection of the trajectories with the hyperplane $Z = 0.8$ is a 3D set. The part of this set is projected on the two-dimensional coordinate plane $(Y; R)$ and is depicted in figure 3a. Analyzing the obtained Poincare section, we see that the torus surface consists of four separated pipes.

Let us choose $\theta = 0.72$ and integrate dynamical system (9) from the same initial data. Using the same section plane, we get another Poincare diagram (fig.3b). The main peculiarities of the diagram are the appearance of the pipe of large radius and the presence of tightly enclosed pipes. The set of curves drown in the diagram looks like a fractal structure, though this has not been studied in detail yet. If parameter θ increases (Figure 4) the structure of the internal region changes most of all. Besides, one can select the regions that the running point visits more frequently (see the pointers in Figure 4).

The analysis of the Poincare sections shows that the trajectories in the four dimensional phase space of dynamical system (9) form a complex object which undergoes bifurcations as the parameter θ increases.

Applying the results of the local asymptotic analysis of the dynamical system (9) we can state, that the complex behavior of the phase space is connected with the reorganization of the homoclinic trajectories and their neighborhoods.

4 Conclusion

In summary, we would like to stress a key role of nonlocal effects, nonlinearity, and oscillating degrees of freedom in the formation of complex wave regimes. When the load applied senses the internal structure of media (and this is the case when the spatio-temporal characteristics of the load and the elements of the internal structure are comparable), then we cannot neglect the dynamics of internal degrees of freedom. Let us stress, that results obtained in this study essentially differ from those predicted by linear models (both local and nonlocal ones) [7].

From the mathematical point of view investigations of the modelling system (1)-(2) are more difficult in comparison with their local analogs, nevertheless, under some additional assumptions they can be treated within the traditional asymptotic techniques. Besides, the variety of observed regimes indicate the existence of another important type of solutions, inherent for essentially nonlocal models.

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