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Numerical Solutions of System of Non-linear ODEs by Euler Modified Method

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Abstract: In this paper, we have proposed Euler's modified method for solving the six coupled system of non-linear ordinary differential equations (ODEs), which are aroused in the reduction of stratified Boussinesq equations. This method can also be called as revised Euler's modified method for solving two simultaneous ODEs. We have obtained the numerical solutions on stable and unstable manifolds. The error between the numerical solution and exact solution is of order 10^{-20} to 10^{-6}. We have coded this programme in C-language.

Keywords: *stratified Boussinesq equation, Euler modified method, integrable systems.*

Mathematics Subject Classification (2010): 34A09, 65L05, 65L99.

1 Introduction

The stratified Boussinesq equations form a system of Partial Differential Equations (PDEs) modelling the movements of planetary atmospheres. It may be noted that literature also refers to Boussinesq approximation as Oberbeck–Boussinesq approximation. For this, one may refer to an interesting article by Rajagopal et al [1] which provides a rigorous mathematical justification for perturbations of the Navier-Stokes equations. Majda & Shefter [2] have chosen certain special solutions of this system of ODEs to demonstrate the onset of instability when the Richardson number is less than 1/4. Majda and Shefter [3] have shown that the analysis, in the special cases considered, reduces to the solutions of Hamiltonian system. These reductions form an interesting coupled system of six non-linear ODEs. Shrinivasan et al [4] have also tested the system for complete integrability by use of first integrals. Further, Desale [6] has incorporated the effect

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of rotation in the same system in the context of basin scale dynamics, while Desale and Sharma [7] have given special solutions of rotating stratified Boussinesq equations. Desale and Patil [8] have tested the system of six coupled nonlinear ODEs by Painleve Test. Burton and Zhang [9] have given the periodic solutions for singular integral equations. Biswas et al [10] have studied the behavior of soliton solutions in the form of KdV partial differential equation in the fiber optics solitons theory in communication engineering.

In this paper, we have given the C-code to find and to test the initial values which lie on the invariant surface given by equation (4). We have implemented Euler Modified method to find the numerical solution of the system (1) passing through the initial values on invariant surface (4). We have discussed the use of this method in the subsection (3.1). We have given the codes for solutions on stable and unstable manifolds of invariant surface which is obtained by four first integrals.

2 Preliminaries

Shrinivasan et al [4] have tested the system (1) as given below for complete integrability. Also, Deasle and Shrinivasan [5] have shown that in the general case, the problem of integration reduces to the integrations of the system of six coupled autonomous ODE's

$$\left.\begin{aligned} \dot{\mathbf{w}} &= \frac{g}{\rho_b}\hat{\mathbf{e_3}} \times \mathbf{b}, \\ \dot{\mathbf{b}} &= \frac{1}{2}\mathbf{w} \times \mathbf{b}, \end{aligned}\right\} \tag{1}$$

where $\mathbf{w} = (w_1, w_2, w_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$ and $\frac{g}{\rho_b}$ is a non-dimensional constant as mentioned by Desale [11] in his Ph. D. thesis.

The above system can be written component-wise as below

$$\left.\begin{aligned} \dot{w_1} &= -\frac{g}{\rho_b}b_2,\ \dot{w_2} = \frac{g}{\rho_b}b_1,\ \dot{w_1} = 0, \\ \dot{b_1} &= \frac{1}{2}(w_2b_3 - w_3b_2),\ \dot{b_2} = \frac{1}{2}(w_3b_1 - w_1b_3),\ \dot{b_3} = \frac{1}{2}(w_1b_2 - w_2b_1). \end{aligned}\right\} \tag{2}$$

The system (1) admits the following four first integrals

$$\left.\begin{aligned} &1)\quad |\mathbf{b}|^2 &&= c_1, \\ &2)\quad \mathbf{w}\cdot\mathbf{b} &&= c_2, \\ &3)\quad \hat{\mathbf{e_3}}\cdot\mathbf{w} &&= c_3, \\ &4)\quad \frac{|\mathbf{w}|^2}{2} + \frac{2g}{\rho_b}\hat{\mathbf{e_3}}\cdot\mathbf{b} &&= c_4, \end{aligned}\right\} \tag{3}$$

with non zero values of c_1, c_2, c_3 and c_4. The possible critical points of the system (1) are $(\pm\hat{\mathbf{e_3}}, \pm\hat{\mathbf{e_3}})$. For $c_1 = 1$ and $w = \pm\hat{\mathbf{e_3}}$, c_3 may assume the values ± 1 (not both). Now we take $c_3 = 1$, so that the possible critical points are $(\hat{\mathbf{e_3}}, \pm\hat{\mathbf{e_3}})$. At the rest points $(\hat{\mathbf{e_3}}, \pm\hat{\mathbf{e_3}})$, the value of c_2 is ± 1.

Remark 2.1 The case $c_2 = -1$ will be surface disjoint from $\mathbf{w}\cdot\mathbf{b} = 1$ and the similar analysis will be carried out if we take $c_2 = -1$. Right now we take $c_1 = 1$, $c_2 = 1$ and $c_3 = 1$. But this forces $\mathbf{b} = \hat{\mathbf{e_3}}$ at a critical point, so with our specific conditions we have only one rest point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ on the invariant surface (3). At this critical point fourth first integral assumes the value $c_4 = \frac{1}{2} + \frac{2g}{\rho_b}$.

With the above specification, we have following four first integrals

$$|\mathbf{b}|^2 = 1, \mathbf{w}\cdot\mathbf{b} = 1, \hat{\mathbf{e_3}}\cdot\mathbf{w} = 1, \frac{|\mathbf{w}|^2}{2} + \frac{2g}{\rho_b}\hat{\mathbf{e_3}}\cdot\mathbf{b} = \frac{1}{2} + \frac{2g}{\rho_b}. \tag{4}$$

A critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ lies on invariant surface and (b_1, b_2, b_3) is on the surface $|\mathbf{b}|^2 = 1$. Therefore we have

$$\left.\begin{array}{rcl} w_1 &=& \dfrac{-b_2 k}{1-b_3} + \dfrac{b_1}{1+b_3}, \\ w_2 &=& \dfrac{b_1 k}{1-b_3} + \dfrac{b_2}{1+b_3}, \\ w_3 &=& 1. \end{array}\right\} \tag{5}$$

where k is a function of b_3, given by the following equation

$$k^2 = \frac{(1-b_3)^2}{(1+b_3)^2}\Big[\frac{4g(1+b_3)-\rho_b}{\rho_b}\Big]. \tag{6}$$

One may refer [4, 5] for more details of this analysis. Since $|\mathbf{b}|^2 = 1$, we can use spherical-polar co-ordinates

$$b_1 = \cos\theta\sin\phi,\ b_2 = \sin\theta\sin\phi,\ b_3 = \cos\phi. \tag{7}$$

Hence,

$$k^2 = \tan^4(\frac{\phi}{2})\Big[\frac{8g}{\rho_b}\cos^2(\frac{\phi}{2}) - 1\Big]. \tag{8}$$

For k to be real , Shrinivasan et al [5] have put up the restriction to ϕ as $0 \le \phi \le 2\cos^{-1}(\sqrt{\frac{\rho_b}{8g}})$. With this limitation k takes the values negative, positive and zero. With these possible choices of k, the invariant surface will be the union of disjoint manifolds corresponding to $k > 0$, is unstable manifold, $k < 0$, is stable manifold and $k = 0$, is a center manifold. Regarding these manifolds, readers are advised to refer to Shrinivasan et al [5].

Now for $k > 0$, the unstable manifold is given by

$$\left.\begin{array}{rcl} w_1 &=& \tan(\frac{\phi}{2})\Big[\cos\theta - \sin\theta\sqrt{\frac{8g}{\rho_b}\cos^2(\frac{\phi}{2}) - 1}\ \Big], \\ w_2 &=& \tan(\frac{\phi}{2})\Big[\cos\theta + \sin\theta\sqrt{\frac{8g}{\rho_b}\cos^2(\frac{\phi}{2}) - 1}\ \Big], \\ w_3 &=& 1, \\ b_1 &=& \cos\theta\sin\phi, \\ b_2 &=& \sin\theta\sin\phi, \\ b_3 &=& \cos\phi, \\ with && \\ k &=& \tan^2(\dfrac{\phi}{2})\Big[\dfrac{8g}{\rho_b}\cos^2(\dfrac{\phi}{2}) - 1\Big]. \end{array}\right\} \tag{9}$$

On this surface, system (1) reduces to

$$\left.\begin{aligned}\frac{d\phi}{dt} &= \tfrac{1}{2}\tan(\tfrac{\phi}{2})\sqrt{\tfrac{8g}{\rho_b}\cos^2(\tfrac{\phi}{2})-1},\\ \frac{d\theta}{dt} &= \tfrac{1}{4}\sec^2(\tfrac{\phi}{2}),\end{aligned}\right\} \tag{10}$$

where as for $k < 0$, the stable manifold is given by

$$\left.\begin{aligned}w_1 &= \tan(\tfrac{\phi}{2})\Big[\cos\theta+\sin\theta\sqrt{\tfrac{8g}{\rho_b}\cos^2(\tfrac{\phi}{2})-1}\;\Big],\\ w_2 &= \tan(\tfrac{\phi}{2})\Big[\cos\theta-\sin\theta\sqrt{\tfrac{8g}{\rho_b}\cos^2(\tfrac{\phi}{2})-1}\;\Big],\\ w_3 &= 1,\\ b_1 &= \cos\theta\sin\phi,\\ b_2 &= \sin\theta\sin\phi,\\ b_3 &= \cos\phi,\\ \textit{with}&\\ k &= -\tan^2(\tfrac{\phi}{2})\Big[\tfrac{8g}{\rho_b}\cos^2(\tfrac{\phi}{2})-1\Big].\end{aligned}\right\} \tag{11}$$

On this surface, system (1) reduces to

$$\left.\begin{aligned}\frac{d\phi}{dt} &= -\tfrac{1}{2}\tan(\tfrac{\phi}{2})\sqrt{\tfrac{8g}{\rho_b}\cos^2(\tfrac{\phi}{2})-1},\\ \frac{d\theta}{dt} &= \tfrac{1}{4}\sec^2(\tfrac{\phi}{2}).\end{aligned}\right\} \tag{12}$$

3 Numerical Solution

In their studies, Shrinivasan et al [5] have shown that the system (1) is completely integrable and solutions exist on invariant surface (3) for all the time. So we are looking for the numerical solution of the system (1) on the invariant surface (3). We find the initial values which satisfy the four first integrals given by (4) and consequently we can find the solutions of system (1) passing through these initial values. We use the following programme to find the initial values so that they satisfy the four first integrals. We use the following programme to test finitely many points.

```
#include<stdio.h>
#include<conio.h>
#include<math.h>
void main()
{ FILE *fp;
  double b10,b20,b30,phi0,theta0;
  double eps=0.0000001,G=39.2;
  double g=9.8,rho_b=2;
  long int i,j,k;
```

```
  double y1,w10,w20,w30;
  double int1,int2,int3L,int3R;
  double diff1,diff2,diff3;
   clrscr();
   fp=fopen("new_02a1.xls","w+");

   fprintf(fp,"\n\t PROGRAMME FOR INITIAL SOLUTIONS
                    SATISFYING FIRST FOUR INTEGRALS");
 b10=0.000001;
 b20=0.000001;
 b30=0.000001;
 printf("\n\t PROGRAMME FOR INITIAL SOLUTIONS
             SATISFYING FIRST FOUR INTEGRALS");
    fprintf(fp,"\n\tb10\tb20\tb30\ttheta0\tphi0\n");
     printf("\nb10\tb20\tb30\ttheta0\tphi0\n");
     for(k=0;k<1000;k++) //b30 loop
       {
          for(j=0;j<1000;j++)//b20 loop
         {
            for(i=0;i<1000;i++) //b10 loop
           {
        theta0=atan(b20/b10);
        phi0=atan(sqrt(b10*b10+b20*b20)/b30);
        y1=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1.0);
        w10=tan(phi0/2.0)*(cos(theta0)-(sin(theta0)*y1));
        w20=tan(phi0/2.0)*(sin(theta0)+(cos(theta0)*y1));
        w30=1.000000;
        int1=b10*b10+b20*b20+b30*b30;
        int2=b10*w10+b20*w20+b30*w30;
   int3L=w10*w10+w20*w20+w30*w30+((4.0*g*b30)/rho_b);
   int3R=1.0+((4.0*g)/rho_b);
   diff1=fabs(int1-1.0);
   diff2=fabs(int2-1.0);
   diff3=fabs(int3L-int3R);
  if(diff1<eps)
    {
   if(diff2<eps)
    {
     if(diff3<eps)
  {
fprintf(fp,"\n\t%.10lf\t%.10lf\t%.10lf\t%.10lf\t%.10lf",
             b10,b20,b30,theta0,phi0);
printf("\n%.10lf\t%.10lf\t%.10lf\t%.10lf\t%.10lf",
            b10,b20,b30,theta0,phi0);
 } } }
 b10=b10+0.000001;
 if(b10>=1.000001) b10=0.000001; }
 b20=b20+0.000001;
```

```
if(b20>=1.000001) b20=0.000001; }
b30=b30+0.000001;
if(b30>=1.000001) b30=0.000001; }
getch(); }
```

With the help of the above programme we get the initial value. After getting the initial value, we decide on which manifold the initial value lies on – that is whether it is stable, unstable or central manifold. Using the above programme, we get the initial value $b_0 = (b_{10}, b_{20}, b_{30})$. From this initial value $b_0 = (b_{10}, b_{20}, b_{30})$, we calculate the value of k, then we conclude whether the initial value is on stable, unstable or on center manifold. Once we confirm, our initial value is either on stable or unstable surface, accordingly we find the numerical solution by Euler modified method. In the following subsection (3.1), we implement the method to calculate the numerical solution. Further, we write the algorithm and encode the programme.

3.1 Implementation of Euler modified method for the numerical solution

We start with the initial condition $t = 0$ and the initial point $\mathbf{b}_0 = (b_{10}, b_{20}, b_{30})$. We calculate the initial value of (ϕ_0, θ_0) as

$$\theta_0 = \tan^{-1}\left(\frac{b_2}{b_1}\right), \quad \phi_0 = \tan^{-1}\left(\frac{\sqrt{b_1^2 + b_2^2}}{b_3}\right). \tag{13}$$

Now, we calculate the value of ϕ_1 and θ_1 by Predictor Formula as

$$\phi_1 = \phi_0 + hf_1(t_0, \phi_0, \theta_0), \quad \theta_1 = \theta_0 + hf_2(t_0, \phi_0, \theta_0), \tag{14}$$

where h is a step size, $f_1 = \frac{1}{2}\tan(\frac{\phi}{2})\sqrt{\frac{8g}{\rho_b}\cos^2(\frac{\phi}{2}) - 1}$, $f_2 = \frac{1}{4}\sec^2(\frac{\phi}{2})$. Since there is an error in ϕ_1 and θ_1, we refine or try to get more accurate values of ϕ_1 and θ_1 by Corrector Formula as below,

$$\phi_1^{(1)} = \phi_0 + \tfrac{h}{2}[f_1(t_0, \phi_0, \theta_0) + f_1(t_0 + h, \phi_1, \theta_1)]. \tag{15}$$

In the above step the error can be reduced to the desired accuracy. Here we have considered the accuracy of 10^{-20}. The error is reduced by repeating the corrector formula as below,

$$\phi_1^{(n+1)} = \phi_0 + \tfrac{h}{2}[f_1(t_0, \phi_0, \theta_0) + f_1(t_0 + h, \phi_1^{(n)}, \theta_1)]. \tag{16}$$

As we get the most correct value of ϕ, we use this value of ϕ for calculating the correct value of θ with the accuracy of 10^{-20} as

$$\theta_1^{(1)} = \theta_0 + \tfrac{h}{2}[f_2(t_0, \phi_0, \theta_0) + f_2(t_0 + h, \phi_1, \theta_1)], \tag{17}$$

$$\theta_1^{(n+1)} = \theta_0 + \tfrac{h}{2}[f_2(t_0, \phi_0, \theta_0) + f_2(t_0 + h, \phi_1, \theta_1^{(n)})], \tag{18}$$

and so on. This gives us the corrected values of θ and ϕ. The exact solutions of (10) are

$$\left.\begin{aligned}
\phi(t) &= 2\sin^{-1}\Big[\frac{2k_1.\sqrt{\frac{G-1}{G}}.e^{-\frac{t}{4}\sqrt{G-1}}}{1 + k_1^2.e^{-\frac{t}{2}\sqrt{G-1}}}\Big],\\
\theta(t) &= \tfrac{t}{4} + \tan^{-1}\Big[\tfrac{\sqrt{G}}{k_1}.e^{\frac{t}{4}\sqrt{G-1}} - \sqrt{G-1}\Big]\\
&\quad - \tan^{-1}\Big[\tfrac{\sqrt{G}}{k_1}.e^{\frac{t}{4}\sqrt{G-1}} + \sqrt{G-1}\Big] + k_2,
\end{aligned}\right\} \tag{19}$$

where k_1, k_2 are constants and $G = 8g/\rho_b$. Now for our calculations, we took $G = 39.2$ with $g = 9.8$ and $\rho_b = 2$. We have compared the corrected values with the exact solutions and we got the minimum error of 10^{-20} and maximum up to 10^{-6}. Now by using the method of back substitution we have obtained the values of $\mathbf{b}(b_1, b_2, b_3)$ and $\mathbf{w}(w_1, w_2, w_3)$.

3.2 Algorithm for numerical solution

Here we give the algorithm for numerical solution by Euler's Modified Method [14, 15]. The details of the algorithm are as given below:

Step 1: Enter the initial values of $t_0, \phi_0, \theta_0, t, g, \rho_b$ and h (step size).

Step 2:Calculate the values of $b_{10}, b_{20}, b_{30}, w_{10}, w_{20}, w_{30}, k_1, k_2$ and k. Here we have obtained the initial values.

Step 3: Calculate the values of ϕ_1 and θ_1 by using Euler's Predictor Formula.

Step 4: Calculate the value of ϕ_1 up to the desired accuracy by using Euler's Corrector Formula.

Step 5: Calculate the value of θ_1 up to the desired accuracy by using Euler's Corrector Formula.

Step 6: Calculate the values of b_1, b_2, b_3, w_1, w_2 and w_3 by using equation (7).

Step 7: Calculate the exact values of ϕ and θ by using equation (9) then calculate the exact values of b_1, b_2, b_3, w_1, w_2 and w_3 by using equation (7).

Step 8: Print the required exact and calculated numerical values.

Step 9: Replace ϕ_1 by ϕ_0, θ_1 by θ_0 and t_0 by $t + h$ and go to Step 3, until the value of ϕ is reached to its maximum for the given unstable manifold.

Step 10: Plot the graphs to see the difference.

Step 11: End.

3.3 Numerical solution on unstable manifold

On this manifold, we have $k > 0$ and the system (1) reduces to (10). Now we use the following programme to find the solution on the unstable manifold.

```
 #include<stdio.h>
 #include<stdlib.h>
 #include<conio.h>
 #include<math.h>
 #include<sys\stat.h>
 void main()
      {
  double f(double p);
  FILE *fp;
  double phi0,phi1,phi10,theta0,theta1,theta10,er_theta,er_phi;
  double h,t,t0,t1,b1,b2,b3,w1,w2,w3,b10,b20,b30,w10,w20,w30;
  double eb1,eb2,eb3,ew1,ew2,ew3,be1,be2,be3,we1,we2;
  double x,y0,y1,z0,z1,diff1,diff2,eps=0.01;
  double etheta,ephi,G=39.2,u,u1,u2,k1,k2,k;
  int i,n; /* g=9.8 , rho_b=2,*/
  clrscr();
printf("\n\n\t\t PROGRAMME FOR EULER MODIFIED METHOD");
```

```
fp=fopen("nrd001.xls","w+"); t0=0.0; t=6.0; h=0.001;
    printf("\n\n\t\t Enter the value of phi0= ");
    scanf("%lf",&phi0);
 printf("\n\n\t\t Enter the value of theta0= ");
    scanf("%lf",&theta0);
    fprintf(fp,"\n The value of phi0=%lf ",phi0);
 fprintf(fp,"\n The value of theta0=%lf ",theta0);
    b10=cos(theta0)*sin(phi0);
    b20=sin(theta0)*sin(phi0);
 b30=cos(phi0);
 y1=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
 w10=tan(phi0/2.0)*(cos(theta0)-sin(theta0)*y1);
 w20=tan(phi0/2.0)*(sin(theta0)+cos(theta0)*y1);
 w30=1.000000;
 fprintf(fp,"\n The value of b10=%lf ",b10);
 fprintf(fp,"\n The value of b20=%lf ",b20);
 fprintf(fp,"\n The value of b30=%lf ",b30);
 fprintf(fp,"\n The value of w10=%lf ",w10);
 fprintf(fp,"\n The value of w20=%lf ",w20);
 fprintf(fp,"\n The value of w30=%lf ",w30);

 /*calculating k1 and k2 for exact solution and
             k for initial solution  */
 u=sin(phi0/2.0);
 k1=(sqrt((G-1.0)/G)+sqrt(((G-1)/G)-u*u))/u;

 k2=theta0-atan((sqrt(G)/k1)-(sqrt(G-1.0)))
         +atan((sqrt(G)/k1)+(sqrt(G-1.0)));
 k=(tan(phi0/2)*tan(phi0/2))*sqrt(G*cos(phi0/2)*cos(phi0/2)-1);
 printf("\n\n\tThe value of k1=%.8f \n\n\tThe value of
        k2=%.8f",k1,k2);
    printf("\n\n\tThe value of k=%.8f ",k);
    fprintf(fp,"\nThe value of k1=%.8f ",k1);
    fprintf(fp,"\nThe value of k2=%.8f ",k2);
    fprintf(fp,"\nThe value of k=%.8f ",k);
         i=0;
printf("\n\n\tPress 'ENTER' to get step by step");
fprintf(fp,"\n t\t b1\t b2\t b3\t w1\t w2\t w3\t theta\tphi
          \tk\t ephi\t etheta");
printf("\n\n\t Error in Theta\t\t Error in Phi \t Value of K");
while(t0<t)
 {
      i++;
      t1=t0+h;
      y0=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
      phi1=phi0+(h/2.0)*tan(phi0/2.0)*y0;
      phi10=phi1;
      theta1=theta0+(0.25*h*f(phi0));
```

```
      theta10=theta1;

  /* Calculation of phi by modified formula */
      do{
        y0=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
        y1=sqrt(39.2*cos(phi10/2.0)*cos(phi10/2.0)-1);
phi1=phi0+(0.25*h)*((tan(phi0/2.0)*y0)+(tan(phi10/2.0)*y1));
       diff1=fabs(phi1-phi10);
       phi10=phi1;
       }while(diff1>eps);
k=(tan(phi1/2)*tan(phi1/2)) *sqrt(G*cos(phi1/2)*cos(phi1/2)-1);

  /* Calculation of theta by modified formula */
    do{
    theta1=theta0+(0.125*h)*(f(phi0)+f(phi1));
    diff2=fabs(theta10-theta1);
      }while(diff2>eps);

 /* Calculation of an approximate solution what we need */
     b1=cos(theta1)*sin(phi1);
     b2=sin(theta1)*sin(phi1);
     b3=cos(phi1);
     y1=sqrt(39.2*cos(phi1/2.0)*cos(phi1/2.0)-1);
     w1=tan(phi1/2.0)*(cos(theta1)-sin(theta1)*y1);
     w2=tan(phi1/2.0)*(sin(theta1)+cos(theta1)*y1);
     w3=1.000000;

 /* calculation  of exact solution */
ephi=2*asin((2*k1*sqrt((G-1)/G)*exp(-(t1/4)*sqrt(G-1)))
       /(1+k1*k1*exp(-(t1/2)*sqrt(G-1))));

   u1=atan((sqrt(G)*exp((t1/4)*sqrt(G-1)))/k1-sqrt(G-1));
   u2=atan((sqrt(G)*exp((t1/4)*sqrt(G-1)))/k1+sqrt(G-1));
   etheta=(t1/4)+u1-u2+k2;
   k=(tan(etheta/2)*tan(etheta/2))
         *sqrt(G*cos(etheta/2)*cos(etheta/2)-1);

 /* calculation of error in theta and phi*/
     er_theta=fabs(theta1-etheta);
     er_phi=fabs(phi1-ephi);

 /*   calculation of B and W */
  be1=cos(etheta)*sin(ephi);
  be2=sin(etheta)*sin(ephi);
  be3=cos(ephi);
  y1=sqrt(39.2*cos(ephi/2.0)*cos(ephi/2.0)-1);
  we1=tan(ephi/2.0)*(cos(etheta)-sin(etheta)*y1);
  we2=tan(ephi/2.0)*(sin(etheta)+cos(etheta)*y1);
```

```
  fprintf(fp,"\n%lf\t%lf\t%lf\t%lf\t%lf\t%lf\t%lf\t%lf\t%lf
           \t%lf\t%lf\t%lf", t1, b1, b2, b3, w1, w2, w3,
           theta1, phi1, k, etheta, ephi);
printf("\n\n\t%.20lf\t%.20lf\t%lf",er_theta,er_phi,k);
        phi0=phi1;
        theta0=theta1;
        t0=t1;
            getch();
         }}
    double f(double p)
       { double p_dash;
          p_dash=(1.0/cos(p/2.0))*(1.0/cos(p/2.0));
          return(p_dash);
       }
```

3.4 Numerical solution on stable manifold

On this manifold, we have $k < 0$ and the system (1) reduces to (12). Now we use the following programme to find the solution on the stable manifold.

```
 #include<stdio.h>
 #include<stdlib.h>
 #include<conio.h>
 #include<math.h>
 #include<sys\stat.h>
 void main()
 {
 double f(double p);
 FILE *fp;
 double phi0,phi1,phi10,theta0,theta1,theta10,er_theta,er_phi;
 double h,t,t0,t1,b1,b2,b3,w1,w2,w3,b10,b20,b30,w10,w20,w30;
 double eb1,eb2,eb3,ew1,ew2,ew3,be1,be2,be3,we1,we2;
 double x,y0,y1,z0,z1,diff1,diff2,eps=0.01;
 double etheta,ephi,G=39.2,u,u1,u2,k1,k2,k;
 int i,n; /* g=9.8 , rho_b=2,*/
clrscr(); printf("\n\n\t\t PROGRAMME FOR EULER MODIFIED METHOD");
fp=fopen("nrd001.xls","w+"); t0=0.0; t=6.0; h=0.001;
printf("\n\n\t\t Enter the value of phi0= ");
scanf("%lf",&phi0);
printf("\n\n\t\t Enter the value of theta0= ");
scanf("%lf",&theta0);
fprintf(fp,"\n The value of phi0=%lf ",phi0);
fprintf(fp,"\n The value of theta0=%lf ",theta0);
   b10=cos(theta0)*sin(phi0);
   b20=sin(theta0)*sin(phi0);
   b30=cos(phi0);
 y1=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
 w10=tan(phi0/2.0)*(cos(theta0)+sin(theta0)*y1);
 w20=tan(phi0/2.0)*(sin(theta0)-cos(theta0)*y1);
```

```
 w30=1.000000;
  fprintf(fp,"\n The value of b10=%lf ",b10);
  fprintf(fp,"\n The value of b20=%lf ",b20);
  fprintf(fp,"\n The value of b30=%lf ",b30);
  fprintf(fp,"\n The value of w10=%lf ",w10);
  fprintf(fp,"\n The value of w20=%lf ",w20);
  fprintf(fp,"\n The value of w30=%lf ",w30);

  /*calculating k1 and k2 for exact solution
    and k for initial solution  */
 u=sin(phi0/2.0);
 k1=(sqrt((G-1.0)/G)+sqrt(((G-1)/G)-u*u))/u;
 k2=theta0-atan((sqrt(G)/k1)-(sqrt(G-1.0)))
          +atan((sqrt(G)/k1)+(sqrt(G-1.0)));
 k= - (tan(phi0/2)*tan(phi0/2))
           *sqrt(G*cos(phi0/2)*cos(phi0/2)-1);
 printf("\n\n\tThe value of k1=%.8f \n\n\tThe value of
          k2=%.8f",k1,k2);
 printf("\n\n\tThe value of k=%.8f ",k);
 fprintf(fp,"\nThe value of k1=%.8f ",k1);
 fprintf(fp,"\nThe value of k2=%.8f ",k2);
 fprintf(fp,"\nThe value of k=%.8f ",k);
     i=0;
printf("\n\n\tPress 'ENTER' to get step by step");
fprintf(fp,"\n t\t b1\t b2\t b3\t w1\t w2\t w3\t theta\t phi
         \tk\t ephi\t etheta");
printf("\n\n\t Error in Theta\t\t Error in Phi \t Value of K");
while(t0<t)
    {
         i++;
         t1=t0+h;
         y0=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
         phi1=phi0-(h/2.0)*tan(phi0/2.0)*y0;
         phi10=phi1;
         theta1=theta0+(0.25*h*f(phi0));
         theta10=theta1;

  /* Calculation of phi by modified formula */
  do{
   y0=sqrt(39.2*cos(phi0/2.0)*cos(phi0/2.0)-1);
   y1=sqrt(39.2*cos(phi10/2.0)*cos(phi10/2.0)-1);
   phi1=phi0-(0.25*h)*((tan(phi0/2.0)*y0)+(tan(phi10/2.0)*y1));
   diff1=fabs(phi1-phi10);
   phi10=phi1;
    }while(diff1>eps);
 k= - (tan(phi1/2)*tan(phi1/2))
          *sqrt(G*cos(phi1/2)*cos(phi1/2)-1);
```

```
  /* Calculation of theta by modified formula */
    do{
       theta1=theta0+(0.125*h)*(f(phi0)+f(phi1));
       diff2=fabs(theta10-theta1);
       }while(diff2>eps);

 /* Calculation of an approximate solution what we need */
    b1=cos(theta1)*sin(phi1);
    b2=sin(theta1)*sin(phi1);
    b3=cos(phi1);
    y1=sqrt(39.2*cos(phi1/2.0)*cos(phi1/2.0)-1);
    w1=tan(phi1/2.0)*(cos(theta1)+sin(theta1)*y1);
    w2=tan(phi1/2.0)*(sin(theta1)-cos(theta1)*y1);
    w3=1.000000;

 /* calculation  of exact solution */
ephi=2*asin((2*k1*sqrt((G-1)/G)*exp(-(t1/4)*sqrt(G-1)))
      /(1+k1*k1*exp(-(t1/2)*sqrt(G-1))));
u1=atan((sqrt(G)*exp((t1/4)*sqrt(G-1)))/k1-sqrt(G-1));
u2=atan((sqrt(G)*exp((t1/4)*sqrt(G-1)))/k1+sqrt(G-1));
etheta=(t1/4)+u1-u2+k2;
k=-(tan(etheta/2)*tan(etheta/2))
   *sqrt(G*cos(etheta/2)*cos(etheta/2)-1);

 /* calculation of error in theta and phi*/
    er_theta=fabs(theta1-etheta);
    er_phi=fabs(phi1-ephi);

 /*   calculation of B and W */
     be1=cos(etheta)*sin(ephi);
     be2=sin(etheta)*sin(ephi);
     be3=cos(ephi);
     y1=sqrt(39.2*cos(ephi/2.0)*cos(ephi/2.0)-1);
     we1=tan(ephi/2.0)*(cos(etheta)+sin(etheta)*y1);
     we2=tan(ephi/2.0)*(sin(etheta)-cos(etheta)*y1);
fprintf(fp,"\n%lf\t%lf\t%lf\t%lf\t%lf\t%lf\t%lf
            \t%lf\t%lf\t%lf\t%lf\t%lf",t1,b1,b2,b3,w1,w2,
            w3, theta1, k, etheta, ephi);
 printf("\n\n\t%.20lf\t%.20lf\t%lf",er_theta,er_phi,k);
     phi0=phi1;
     theta0=theta1;
     t0=t1;
      getch();
        } }
 double f(double p)
        { double p_dash;
          p_dash=(1.0/cos(p/2.0))*(1.0/cos(p/2.0));
          return(p_dash);  }
```

4 Experimental Results

We have written the code for the above algorithm in C-programming. We have plotted the graphs by using Matlab. Here we have considered the initial solution as $\phi_0 = 0.100$ and $\theta_0 = 0.000$ for $k > 0$. Since at $\phi = 2.820649$ the value of k becomes negative, we have considered $\phi_0 = 2.820649$ and $\theta_0 = 0.000$ for $k > 0$.

In each figure, the first graph shows the numerical value calculated by us, the second graph shows the exact solution and the third graph shows the comparison of the first and the second graphs as shown in Figure 1 to Figure 16.

4.1 Figures for numerical solution on unstable manifold

Here we consider $k > 0$. Here are Figures from 1 to 8.

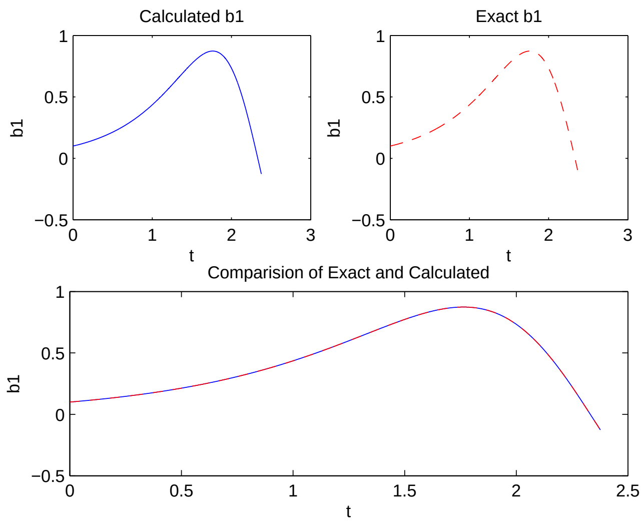

Figure 1: Graphs for b_1.

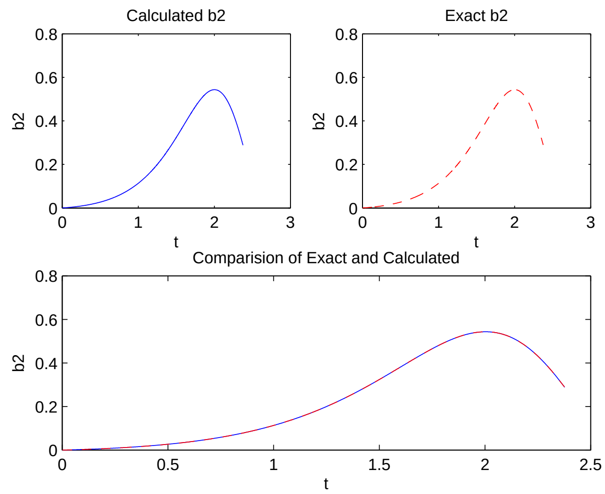

Figure 2: Graphs for b_2.

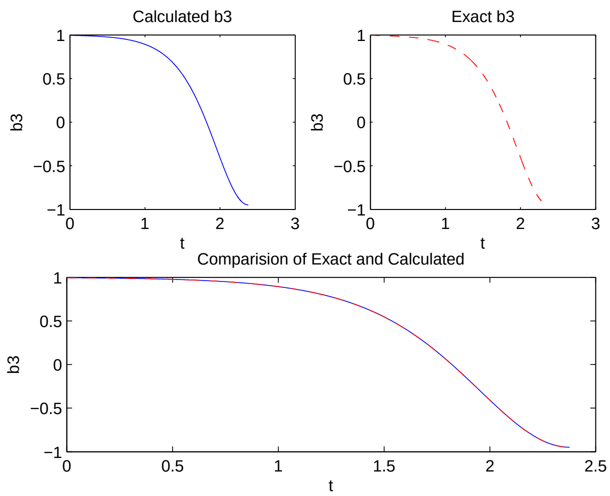

Figure 3: Graphs for b_3.

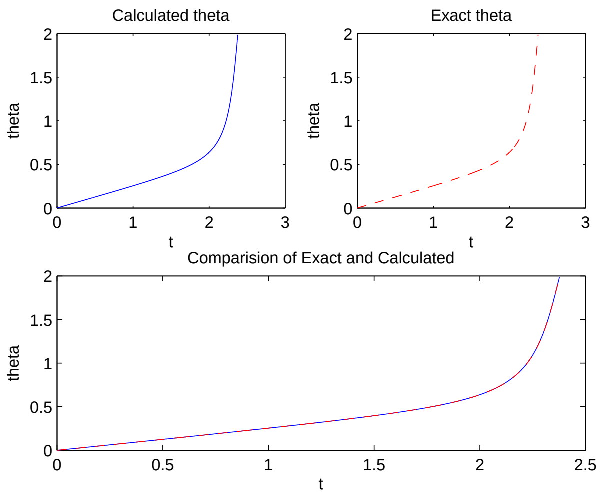

Figure 4: Graphs for θ.

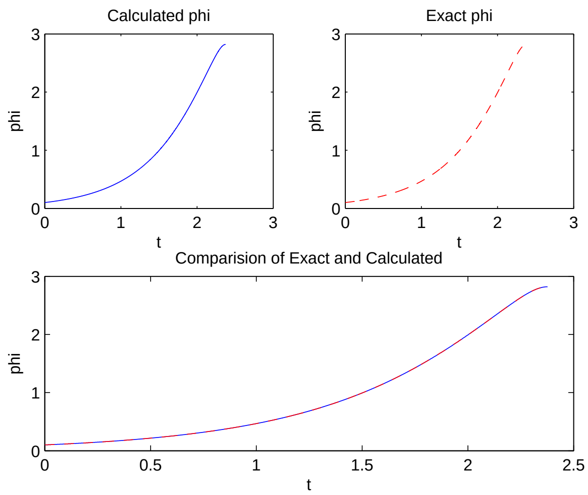

Figure 5: Graphs for ϕ.

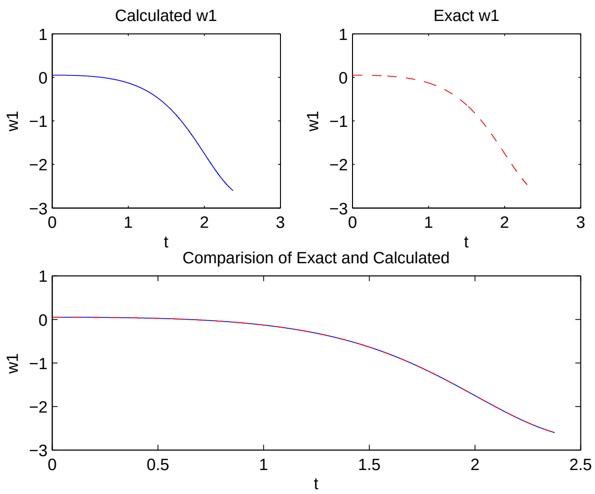

Figure 6: Graphs for w_1.

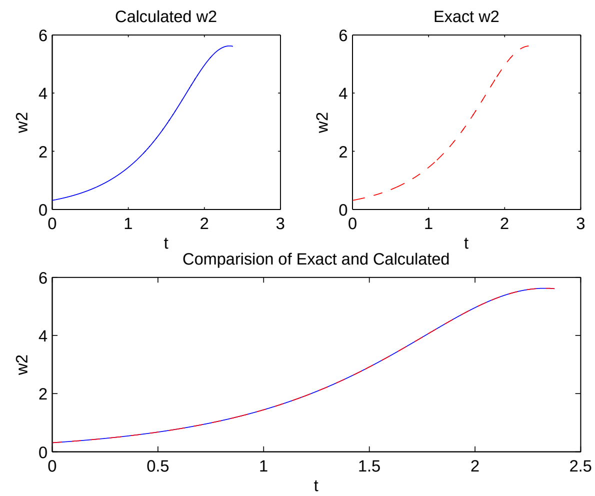

Figure 7: Graphs for w_2.

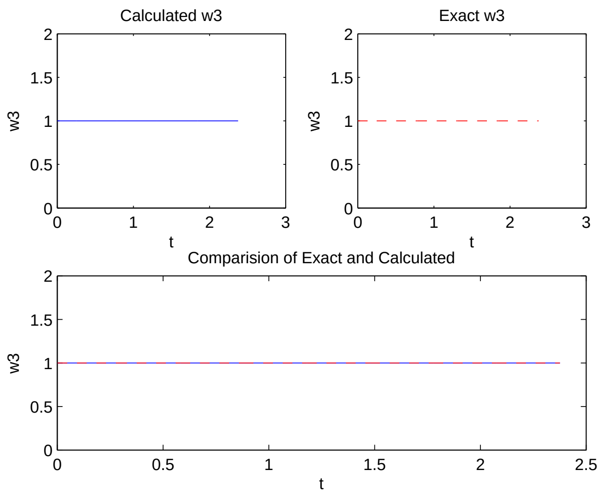

Figure 8: Graphs for w_3.

4.2 Numerical solution on stable manifold

Here we consider $k < 0$. Here are Figures from 9 to 16.

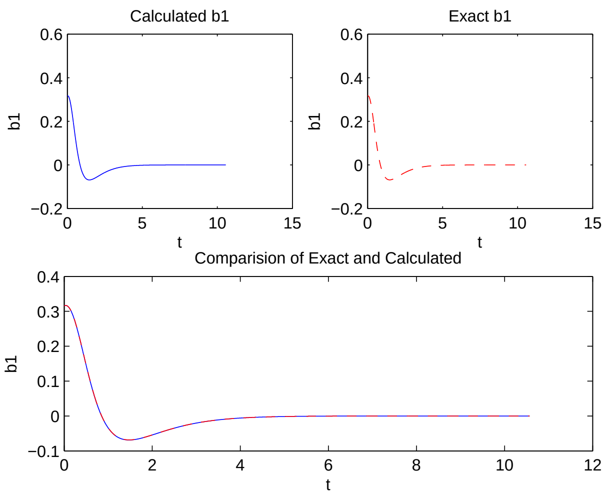

Figure 9: Graphs for b_1.

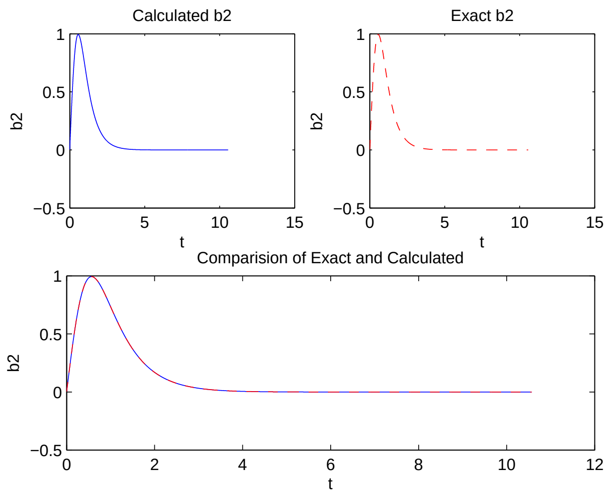

Figure 10: Graphs for b_2.

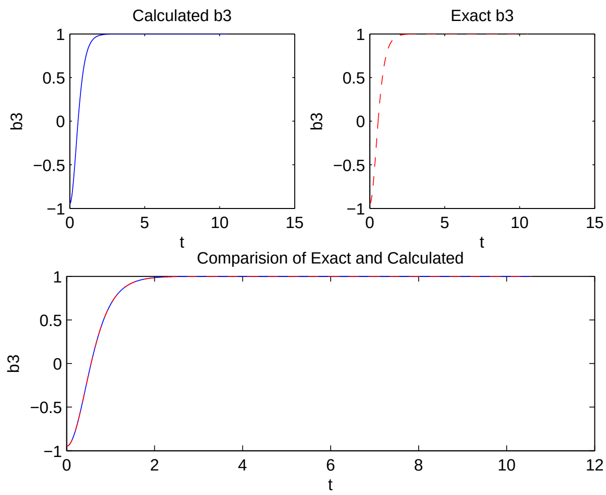

Figure 11: Graphs for b_3.

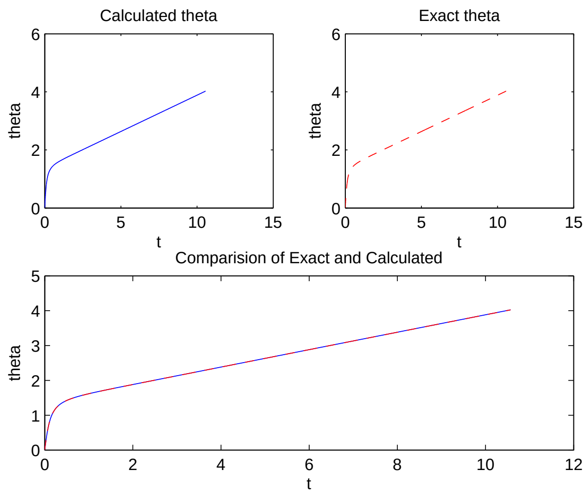

Figure 12: Graphs for θ.

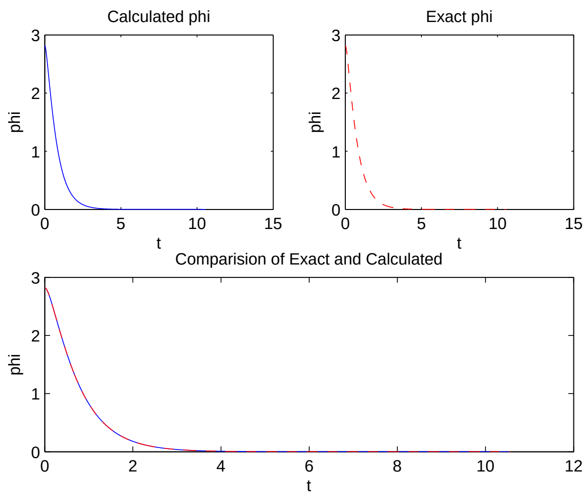

Figure 13: Graphs for ϕ.

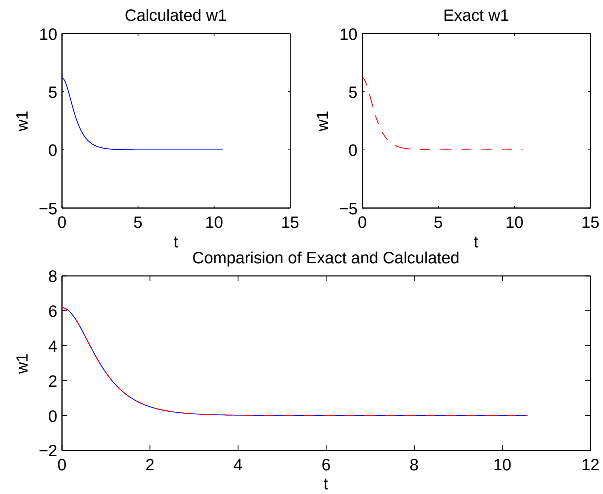

Figure 14: Graphs for w_1.

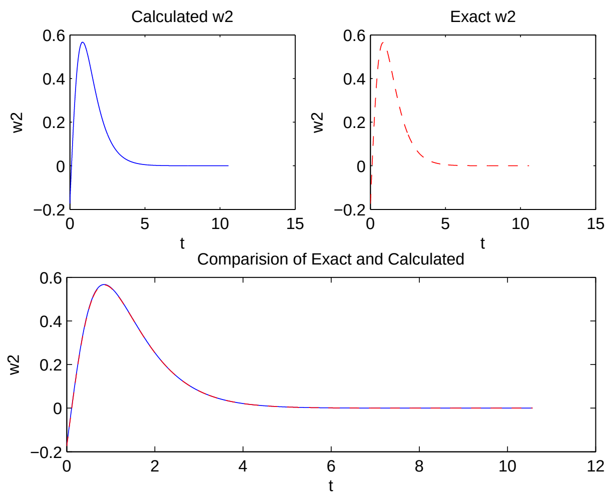

Figure 15: Graphs for w_2.

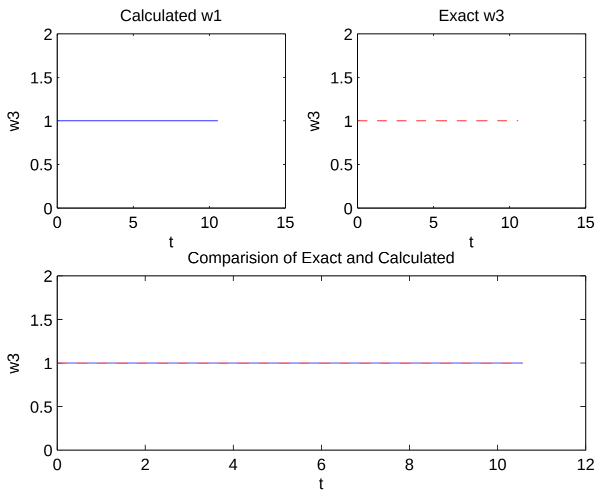

Figure 16: Graphs for w_3.

5 Conclusion

Here we have presented the scheme of Euler Modified Method for the numerical solution of the system of non-linear six coupled ODE's (1), with the error of 10^{-6}. Initially we have an error of 10^{-20} in the solution. It can be reduced as we reduce the step size. This error increases but it is up to 10^{-6} which is the upper bound. In future we will attempt to minimize the error and sharpen the accuracy of the solution.

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Euler Solutions for Integro Differential Equations with Retardation and Anticipation

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Abstract: In this paper, we obtain results for Euler solution for integro differential equation with retardation and anticipation.

Keywords: *integro differential equations with retardation and anticipation; Euler solutions.*

Mathematics Subject Classification (2010): 45J99, 47G20.

1 Introduction

Integro differential equations arise quite frequently as mathematical models in diverse disciplines. The study of integro differential equations has been attracting the attention of many scientific researchers due to its potential as a better model to represent physical phenomena in various disciplines. Much work has been done in the existence and uniqueness of solutions for integro differential equations see [2, 3, 6, 7, 8, 12]. All these results are abstract in the sense that there is no specific procedure to obtain a solution of the considered equations, so the Euler solutions for integro differential equations are studied [4].

In many physical phenomena the both past history and future play an important role along with the present state and hence an appropriate model of the phenomena will be one that involves past history and future expectation also. This led to the study of systems involving both retardation and anticipation, for example, see [1]. The existence of Euler solutions have been studied for set differential equations [11], for causal differential equations [10], for delay differential equations [5], due to the inherited simplicity in its idea which paves a path for obtaining a solution of the given system. In this paper,

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we give an approach to obtaining the solution of the integro differential equation with retardation and anticipation under continuity conditions.

In this paper we consider the integro differential equations with retardation and anticipation of the type

$$x' = f(t, x, Sx, x_t, x^t), \quad t \in I = [t_0, T], \tag{1}$$

$$x_{t_0}(0) = \phi_0(0), \quad x^T(0) = \psi_0(0), \tag{2}$$

where the retardation function x_t is defined as $x_t \in C_0 = C[[-h_1, 0], \ \mathbb{R}]$ such that $x_t(s) = x(t+s)$, $s \in [-h_1, 0]$ and the anticipation is defined as $x^t \in C_1 = C[[0, h_2], \ \mathbb{R}]$ such that $x^t(\sigma) = x(t+\sigma)$ where $\sigma \in [0, \ h_2]$ and construct Euler solution for the fore mentioned integro differential equation with retardation and anticipation.

2 Preliminaries

In this section we begin with the integro differential equation given by

$$x' = f(t, x) + \int_{t_0}^{t} K(t, s, x(s))ds, \tag{3}$$

$$x(t_0) = x_0. \tag{4}$$

We begin with the following known results corresponding to integro differential equations which are prerequisite to obtain the Euler solutions for integro differential equations with retardation and anticipation. These results are from [9].

Theorem 2.1 *Assume that*
A(1) $g \in C[\mathbb{R}_+ \times \mathbb{R}, \ \mathbb{R}]$, $H \in C[\mathbb{R}_+^2 \times \mathbb{R}, \ \mathbb{R}]$ *and* $H(t, s, u)$ *is monotone non decreasing in* u *for each* $(t, s) \in \mathbb{R}_+^2$;
A(2) $v' \leq g(t, v) + \int_{t_0}^{t} H(t, s, v(s))ds$ *and* $w' \geq g(t, w) + \int_{t_0}^{t} H(t, s, w(s))ds$;
A(3) for $(t, s) \in \mathbb{R}_+^2$, $x \geq y$ *and* $L \geq 0$,

$$g(t, x) - g(t, y) \leq L(x - y), \quad H(t, s, x) - H(t, s, y) \leq L^2(x - y).$$

Then we have $v(t) \leq w(t)$, *for* $t \geq t_0$, *provided* $v(t_0) \leq w(t_0)$.

Next we state the following result which gives existence of extremal solutions.

Theorem 2.2 *Assume that* $g \in C[[t_0, \ t_0 + a] \times \mathbb{R}, \ \mathbb{R}]$, $H \in C[[t_0, \ t_0 + a] \times [t_0, \ t_0 + a] \times \mathbb{R}, \ \mathbb{R}]$, $H(t, s, u)$ *is non decreasing in* u *for each* (t, s) *and* $\int_t^s |H(\sigma, s, u(s))| \ d\sigma \leq N$ *for* $t_0 \leq s \leq t \leq t_0 + a$, $u \in \Omega^0 = \{u \in C[[t_0, \ t_0 + a], \ \mathbb{R}] : \ |u(t) - u_0| \leq b\}$. *Then there exists a maximal and minimal solutions for the scalar IVP*

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s)) \ ds, \tag{5}$$

$$u(t_0) = u_0. \tag{6}$$

on $[t_0, t_0 + \alpha]$, *for some* $0 < \alpha < a$.

We now give the comparison theorem, which is used in the proof of our main result.

Theorem 2.3 *Assume that* $g \in C[\mathbb{R}_+^2,\ \mathbb{R}]$, $H \in C[\mathbb{R}_+^3,\ \mathbb{R}]$, $H(t,s,u)$ *is non decreasing in* u *for each* (t,s) *and for* $t \geq t_0$, $D_-m(t) \leq g(t,m(t)) + \int_{t_0}^t H(t,s,m(s))ds$, *where* $m \in C[\mathbb{R}_+,\ \mathbb{R}]$ *and* $D_-m(t) = lim_{h\to 0^-} inf[\ \frac{m(t+h)-m(t)}{h}]$. *Suppose that* $\gamma(t)$ *is the maximal solution of* $u' = g(t,u(t)) + \int_{t_0}^t H(t,s,u(s))ds$, $u(t_0) = u_0 \geq 0$, *existing on* $[t_0,\infty)$. *Then* $m(t) \leq \gamma(t)$, *for* $t \geq t_0$, *provided* $m(t_0) \leq u_0$.

Before we proceed further, we state the following known result relating to integro differential equations, which is indirectly used in our work.

Theorem 2.4 *Let* E_1 *be an open* (t,u)*-set in* $\mathbb{R}^{n+1}$ *and let* $f \in C[E_1,\mathbb{R}^n]$, $K \in C[E_1 \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ *and* $x(t)$ *be a solution of* (3) *and* (4)*on some interval* $t_0 \leq t \leq a_0$. *Then* $x(t)$ *can be extended as a solution to the boundary of* E_1.

We now present a theorem relating to the largest interval of existence of maximal solutions in a particular setup.

Theorem 2.5 *Let the hypothesis of Theorem* 2.2 *hold. Suppose that the largest interval of existence of the maximal solution* $r(t)$ *of* (5) *and* (6) *is* $[t_0,\ t_0+a)$. *Then there is an* $\epsilon_0 > 0$ *such that* $0 < \epsilon < \epsilon_0$, *the maximal solution* $r(t,\epsilon)$ *of*

$$u' = g(t,u) + \int_{t_0}^t H(t,s,u(s))\ ds + \epsilon, \tag{7}$$

$$u(t_0) = u_0 + \epsilon \geq 0, \tag{8}$$

exists over $[t_0,t_1] \subset [t_0,t_0+a)$ *and* $\lim_{\epsilon\to 0} r(t,\epsilon) = r(t)$ *uniformly on* $[t_0,t_1]$.

3 Comparison Theorems

In order to construct the Euler solutions for the integro differential equation with retardation and anticipation. We need the following comparison theorems. We begin with the following result which deals with the existence of maximal solution in our setup, which is required for our main result.

Theorem 3.1 *Let* E *be the product space* $[t_0,t_0+a)\times\mathbb{R}^2$ *and* $g \in C[E,\ \mathbb{R}]$, $H \in C[[t_0,t_0+a)\times[t_0,t_0+a)\times\mathbb{R},\ \mathbb{R}]$. *Assume that* $g(t,u,v)$ *is non decreasing in* v *for each* (t,u), *and* $H(t,s,u)$ *is non decreasing in* u *for each* (t,s). *Suppose that* $r(t)$ *is the maximal solution of the integro differential equation*

$$u' = g(t,u,u) + \int_{t_0}^t H(t,s,u(s))ds, \tag{9}$$

$$u(t_0) = u_0 \geq 0, \tag{10}$$

existing on $[t_0,t_0+a)$ *and*

$$r(t) \geq 0, \tag{11}$$

on $[t_0,t_0+a)$. *Then the maximal solution* $r_1(t)$ *of*

$$u' = g_1(t,u) + \int_{t_0}^t H(t,s,u(s))ds, \tag{12}$$

$$u(t_0) = u_0 \geq 0, \tag{13}$$

where $g_1(t,u) = g(t,u,r(t))$ *exists on* $[t_0,t_0+a)$ *and* $r_1(t) = r(t)$ *for* $t \in [t_0,t_0+a)$, $\int_s^t |H(\sigma,s,u(s))|d\sigma \leq N$ *for* $t_0 \leq s \leq t \leq t_0+a$.

Proof. Consider the scalar integro differential equation (12) and (13). By Theorem 2.2 there exists a maximal solution $r_1(t)$ of (12) and (13) in the interval $[t_0, t_0+\alpha)$, where $0 < \alpha < a$ and by Theorem 2.4 this maximal solution can be extended from $[t_0, t_0+\alpha)$ to $[t_0, t_0+a)$. This implies that either $r_1(t)$ is defined over $[t_0, t_0+a)$ or there exists a $t_1 < t_0 + a$ such that

$$|r_1(t_k)| \to \infty, \tag{14}$$

for a certain sequence $\{t_k\}$, such that $t_k \to t_1^-$ as $k \to \infty$. Observe that

$$r'(t) = g(t, r(t), r(t)) + \int_{t_0}^{t} H(t, s, r(s))ds = g_1(t, r(t)) + \int_{t_0}^{t} H(t, s, r(s))ds,$$

and Theorem 2.3 yields that

$$r(t) \le r_1(t), \tag{15}$$

as far as $r_1(t)$ exists. Now using the relations (11), (14) and (15), we have

$$|r_1(t_k)| \to +\infty \tag{16}$$

for some sequence $\{t_k\}$, such that $t_k \to t_1^-$ as $k \to \infty$. We shall prove that (16) does not hold. Since the largest interval of existence of maximal solution $r(t)$ of the scalar integro differential equaiton (9) and (10) is $[t_0, t_0+a)$, so by Theorem 2.5 there is an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ and the maximal solution $r(t, \epsilon)$ of

$$u' = g(t, u, u) + \int_{t_0}^{t} H(t, s, u(s))\ ds + \epsilon, \tag{17}$$

$$u(t_0) = u_0 + \epsilon \ge 0, \tag{18}$$

exists over $[t_0,\ t_1+\nu] \subset [t_0,\ t_0+a),\ \ \nu > 0,\ \ t_1+\nu < t_0+a$. From the relations (17), (18) we get

$$r'(t, \epsilon) > g(t, r(t, \epsilon), r(t, \epsilon)) + \int_{t_0}^{t} H(t, s, r(s, \epsilon))ds$$

and $r(t_0) = u_0 < u_0 + \epsilon = r(t_0, \epsilon)$. So

$$r(t_0) < r(t_0, \epsilon).$$

Now applying Theorem 2.1 we conclude that

$$r(t) < r(t, \epsilon), \tag{19}$$

for $t \in [t_0, t_1+\nu]$. Since g is non decreasing in v, we arrive at $r'(t, \epsilon) > g_1(t, r(t, \epsilon)) + \int_{t_0}^{t} H(t, s, r(s, \epsilon))ds$, for $t \in [t_0, t_1+\nu]$. But

$$r_1'(t) = g_1(t, r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds,$$

for $t \in [t_0, t_1]$ and $r_1(t_0) = u_0 < u_0 + \epsilon = r(t_0, \epsilon)$, so

$$r_1(t) < r(t, \epsilon),$$

for $t \in [t_0, t_1]$. Since $r(t, \epsilon)$ exists on $[t_0, t_1 + \nu]$, $\nu > 0$. This leads to a contradiction to (16). Hence $r_1(t)$ exists on $[t_0, t_0 + a)$. Thus $r(t) \leq r_1(t)$ for $t \in [t_0, t_0 + a)$. Furthermore,

$$\begin{aligned} r_1'(t) &= g_1(t, r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds \\ &= g(t, r_1(t), r(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds. \end{aligned}$$

From the monotonic character of g in v, and from the relation (15), we get

$$\begin{aligned} r_1'(t) &= g(t, r_1(t), r(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds \\ &\leq g(t, r_1(t), r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds. \end{aligned}$$

Now using Theorem 2.3, we find that

$$r_1(t) \leq r(t) \tag{20}$$

on $t \in [t_0, t_0 + a)$, which implies along with the relation (15) that $r_1(t) = r(t)$ for $t \in [t_0, t_0 + a)$.

We need the following known result in suitable form.

Theorem 3.2 *Let the hypothesis of Theorem* 3.1 *hold and* $m \in C[[t_0,\ t_0 + a),\ \mathbb{R}]$ *such that* $(t, m(t), \nu) \in E$, $t \in [t_0,\ t_0 + a)$ *and* $m(t_0) \leq u_0$. *Assume that for a fixed Dini Derivative the inequality* $Dm(t) \leq g(t, m(t), \nu) + \int_{t_0}^{t} H(t, s, m(s))ds$, *is satisfied for* $t \in [t_0,\ t_0 + a) - S$, *where* S *denotes an at most countable subset of* $[t_0, t_0 + a)$. *Then for all* $\nu \leq r(t)$, $t \in [t_0, t_0 + a)$, *we have* $m(t) \leq r(t)$, *for* $t \in [t_0, t_0 + a)$.

Proof. Since the hypothesis of Theorem 3.1 holds, so there exists a maximal solution $r_1(t)$ of the scalar integro differential equation (12) and (13) with $g_1(t, u) = g(t, u, r(t))$ exists on $[t_0, t_0 + a)$ and $r(t) = r_1(t)$ for $t \in [t_0, t_0 + a)$. Let $\nu \leq r(t)$, $t \in [t_0, t_0 + a)$. Then using the monotonicity of g in ν we get

$$\begin{aligned} Dm(t) &\leq g(t, m(t), \nu) + \int_{t_0}^{t} H(t, s, m(s))ds \\ &\leq g(t, m(t), r(t)) + \int_{t_0}^{t} H(t, s, m(s))ds \\ Dm(t) &\leq g_1(t, m(t)) + \int_{t_0}^{t} H(t, s, m(s))ds, \end{aligned}$$

for $t \in [t_0, t_0 + a) - S$, which on using Theorem 2.3 gives $m(t) \leq r(t)$, for $t \in [t_0, t_0 + a)$.

The following theorem is needed before we proceed further.

Theorem 3.3 *Assume that* $m \in C[I,\ \mathbb{R}_+]$, $g \in C[I \times \mathbb{R}_+,\ \mathbb{R}_+]$, $H \in C[I \times I \times \mathbb{R}_+,\ \mathbb{R}_+]$, H *is non decreasing in* u *for each* (t, s) *and for* $t \in I = [t_0, T]$,

$$D_-m(t) \leq g(t, |m|_0(t)) + \int_{t_0}^{t} H(t, s, |m|(s))ds, \tag{21}$$

where $|m|_0(t) = sup_{t_0 \leq s \leq t}|m(s)|$. *Suppose that* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s))ds, \tag{22}$$

$$u(t_0) = u_0 \geq 0, \tag{23}$$

existing on $[t_0, T)$. *Then* $m(t) \leq r(t), \quad t \geq t_0$, *provided* $| \, m(t_0) \, |_0 \leq u_0$.

Proof. Since the largest interval of existence of maximal solution is $[t_0, T)$ for the integro differential equation (22) so there exists an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, t_0, u_0, \epsilon)$ of

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s))ds + \epsilon, \tag{24}$$

$$u(t_0) = u_0 + \epsilon \geq 0, \tag{25}$$

existing on $[t_0, \; t_1] \subset [t_0, \; T)$, for $t_1 < T$ and $lim_{\epsilon \to 0} r(t, t_0, u_0, \epsilon) = r(t, t_0, u_0)$ uniformly on $[t_0, t_1]$. To prove the conclusion of the theorem, it is sufficient to show that

$$m(t) < r(t, t_0, u_0, \epsilon), \tag{26}$$

for $t_0 \leq t \in I$. Suppose that the relation (26) does not hold then there exists $t_\alpha > t_0$ such that $m(t_\alpha) = r(t_\alpha, t_0, u_0, \epsilon) \;\; and \;\; m(t) < r(t, t_0, u_0, \epsilon)$ for $t_0 \leq t < t_\alpha$. this yields on computation,

$$D_{-}m(t_\alpha) > g(t_\alpha, r(t_\alpha, t_0, u_0, \epsilon)) + \int_{t_0}^{t} H(t_\alpha, s, r(t_\alpha, t_0, u_0, \epsilon))ds \tag{27}$$

which is contradiction. Observe that we have used the fact that $g(t, u) \geq 0, \;\; H(t, s, u) \geq 0$ implies that $r(t_\alpha, t_0, u_0, \epsilon)$ is non decreasing in t and

$$|m|_0(t_\alpha) = sup_{t_0 \leq s \leq t_\alpha}|m(s)| = r(t_\alpha, t_0, u_0, \epsilon) = m(t_\alpha),$$

which yields

$$\begin{aligned} D_{-}m(t_\alpha) &\leq g(t_\alpha, |m|_0(t_\alpha)) + \int_{t_0}^{t_\alpha} H(t_\alpha, s, |m|_0(s))ds, \\ &= g(t_\alpha, r(t_\alpha, t_0, u_0, \epsilon)) + \int_{t_0}^{t} H(t_\alpha, s, r(t_\alpha, t_0, u_0, \epsilon))ds \end{aligned}$$

which is contradiction to (27), and the proof is complete.

4 Euler Solutions

In this section we define an Euler solution and prove a result for its existence of integro differential equation with retardation and anticipation. Further we give a result which gives conditions under which the Euler solution becomes a solution of the IVP of the integro differential equation with retardation and anticipation.

Consider the integro differential equation with retardation and anticipation:

$$x' = f(t, x, Sx, x_t, x^t), \tag{28}$$

$$x_{t_0}(0) = \phi_0(0), \quad x^T(0) = \psi_0(0), \tag{29}$$

where $t \in I = [t_0,\ T]$, $\phi_0 \in C_0$, $\psi_0 \in C_1$, $f \in C[I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1,\ \mathbb{R}]$, $Sx(t) = \int_{t_0}^{t} K(t, s, x)ds$, $K(t, s, x) \in C[I^2 \times \mathbb{R},\ \mathbb{R}_+]$ and $C_0 = C[[-h_1, 0],\ \mathbb{R}]$, $C_1 = C[[0, h_2],\ \mathbb{R}]$.

In order to construct the Euler Solution we consider a partition π of the interval I and on each subinterval of the partition, we obtain a differential equation where the right hand side is a constant. This will help us to define Euler solution as a limit of a sequence of polygonal arcs.

In order to do so we have to find a reasonable estimate of x^t in the right hand side of the differential equation (28). For this we take the anticipation as

$$z(t) = \begin{cases} x^t(0), \ \textit{wherever} \quad \mid \xi^t(0) - \phi_0(0) \mid < M, \\ x^t(0) + \dfrac{\xi(t)}{j}, \end{cases} \tag{30}$$

where j is the number of points in the partition π and

$$\xi(t) = \begin{cases} \phi_0(0), \quad t \in [t_0 - h_1,\ t_0], \\ \phi_0(0) + \dfrac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t - t_0), \quad t \in [t_0,\ T], \\ \psi_0(0), \quad t \in [T,\ T + h_2]. \end{cases} \tag{31}$$

With this approximation the integro differential equation with retardation and anticipation reduces to the integro differential equation with retardation only, i.e.,

$$x' = f(t, x, Sx, x_t, z(t)), \tag{32}$$

$$x_{t_0}(0) = \phi_0(0), \quad z(T) = \psi_0(0), \tag{33}$$

for $t \in I = [t_0, T]$. Let partition of the interval $[t_0, T]$ be given by

$$\pi = \{t_0, t_1, t_2, ..., t_N = T\}. \tag{34}$$

Consider the sub interval $[t_0, t_1]$ and the differential equation (32), in that subinterval. In the right hand side of (32) replace t by t_0, x by x_0, x_t by $\phi_0(0)$, $z(t)$ by $z(t_0)$ and Sx by $(Sx(t_0),\ t_0)$ ie., in the integral replace t with t_0, s with t_0, x with x_0, so (32) reduces to

$$x' = f(t_0, x_0, (Sx(t_0), t_0), \phi_0(0), z(t_0)). \tag{35}$$

Then the right hand side of the differential equation (35) is a constant and hence (35) posses a unique solution $x(t) = x(t, t_0, \phi_0(0))$ on $[t_0, t_1]$.

Set $x_1 = x(t_1) = x(t_1, t_0, \phi_0(0))$. We now choose the next subinterval $[t_1, t_2]$ and consider the differential equation (32) by setting $t = t_1$, $x = x_1$, $x_t = \phi_1(t_1)$, $z(t) = z(t_1)$ and $Sx = (Sx(t_1),\ t_1)$, i.e., in the integral replace t with t_1, s with t_1, x with x_1. Then the system (32) reduces to

$$x' = f(t_1, x_1, (Sx(t_1), t_1), \phi_1(t_1), z(t_1)), \tag{36}$$

where

$$\phi_1(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ x(t, t_0, \phi_0(0)), & t \in [t_0, t_1], \end{cases} \tag{37}$$

$$z(t) = \begin{cases} x^{t_1}(0), & \mid \xi^{t_1}(0) - \phi_0(0) \mid < M, \\ x^{t_1}(0) + \dfrac{\xi(t_1)}{N+1}, \end{cases}$$

$$\xi(t_1) = \phi_0(0) + \frac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t_1 - t_0). \tag{38}$$

Clearly the right hand side of (36) is a constant hence there exists a unique solution $x(t) = x(t, t_1, \phi_1(t_1))$ on $[t_1, t_2]$.

Set $x_2 = x(t_2) = x(t_2, t_1, \phi_1(t_1))$. Again consider the integro differential equation with retardation (32) on $[t_2, t_3]$ and as earlier replacing t by t_2, x by x_2, x_t by $\phi_2(t_2)$, $z(t) = z(t_2)$ and Sx by $(Sx(t_2),\ t_2)$, i.e., in the integral replace t by t_2, s by t_2, x by x_2. Then the system (32) reduces to

$$x' \ = \ f(t_2, x_2, (Sx(t_2), t_2), \phi_2(t_2), z(t_2)), \tag{39}$$

where

$$\phi_2(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ \phi_1(t), & t \in [t_0, t_1], \\ x(t, t_1, \phi_1(t_1)), & t \in [t_1, t_2], \end{cases} \tag{40}$$

$$z(t) = \begin{cases} x^{t_2}(0), & \mid \xi^{t_2}(0) - \phi_0(0) \mid < M, \\ x^{t_2}(0) + \dfrac{\xi(t_2)}{N+1}, \end{cases}$$

$$\xi(t_2) = \phi_0(0) + \frac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t_2 - t_0). \tag{41}$$

We observe that the right hand side of (39) is a constant and proceeding as earlier we get a solution $x(t, t_2, \phi_2(t_2))$ in the interval $[t_2, t_3]$. Set $x_3 = x(t_3) = x(t_3, t_2, \phi_2(t_2))$.

Now proceeding in this fashion, we construct a sequence of arcs $x(t, t_0, \phi_0(0))$, $x(t, t_1, \phi_1(t_1)), ..., x(t, t_{N-1}, \phi_{N-1}(t_{N-1}))$ on the sub intervals $[t_0, t_1]$, $[t_1, t_2]$, $..., [t_{N-1}, t_N]$ respectively, which is the Euler polygonal arcs defined on the partition $\pi = \{t_0, t_1, t_2, ..., t_N = T\}$. Thus the entire arc on I is defined by

$$x_\pi = x_\pi(t) = \{x(t, t_i, \phi_i(t_i)) : t_i \le t \le t_{i+1}, \ \ i = 0, 1, 2, ..., N-1\}, \tag{42}$$

where

$$\phi_i(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ \phi_1(t), & t \in [t_0, t_1], \\ \vdots \\ x(t, t_{i-1}, \phi_{i-1}(t_{i-1})), & t \in [t_{i-1}, t_i]. \end{cases} \tag{43}$$

In (42) the notation emphasizes the fact that the arc corresponds to the partition π. The diameter μ_π of the partition π is given by

$$\mu_\pi = \max\{t_i - t_{i-1} : 1 \le i \le N\}. \tag{44}$$

Definition 4.1 An Euler solution for the integro differential equation with retardation and anticipation (28), (29) is any arc $x = x(t)$ which is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding to some sequence π_j such that $\pi_j \to 0$, as the diameter $\mu_{\pi_j} \to 0$, as $j \to \infty$.

Remark 4.1 Observe that the number of points N_j of the partition π_j must tend to ∞ as $\pi_j \to 0$ and also that the Euler arc satisfies the conditions $x_{t_0}(0) = \phi_0(0)$, $x^T(0) = \psi_0(0)$.

We now state a result which guarantees the existence of an Euler solution.

Theorem 4.1 *Assume that*

$$\mid f(t, x, Sx, x_t, z^t) \mid \leq g(t, \mid x \mid_0 (t), \mid z(t) \mid) + \int_{t_0}^{t} H(t, s, \mid x(s) \mid) ds, \tag{45}$$

where $f : I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1 \to \mathbb{R},\ K : I^2 \times \mathbb{R} \to \mathbb{R}_+,\ g \in C[I \times \mathbb{R}_+ \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (u, v), *is non decreasing in* u *for each* (t, v), *is non decreasing in* v *for each* (t, u) ,$H \in C[I^2 \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (s, u), *is non decreasing in* s *for each* (t, u), *is non decreasing in* u *for each* (t, s), $\mid x \mid_0 (t) = max_{t-h_1 \leq t+s \leq t} \mid x(t + s) \mid$ *and* $r(t, t_0, u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t, u, u) + \int_{t_0}^{t} H(t, s, u) ds, \tag{46}$$

$$u(t_0) = u_0, \quad u(T) = \psi_0(0), \tag{47}$$

existing on $[t_0, T]$ *and* $|z(t)| \leq r(t)$, *and* z^t *is the reasonable estimate of* x^t. *Then,*
(a) there exists at least one Euler solution $x(t) = x(t, t_0, \phi_0(0))$ *of the IVP* (28), (29) *which satisfies the Lipschitz condition;*
(b) any Euler solution $x(t)$ *of* (28), (29) *satisfies the relation*

$$\mid x(t) - \phi_0(0) \mid \leq r(t, t_0, u_0) - u_0, \quad t \in [t_0, T], \tag{48}$$

where $u_0 = \mid \phi_0 \mid$.

Proof. Let π be the partition of $[t_0, T]$ defined by (34) and let $x_\pi = x_\pi(t)$ denote the corresponding arc with nodes of x_π represented by $x_1, x_2, x_3, ..., x_N$. Writing $x_\pi(t) = x_i(t) = \ x(t, t_i, \phi_i(t_i)), \quad t_i \leq t \leq t_{i+1},\ i = 0, 1, 2, ..., N - 1$, where $\phi_i(t_i)$ is given by (43) and observe that $x_i(t_i) = x_i, \quad i = 0, 1, 2, ..., N - 1$. Further for any $t \in [t_i, t_{i+1}]$, we have from the definition of Euler solution

$$\begin{aligned} \mid x'_\pi(t) \mid &= \mid f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) \mid \\ &\leq g(t_i, \mid x_{t_i}(0) \mid, \mid z(t_i) \mid) + \int_{t_0}^{t_i} H(t_i, t_i, \mid x(t_i) \mid) ds \end{aligned}$$

thus

$$\mid x'_\pi(t) \mid \leq g(t_i, \mid x_{t_i}(0) \mid, \mid z(t_i) \mid) + \int_{t_0}^{t_i} H(t_i, t_i, \mid x(t_i) \mid) ds, \qquad i = 0, 1, 2, ..., N - 1. \tag{49}$$

Consider the interval $[t_0, t_1]$ and applying the properties of norm, integral and the non decreasing nature of g and H, along with the fact that both g and H are non-negative, we get

$$\begin{aligned}
| x_1(t) - \phi_0(0) | = & \ | \phi_0(0) + \int_{t_0}^{t} f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0))ds - \phi_0(0) | \\
& \leq \int_{t_0}^{t} | f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0)) | ds \\
& \leq \int_{t_0}^{t} [g(s, r(s), r(s)) + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
& \leq r(T, t_0, | \phi_0 |) - | \phi_0 | = \psi_0(0) - \phi_0(0) = M \ (say).
\end{aligned}$$

Next consider the interval $[t_1, t_2]$ again as before, using the properties of norm and integral, the monotone character of g and H and the fact that both g and H are non negative, we obtain,

$$\begin{aligned}
| x_2(t) - \phi_0(0) | = & | x_1(t_1) + \int_{t_1}^{t} f(t_1, x_1, Sx_1, x_{t_1}(0), z(t_1))ds - \phi_0(0) | \\
& \leq \int_{t_0}^{t_1} | f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0)) | ds \\
& + \int_{t_1}^{t} | f(t_1, x_1, Sx_1, x_{t_1}(0), z(t_1)) | ds \\
& = \int_{t_0}^{t} [g(s, r(s), r(s))ds + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
& \leq r(T, t_0, | \phi_0 |) - | \phi_0 | = \psi_0(0) - \phi_0(0) = M \ (say).
\end{aligned}$$

Proceeding in this manner, on each subinterval $[t_i, t_{i+1}]$, we arrive at

$$| x_i(t) - \phi_0(0) | \leq r(T, t_0, | \phi_0 |) - | \phi_0 | = M.$$

Thus combining the relations of all polygonal arcs over the partition π, we deduce that

$$| x_\pi(t) - \phi_0(0) | \leq r(T, t_0, | \phi_0 |) - | \phi_0 | = M, \tag{50}$$

on $[t_0, T]$. Now from the relation (49), we have

$$\begin{aligned}
| x'_\pi(t) | & \leq g(t_i, | x_{t_i}(0) |, | z(t_i) |) + \int_{t_0}^{t_i} H(t_i, t_i, | x(t_i) |)ds \\
& \leq g(T, r(T), r(T)) + \int_{t_0}^{t} H(t, s, r(s))ds \\
& = r'(T, t_0, | \phi_0 |) = L \ (say).
\end{aligned}$$

We next show that x_π is Lipschitz. For this consider $t_0 \leq l \leq t \leq T$, where $l \in [t_i, t_{i+1}]$

and $t \in [t_k, t_{k+1}], \quad i < k$. Then

$$\begin{aligned}
\mid x_\pi(t) - x_\pi(l) \mid =& \mid x_k + \int_{t_k}^{t} f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k))ds \\
& - \{x_i + \int_{t_i}^{l} f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i))ds\} \mid \\
& + ... + \int_{t_{k-1}}^{t_k} f(t_{k-1}, x_{k-1}, Sx_{k-1}, x_{t_{k-1}}(0), z(t_{k-1}))ds \\
& + \int_{t_k}^{t} f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k))ds \\
& - \{x_i + \int_{t_i}^{l} f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i))ds\} \mid \\
& \le \int_{t_i}^{t_{i+1}} \mid f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) \mid ds \\
& + ... + \int_{t_{k-1}}^{t_k} \mid f(t_{k-1}, x_{k-1}, Sx_{k-1}, x_{t_{k-1}}(0), z(t_{k-1})) \mid ds \\
& + \int_{t_k}^{t} \mid f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k)) \mid ds \\
& - \int_{t_i}^{l} \mid f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) \mid ds \\
& = \int_{l}^{t} [g(s, r(s), r(s)) + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
& = \int_{l}^{t} r'(s, t_0, u_0)ds \le L(t - l),
\end{aligned}$$

for some $\xi \in (l, t)$. This follows using the relations (45), (46), (47) along with the fact that $g(t, u, v),\ H(t, s, u),\ r(t)$ are positive and non decreasing. Thus x_π satisfies the Lipschitz condition with some constant L on $[t_0, T]$. Now let π_j be a sequence of partitions of $[t_0, T]$ such that $\pi_j \to 0$ as $j \to \infty$. Thus from the earlier construction, we get a sequence of polygonal arcs x_{π_j} on $[t_0, T]$ corresponding to each partition π_j satisfying

$$x_{\pi_j}(t_0) = \phi_0(0), \quad \mid x_{\pi_j}(t) - \phi_0(0) \mid \le M, \quad \mid x'_{\pi_j}(t) \mid \le L.$$

Hence the family $\{x_{\pi_j}\}$ is equicontinuous and uniformly bounded. Then the family $\{x_{\pi_j}\}$ satisfies the hypothesis of the Ascoli–Arzela Theorem and hence we obtain a subsequence which converges uniformly to a continuous function $x(t)$ on $[t_0, T]$ which is absolutely continuous on $[t_0, T]$. Now using the definition of the Euler solution, we conclude that $x(t)$ is an Euler solution for (28), (29) on $[t_0, T]$. To prove the relation in (b), it suffices to observe that $x(t)$ is the uniform limit of the polygonal arcs that satisfy the relation (48) and thus inherits the property. Thus the proof is complete.

Remark 4.2 If f and K are continuous and $K(t, s, x)$ is non decreasing in t for each (s, x), we can show that the Euler solution is a solution. This is the essence of the next result.

Theorem 4.2 *Assume that*

$$| f(t,x,Sx,x_t,z^t) | \leq g(t, | x |_0 (t), | z(t) |) + \int_{t_0}^{t} H(t,s, | x(s) |)ds, \tag{51}$$

where $g \in C[I \times \mathbb{R}_+ \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (u,v)*, is non decreasing in* u *for each* (t,v)*, is non decreasing in* v *for each* (t,u) *,*$H \in C[I^2 \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (s,u)*, is non decreasing in* s *for each* (t,u)*, is non decreasing in* u *for each* (t,s)*,* $| x |_0\ (t) = max_{t-h_1 \leq t+s \leq t} | x(t+s) |$ *and* $r(t,t_0,u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t,u,u) + \int_{t_0}^{t} H(t,s,u)ds, \tag{52}$$

$$u(t_0) = u_0, \quad u(T) = \psi_0(0), \tag{53}$$

existing on $[t_0,T]$*,* $|z(t)| \leq r(t)$*, and* $z(t)$ *is the reasonable estimate of* x^t*. Further suppose that* $f \in C[I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1,\ \mathbb{R}]$*,* $K \in C[I^2 \times \mathbb{R}, \mathbb{R}_+]$ *is non decreasing in* t *for each* (s,x)*,* $max_{t,s\in[t_0,T]}K(t,s,x) = k_1 \leq \frac{M+\phi_0(0)}{T-t_0}$*. Then the Euler solution* $x(t)$ *is a solution of* (28)*,* (29)*.*

Proof. Since the hypothesis of Theorem 4.1 is satisfied so we obtain a sequence $\{x_{\pi_j}\}$ of polygonal arcs for the integro differential equation with retardation and anticipation (28), (29) that converge uniformly to an Euler solution $x(t)$ on $[t_0,T]$.

Let $\widehat{B}(\phi_0(0),M) = \{(x,Sx,x_t,x^t) :\ x \in C[I,\mathbb{R}],\ \ | x(t) - \phi_0(0) | \leq M,$ $| Sx(t) - \phi_0(0) | \leq k_1(T-t_0) - | \phi_0(0) | \leq M,\ \ \sup_{-h_1 \leq s \leq 0} | x(t+s) - \phi_0(0) | \leq M,$ $\sup_{\sigma\in[0,h_2]} | x(t+\sigma) - \phi_0(0) | \leq M,\ \ t \in [t_0,T]\}$. Then, we observe that all the Euler polygonal arcs belongs to the ball $\widehat{B}(\phi_0(0),M)$, from the proof of Theorem 4.1, also we conclude that all these Euler arcs satisfy Lipschitz condition with some constant L. Now since f is continuous implies that it is uniformly continuous on compact sets $I \times \widehat{B}$. Hence for any given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$| t - t^* | < \delta, \quad | x(t) - x(t^*) | < \delta, \quad | Sx(t) - Sx(t^*) | < \delta, \quad | x_t - x_{t^*} | < \delta, \quad | x^t - x^{t^*} | < \delta,$$

implies

$$| f(t,x,Sx,x_t,x^t) - f(t^*,x^*,Sx^*,x_{t^*},x^{t^*}) | < \epsilon,$$

for any $t,\ t^* \in [t_0,T]$ and $x, x^* \in C[[t_0,T],\mathbb{R}]$ such that $(x,Sx,x_t,x^t) \in \widehat{B}(\phi_0(0),M)$. Let j be sufficiently large so that the diameter of μ_{π_j} corresponding to that j which satisfies $\mu_{\pi_j} < \delta$ and $L\mu_{\pi_j} < \delta,\ \ k_1\mu_{\pi_j} < \delta,\ \ (L + \frac{M}{j(T-t_0)})\mu_{\pi_j} < \delta$. Let $\pi_j = \{t_0,t_1,t_2,...,T\}$. Now for any t, which is not one of the infinitely many points at which $x_{\pi_j}(t)$ is a node, then we have $x'_{\pi_j}(t) = f(\widehat{t}, x_{\pi_j}(\widehat{t}), Sx_{\pi_j}(\widehat{t}), x_{\pi_j\widehat{t}}, z(\widehat{t}))$ for some $\widehat{t}$ with in $\mu_{\pi_j} < \delta$ of t. We have $| t - \widehat{t} | < \delta$, using the fact that x_{π_j} is Lipschitz, we get $| x_{\pi_j}(t) - x_{\pi_j}(\widehat{t}) | \leq L(t-\widehat{t}) \leq L\mu_{\pi_j} < \delta$,

$$\begin{aligned} | Sx_{\pi_j}(t) - Sx_{\pi_j}(\widehat{t}) | = &\ \ | \int_{t_0}^{t} K(t,s,x_{\pi_j}(s)ds - \int_{t_0}^{\widehat{t}} K(\widehat{t},s,x_{\pi_j}(s)ds | \\ \leq &\ \int_{t_0}^{t} | K(t,s,x_{\pi_j}(s) | ds < \delta. \end{aligned}$$

Now consider $\mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid$ for $t - h_1 \leq t+s \leq t$. Then

$$\mid x_{\pi_{jt}}(s) - x_{\pi_{j\widehat{t}}}(s) \mid = \mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid < \delta,$$

$$\mid x_{\pi_{jt}} - x_{\pi_{j\widehat{t}}} \mid = sup_{t_0+h_1 \leq t+s \leq t} \mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid \leq L\mu_{\pi_j} < \delta.$$

Also if $\mid \xi^t(0) - \phi_0(0) \mid < M$ then $\mid x^t_{\pi_j}(0) - x^{\widehat{t}}_{\pi_j}(0) \mid = \mid x_{\pi_j}(t) - x_{\pi_j}(\widehat{t}) \mid < \delta$ otherwise

$$\mid z(t) - z(t_1) \mid = \mid x^t_{\pi_j}(0) + \frac{z(t)}{j} - \frac{z(\widehat{t})}{j} - x^{\widehat{t}}_{\pi_j}(0) \mid \leq [L + \frac{M}{j(T-t_0)}]\mu_{\pi_j} < \delta.$$

Hence we have $\mid z(t) - z(\widehat{t}) \mid < \delta$. Thus by uniform continuity of f on compact sets $\mid x'_{\pi_j}(t) - f(t, x_{\pi_j}(t), Sx_{\pi_j}(t), x_{\pi_{jt}}, z(t)) \mid$

$$= \mid f(\widehat{t}, x_{\pi_j}(\widehat{t}), Sx_{\pi_j}(\widehat{t}), x_{\pi_{j\widehat{t}}}, z(\widehat{t})) - f(t, x_{\pi_j}(t), Sx_{\pi_j}(t), x_{\pi_{jt}}, z(t)) \mid < \epsilon.$$

Now for any $t \in [t_0, T]$, consider

$$\begin{aligned} \mid x_{\pi_j}(t) - \phi_0(0) - &\int_{t_0}^{t} f(s, x_{\pi_j}(s), Sx_{\pi_j}(s), x_{\pi_{js}}, z(s))ds \mid \\ &\leq \int_{t_0}^{t} \mid x'_{\pi_j}(s) - f(s, x_{\pi_j}(s), Sx_{\pi_j}(s), x_{\pi_{js}}, z(s)) \mid ds \leq \epsilon(T - t_0). \end{aligned}$$

Letting $j \to \infty$ in the above inequality, we get

$$\mid x(t) - \phi_0(0) - \int_{t_0}^{t} f(s, x(s), Sx(s), x_s, x^s)ds \mid \quad < \epsilon(T - t_0).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$x(t) = \phi_0(0) + \int_{t_0}^{t} f(s, x(s), Sx(s), x_s, x^s)ds$$

which implies that $x(t)$ is continuously differentiable and hence

$$x'(t) = f(t, x, Sx, x_t, x^t)$$

and $x_{t_0}(0) = \phi_0(0),\ x^T(0) = z(T) = \psi_0(0), \quad t_0 \in [t_0, T]$. Thus the proof is complete.

5 Conclusion

The concepts of anticipation and retardation arise naturally when modeling any goal oriented physical phenomena. Recently, integro differential equations including these concepts, have been studied in [6, 12]. In this paper we provided an existence result, using the concept of Euler solutions and gave criteria under which this Euler solution becomes a solution. In future, we propose to develop the necessary tools to obtain numerical solutions of the considered problem.

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Representation of the Solution for Linear System of Delay Equations with Distributed Parameters

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Abstract: The first boundary value problem for an autonomous system of linear delay partial differential equations of the second order has been solved. The solution is presented in an analytical form of formal series for the case, when matrices of coefficients are commutative and their eigenvalues are real and different. The obtained solution is studied on convergence and differentiability.

Keywords: *delay partial differential equation; first boundary value problem; time delay argument.*

Mathematics Subject Classification (2010): 35C10, 35G45, 35R10.

1 Introduction

Usually, when systems of differential equations are investigated, the main attention is paid to systems of ordinary differential equations (e.g., [1, 2]) or systems of partial differential equations [3]– [7]. Aside remains the analysis of systems of partial differential equations with delay. Their investigation is extremely rare [8]– [10].

Autonomous second-order systems of linear differential equations of with constant delay are considered in this paper:

$$\left\{ \begin{array}{l} \frac{\partial u(x,t)}{\partial t} = a_{11}\frac{\partial^2 u(x,t)}{\partial x^2} + a_{12}\frac{\partial^2 v(x,t)}{\partial x^2} + b_{11}u\left(x,t-\tau\right) + b_{12}v\left(x,t-\tau\right), \\ \frac{\partial v(x,t)}{\partial t} = a_{21}\frac{\partial^2 u(x,t)}{\partial x^2} + a_{22}\frac{\partial^2 v(x,t)}{\partial x^2} + b_{21}u\left(x,t-\tau\right) + b_{22}v\left(x,t-\tau\right). \end{array} \right. \tag{1}$$

We assume that matrices

$$A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right], B = \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right]$$

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are normal, i.e. $AA^* = A^*A$, $BB^* = B^*B$, where A^* is the conjugate transpose of A, B^* is the conjugate transpose of B; and they satisfy the commutativity condition, i.e.,

$$AB = BA.$$

Functions $u\,(x,t)$, $v\,(x,t)$ are defined in a semistrip $t \geq -\tau$, $0 \leq x \leq l$, where l is a positive constant, and the initial and boundary conditions are

$$\begin{aligned} u\,(0,t) = \mu_1\,(t)\,, u\,(l,t) = \mu_2\,(t)\,, v\,(0,t) = \theta_1\,(t)\,, v\,(l,t) = \theta_2\,(t)\,, t \geq -\tau, \\ u\,(x,t) = \varphi\,(x,t)\,, v\,(x,t) = \psi\,(x,t)\,, 0 \leq x \leq l, -\tau \leq t \leq 0. \end{aligned} \tag{2}$$

Compatibility conditions are fulfilled:

$$\mu_1\,(t) = \varphi\,(0,t)\,, \mu_2\,(t) = \varphi\,(l,t)\,, \theta_1\,(t) = \psi\,(0,t)\,, \theta_2\,(t) = \psi\,(l,t)\,, -\tau \leq t \leq 0.$$

A solution of the first boundary value problem has been obtained for the case, when eigenvalues of the matrices A and B are real and different.

2 Representation of Solution for Delay System

If the matrices A and B are normal and satisfy the commutativity condition, then, according to [11]– [13], there always exists a nonsingular matrix S, which simultaneously reduces matrices A and B to the Jordan forms Λ_1 and Λ_2:

$$\begin{aligned} &S^{-1}AS = \Lambda_1, \quad S^{-1}BS = \Lambda_2, \\ &S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}, \; S^{-1} = \frac{1}{\Delta}\begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix}, \; \Delta = s_{11}s_{22} - s_{12}s_{22}. \end{aligned} \tag{3}$$

Therefore by a transformation

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = S \begin{pmatrix} \xi(x,t) \\ \eta(x,t) \end{pmatrix}$$

system (1) can be reduced to a form

$$\begin{pmatrix} \frac{\partial \xi(x,t)}{\partial t} \\ \frac{\partial \eta(x,t)}{\partial t} \end{pmatrix} = \Lambda_1 \begin{pmatrix} \frac{\partial^2 \xi(x,t)}{\partial x^2} \\ \frac{\partial^2 \eta(x,t)}{\partial x^2} \end{pmatrix} + \Lambda_2 \begin{pmatrix} \xi\,(x,t-\tau) \\ \eta\,(x,t-\tau) \end{pmatrix}, \tag{4}$$

where Λ_1 is the Jordan form of the matrix A and Λ_2 is the Jordan form of the matrix B. The initial and boundary conditions will be

$$\begin{aligned} \xi\,(0,t) = \overline{\mu}_1\,(t)\,, \xi\,(l,t) = \overline{\mu}_2\,(t)\,, \eta\,(0,t) = \overline{\theta}_1\,(t)\,, \eta\,(l,t) = \overline{\theta}_2\,(t)\,, t \geq -\tau, \\ \xi\,(x,t) = \overline{\varphi}\,(x,t)\,, \eta\,(x,t) = \overline{\psi}\,(x,t)\,, 0 \leq x \leq l, -\tau \leq t \leq 0, \end{aligned} \tag{5}$$

where

$$\begin{pmatrix} \overline{\mu}_1\,(t) \\ \overline{\theta}_1\,(t) \end{pmatrix} = S^{-1} \begin{pmatrix} \mu_1\,(t) \\ \theta_1\,(t) \end{pmatrix}, \begin{pmatrix} \overline{\mu}_2\,(t) \\ \overline{\theta}_2\,(t) \end{pmatrix} = S^{-1} \begin{pmatrix} \mu_2\,(t) \\ \theta_2\,(t) \end{pmatrix},$$

$$\begin{pmatrix} \overline{\varphi}\,(x,t) \\ \overline{\psi}\,(x,t) \end{pmatrix} = S^{-1} \begin{pmatrix} \varphi\,(x,t) \\ \psi\,(x,t) \end{pmatrix}.$$

We will consider the representation of solution of the first boundary value problem for the system (1), (2), when roots of the characteristic equations λ_1, λ_2, ς_1, ς_2 of the both matrices A and B are real and different, i.e. $\lambda_1 \neq \lambda_2$, $\varsigma_1 \neq \varsigma_2$. In this case, after the transformation the system (4) decouples into two independent equations:

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right), \quad \frac{\partial \eta(x,t)}{\partial t} = \lambda_2 \frac{\partial^2 \eta(x,t)}{\partial x^2} + \varsigma_2 \eta\left(x, t-\tau\right). \tag{6}$$

We will consider the first equation of system (6)

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) \tag{7}$$

with initial and boundary conditions

$$\xi\left(0,t\right) = \overline{\mu}_1\left(t\right), \xi\left(l,t\right) = \overline{\mu}_2\left(t\right), t \geq -\tau, \xi\left(x,t\right) = \overline{\varphi}\left(x,t\right), 0 \leq x \leq l, -\tau \leq t \leq 0.$$

A solution will be in the form

$$\xi\left(x,t\right) = \xi_0\left(x,t\right) + \xi_1\left(x,t\right) + \overline{\mu}_1\left(t\right) + \frac{x}{l}\left[\overline{\mu}_2\left(t\right) - \overline{\mu}_1\left(t\right)\right], \tag{8}$$

where

- $\xi_0\left(x,t\right)$ is a solution of homogeneous equation

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) \tag{9}$$

with zero boundary $\xi\left(0,t\right) = 0$, $\xi\left(l,t\right) = 0$ and nonzero initial conditions $\xi\left(x,t\right) = \Phi\left(x,t\right)$, $\Phi\left(x,t\right) = \bar{\varphi}\left(x,t\right) - \bar{\mu}_1\left(t\right) - \frac{x}{l}\left[\bar{\mu}_2\left(t\right) - \bar{\mu}_1\left(t\right)\right]$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$.

- $\xi_1\left(x,t\right)$ is a solution of inhomogeneous equation

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi\left(x, t-\tau\right) + F\left(x,t\right), \tag{10}$$

$$F\left(x,t\right) = \varsigma_1 \left\{\overline{\mu}_1\left(t-\tau\right) + \frac{x}{l}\left[\overline{\mu}_2\left(t-\tau\right) - \overline{\mu}_1\left(t-\tau\right)\right]\right\} - \dot{\overline{\mu}}_1\left(t\right) - \frac{x}{l}\left[\dot{\overline{\mu}}_2\left(t\right) - \dot{\overline{\mu}}_1\left(t\right)\right]$$

with zero boundary $\xi\left(0,t\right) = 0$, $\xi\left(l,t\right) = 0$, $t \geq -\tau$ and zero initial conditions $\xi\left(x,t\right) = 0$, $-\tau \leq t \leq 0$, $0 \leq x \leq l$.

2.1 Homogeneous equation

For finding the solution $\xi_0\left(x,t\right)$ we will use the method of separation of variables. According to this method, the solution will be in a form of product of two functions $\xi_0\left(x,t\right) = X\left(x\right) T\left(t\right)$. After substitution in the equation (7) we obtain

$$X\left(x\right) T'(t) = \lambda_1 X''\left(x\right) T\left(t\right) + \varsigma_1 X\left(x\right) T\left(t-\tau\right).$$

Separating variables, we have

$$\frac{T'\left(t\right) - \varsigma_1 T\left(t-\tau\right)}{\lambda_1 T\left(t\right)} = \frac{X''\left(x\right)}{X\left(x\right)} = -k^2,$$

where k is an arbitrary constant. We will divide the obtained expression into two equations

$$T'(t) + \lambda_1 k^2 T(t) - \varsigma_1 T(t-\tau) = 0, \quad X''(x) + k^2 X(x) = 0. \tag{11}$$

Solutions of the second equation from (11), which is not identically zero and satisfies zero boundary conditions $X(0) = 0$, $X(l) = 0$, are

$$X_n(x) = A_n \sin \frac{\pi n}{l} x, \; k_n^2 = \left(\frac{\pi n}{l}\right)^2, \; n = 1, 2, ...$$

where A_n are arbitrary constants.

Now we will consider the first of equations from (11)

$$T_n'(t) = -\lambda_1 \left(\frac{\pi n}{l}\right)^2 T_n(t) + \varsigma_1 T_n(t-\tau), \; n = 1, 2, \ldots. \tag{12}$$

To obtain initial conditions for each of the equations (12) we will expand the corresponding initial condition $\Phi(x,t)$ into series under solutions of the second equation

$$\Phi(x,t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin \frac{\pi n}{l} x, \tag{13}$$

$$\Phi_n(t) = \frac{2}{l} \int_0^l \overline{\varphi}(s,t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n} \left[(-1)^n \bar{\mu}_2(t) - \bar{\mu}_1(t)\right], n = 1, 2, ...$$

Preliminary we should consider some results on linear homogeneous equations with constant delay

$$\dot{x}(t) = bx(t-\tau) \tag{14}$$

with an initial condition $x(t) = \beta(t)$, $-\tau \le t \le 0$, $b \in \mathbb{R}$.

Definition 2.1 [14] A delay exponential function $\exp_\tau\{b,t\}$ is a function which can be written as

$$\exp_\tau\{b,t\} = \begin{cases} 0, \quad \text{if} \quad -\infty < t < -\tau, \\ 1, \quad \text{if} \quad -\tau \le t < 0, \\ 1 + b\frac{t}{1!}, \quad \text{if} \quad 0 \le t < \tau, \\ \ldots \\ 1 + b\frac{t}{1!} + b^2\frac{(t-\tau)^2}{2!} + ... + b^k \frac{[t-(k-1)\tau]^k}{k!}, \quad \text{if} \quad (k-1)\tau \le t < k\tau, \end{cases} \tag{15}$$

a k-degree polynomial on intervals $(k-1)\tau < t \le k\tau$ "merged" in points $t = k\tau$, $k = 0, 1, 2, ...$, $b = const$.

Lemma 2.1 *A rule of differentiation for the delay exponential function can be formulated in the following way:*

$$\frac{d}{dt} \exp_\tau\{b,t\} = b \exp_\tau\{b, t-\tau\}. \tag{16}$$

I.e., the delay exponential function is a solution of the equation (14) with unitary initial conditions $x(t) \equiv 1$, $-\tau \le t \le 0$.

Proof. Within an interval $(k-1)\tau < t \leq k\tau$ the delay exponential function is represented as follows

$$\exp_\tau\{b,t\} = 1 + b\frac{t}{1!} + b^2\frac{(t-\tau)^2}{2!} + b^3\frac{(t-2\tau)^3}{3!} + ... + b^k\frac{[t-(k-1)\tau]^k}{k!}.$$

Differentiating this function we will obtain

$$\frac{d}{dt}\exp_\tau\{b,t\} = b + b^2\frac{t-\tau}{1!} + b^3\frac{(t-2\tau)^2}{2!} + b^4\frac{(t-3\tau)^3}{3!} + ... + b^k\frac{[t-(k-1)\tau]^{k-1}}{(k-1)!} =$$

$$= b\left\{1 + b\frac{t-\tau}{1!} + b^2\frac{(t-2\tau)^2}{2!} + b^3\frac{(t-3\tau)^3}{3!} + ... + b^{k-1}\frac{[t-(k-1)\tau]^{k-1}}{(k-1)!}\right\} =$$

$$= b\exp_\tau\{b,t-\tau\},$$

Q.E.D. □

Theorem 2.1 *A solution of the equation (14), which satisfies the initial condition $x(t) = \beta(t)$, $-\tau \leq t \leq 0$, can be presented as follows*

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds. \tag{17}$$

Proof. As the expression (17) is a linear functional of the delay exponential function $\exp_\tau\{b,t\}$ which, as it was shown in Lemma 2.1, is the solution of the equation (14), then the functional (17) is a solution of the homogeneous equation (14) for any function $\beta(t)$. We will show that initial conditions are satisfied, i.e. for $-\tau \leq t \leq 0$ the following identity is correct:

$$\beta(t) \equiv \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds.$$

Then we will divide an integral from the expression (17) into two integrals:

$$x(t) = \exp_\tau\{b,t\}\beta(-\tau) + \int_{-\tau}^{t}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds +$$

$$+ \int_{t}^{0}\exp_\tau\{b,t-\tau-s\}\beta'(s)\,ds.$$

Using the definition of the delay exponential function, we can obtain that
- $\exp_\tau\{b,t\} \equiv 1$ at $-\tau \leq t \leq 0$;
- $\exp_\tau\{b,t-\tau-s\} \equiv 1$ at $-\tau \leq s \leq t$;
- $\exp_\tau\{b,t-\tau-s\} \equiv 0$ at $t < s \leq 0$.

Therefore,

$$x(t) = \beta(-\tau) + \int_{-\tau}^{t}\beta'(s)\,ds = \beta(-\tau) + \beta(t) - \beta(-\tau) = \beta(t),$$

Q.E.D. □

Remark 2.1 Under the hypothesis of the theorem, continuous differentiability of the initial function $\beta(t)$ is required. Computing the integral in (17) by parts we obtain

$$x(t) = \exp_\tau \{b, t-\tau\} \beta(0) + b \int_{-\tau}^{0} \exp_\tau \{b, t-2\tau-s\} \beta(s)\, ds. \tag{18}$$

The equality (18) is an integral representation of the solution under the assumption of only continuity of the function $\beta(t)$.

Further we will consider the differential equation

$$\dot{x}(t) = ax(t) + bx(t-\tau) \tag{19}$$

with an initial condition $x(t) = \beta(t)$, $-\tau \le t \le 0$, $a, b \in \mathbb{R}$.

Theorem 2.2 *A solution of the equation (19), which satisfies initial condition $x(t) = \beta(t)$, $-\tau \le t \le 0$, can be presented as*

$$\begin{gathered} x(t) = \exp_\tau \{b_1, t\} e^{a(t+\tau)} \beta(-\tau) + \int_{-\tau}^{0} \exp_\tau \{b_1, t-\tau-s\} e^{a(t-s)} \left[\beta'(s) - a\beta(s)\right] ds, \\ b_1 = be^{-a\tau}. \end{gathered} \tag{20}$$

Proof. We will make a substitution $x(t) = e^{at} y(t)$, where $y(t)$ is a new unknown function

$$\begin{gathered} ae^{at} y(t) + e^{at} \dot{y}(t) = ae^{at} y(t) + be^{a(t-\tau)} y(t-\tau), \\ \dot{y}(t) = b_1 y(t-\tau), \quad b_1 = be^{-a\tau}. \end{gathered} \tag{21}$$

Correspondingly, the initial condition for the equation (21) is

$$y(t) = e^{-at} \beta(t).$$

As follows from (17) a solution of the corresponding Cauchy problem for the equation (21) will be

$$y(t) = \exp_\tau \{b_1, t\} e^{a\tau} \beta(-\tau) + \int_{-\tau}^{0} \exp_\tau \{b_1, t-\tau-s\} \left[e^{-as} \beta'(s) - ae^{-as} \beta(s)\right] ds.$$

Again, using a substitution $x(t) = e^{at} y(t)$, we obtain

$$x(t) = \exp_\tau \{b_1, t\} e^{a(t+\tau)} \beta(-\tau) + \int_{-\tau}^{0} \exp_\tau \{b_1, t-\tau-s\} e^{a(t-s)} \left[\beta'(s) - a\beta(s)\right] ds,$$

i.e. the statement of Theorem 2.2. □

Using the results obtained above, we will solve each of the equations (12). According to the equality (20), solutions of (12) will be

$$\begin{aligned} T_n(t) = &\exp_\tau \{r_1, t\} e^{q_1(t+\tau)} \Phi_n(-\tau) + \\ &+ \int_{-\tau}^{0} \exp_\tau \{r_1, t-\tau-s\} e^{q_1(t-s)} \left[\Phi_n'(s) - q_1 \Phi_n(s)\right] ds, \end{aligned}$$

$$r_1 = \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, q_1 = -\lambda_1 \left(\frac{\pi n}{l}\right)^2.$$

Thus, the solution $\xi_0(x,t)$ of the homogeneous equation (9), which satisfies zero boundary $\xi(0,t) = 0$, $\xi(l,t) = 0$ and nonzero initial conditions $\xi(x,t) = \Phi(x,t)$, $-\tau \le t \le 0$, $0 \le x \le l$, is

$$\xi_0(x,t) = \sum_{n=1}^{\infty} \left\{ \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t+\tau)} \Phi_n(-\tau) + \right.$$

$$\left. + \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t - \tau - s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \left[\Phi_n'(s) + \lambda_1 \left(\frac{\pi n}{l}\right)^2 \Phi_n(s) \right] ds \right\} \times$$

$$\times \sin \frac{\pi n}{l} x,$$

$$\Phi_n(t) = \frac{2}{l} \int_0^l \overline{\varphi}(s,t) \sin \frac{\pi n}{l} s ds + \frac{2}{\pi n} \left[(-1)^n \bar{\mu}_2(t) - \bar{\mu}_1(t) \right], n = 1, 2, ...$$

2.2 Inhomogeneous equation

Further we will consider the inhomogeneous equation (10)

$$\frac{\partial \xi(x,t)}{\partial t} = \lambda_1 \frac{\partial^2 \xi(x,t)}{\partial x^2} + \varsigma_1 \xi(x, t-\tau) + F(x,t),$$

$$F(x,t) = \varsigma_1 \left\{ \overline{\mu}_1(t-\tau) + \frac{x}{l} \left[\overline{\mu}_2(t-\tau) - \overline{\mu}_1(t-\tau) \right] \right\} - \dot{\overline{\mu}}_1(t) - \frac{x}{l} \left[\dot{\overline{\mu}}_2(t) - \dot{\overline{\mu}}_1(t) \right]$$

with zero boundary $\xi(0,t) = 0$, $\xi(l,t) = 0$, $t \ge -\tau$ and zero initial conditions $\xi(x,t) = 0$, $-\tau \le t \le 0$, $0 \le x \le l$. We will try to find a solution in the form of series expansion in terms of the functions from the previous problem, i.e. in the form

$$\xi_1(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{\pi n}{l} x.$$

After substituting the series in the equation (10) and having equated coefficients of the same terms, we obtain a system of the equations

$$T_n'(t) = -\lambda_1 \left(\frac{\pi n}{l}\right)^2 T_n(t) + \varsigma_1 T_n(t-\tau) + f_n(t), \quad n = 1, 2, ..., \tag{22}$$

where

$$f_n(t) = \frac{2}{l} \int_0^l F(s,t) \sin \frac{\pi n}{l} s ds =$$

$$= \frac{2}{\pi n} \left[\varsigma_1 \left((-1)^{n+1} \overline{\mu}_2(t-\tau) + \overline{\mu}_1(t-\tau) \right) - \left((-1)^{n+1} \dot{\overline{\mu}}_2(t) + \dot{\overline{\mu}}_1(t) \right) \right], n = 1, 2, ...$$

Preliminary we will consider a linear inhomogeneous equation with a constant delay:

$$\dot{x}(t) = ax(t) + bx(t-\tau) + f(t). \tag{23}$$

We will solve the Cauchy problem for (22) with a zero initial condition $x(t) \equiv 0$, $-\tau \le t \le 0$, where $a, b \in \mathbb{R}$, $f : [0, \infty) \to \mathbb{R}$.

Theorem 2.3 *A solution of the inhomogeneous equation (23), which satisfies zero initial conditions* $x(t) \equiv 0$, $-\tau \le t \le 0$, *will be*

$$x(t) = \int_0^t \exp_\tau \{b_1, t-\tau-s\} e^{a(t-s)} f(s)\, ds,\ b_1 = be^{-a\tau}. \tag{24}$$

Proof. As in the previous case, we apply the substitution $x(t) = e^{at} y(t)$ and obtain a differential equation

$$ae^{at} y(t) + e^{at} \dot{y}(t) = ae^{at} y(t) + be^{a(t-\tau)} y(t-\tau) + f(t).$$

It will be adduced to

$$\dot{y}(t) = b_1 y(t-\tau) + e^{-at} f(t),\ b_1 = be^{-a\tau}. \tag{25}$$

We will show that the solution of the inhomogeneous equation (25), which satisfies zero initial condition, is

$$y(t) = \int_0^t \exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\, ds. \tag{26}$$

Substituting (26) in the equation (25)

$$\exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} + b_1 \int_0^t \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds =$$

$$= b_1 \int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds + e^{-at} f(t),$$

considering that

$$\exp_\tau \{b_1, t-\tau-s\} e^{-as} f(s)\big|_{s=t} = \exp\{b_1, -\tau\} e^{-at} f(t) = e^{-at} f(t),$$

and dividing the second integral into two, we obtain

$$e^{-at} f(t) + b_1 \left(\int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) +$$

$$+ b_1 \left(\int_{t-\tau}^{t} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) =$$

$$= b_1 \left(\int_0^{t-\tau} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) + e^{-at} f(t).$$

Hence

$$e^{-at} f(t) + b_1 \left(\int_{t-\tau}^{t} \exp_\tau \{b_1, t-2\tau-s\} e^{-as} f(s)\, ds \right) = e^{-at} f(t).$$

After substitution

$$t-2\tau-s = \omega, s = t-\tau \Rightarrow \omega = -\tau, s = t \Rightarrow \omega = -2\tau$$

we obtain

$$be^{-a\tau}\left(\int_{-2\tau}^{-\tau}\exp_{\tau}\{b_1,\omega\}e^{a(\omega-t+2\tau)}f(t-2\tau-\omega)\,d\omega\right)=0$$

and identity

$$e^{-at}f(t)=e^{-at}f(t),$$

which proves correctness of the equality (26). Hence

$$x(t)=e^{at}y(t)=\int_0^t\exp_{\tau}\{b_1,t-\tau-s\}e^{a(t-s)}f(s)\,ds,$$

Q.E.D. □

Corollary 2.1 *A solution of the inhomogeneous equation (23) with initial condition* $x(t)\equiv\beta(t)$, $-\tau\le t\le 0$ *is*

$$\begin{aligned}x(t)=&\exp_{\tau}\{b_1,t\}e^{a(t+\tau)}\beta(-\tau)+\int_{-\tau}^{0}\exp_{\tau}\{b_1,t-\tau-s\}e^{a(t-s)}\left[\beta'(s)-a\beta(s)\right]ds+\\&+\int_0^t\exp_{\tau}\{b_1,t-\tau-s\}e^{a(t-s)}f(s)\,ds,\quad b_1=be^{-a\tau}.\end{aligned}\tag{27}$$

Proof. Proof is based on statements of the previous Theorems 2.2 and 2.3. □

Using the results obtained above, a solution of each of the equations (22)

$$T_n'(t)=-\lambda_1\left(\frac{\pi n}{l}\right)^2T_n(t)+\varsigma_1T_n(t-\tau)+f_n(t),n=1,2,...$$

can be written as

$$T_n(t)=\int_0^t\exp_{\tau}\{r_1,t-\tau-s\}e^{q_1(t-s)}f_n(s)\,ds,\ r_1=\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\ q_1=-\lambda_1\left(\frac{\pi n}{l}\right)^2.\tag{28}$$

Hence, a solution of the inhomogeneous equation (10) with zero boundary $\xi(0,t)=0$, $\xi(l,t)=0$, $t\ge-\tau$ and zero initial conditions $\xi(x,t)=0$, $-\tau\le t\le 0$, $0\le x\le l$, is

$$\xi_1(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t\exp_{\tau}\left\{\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}f_n(s)\,ds\right\}\sin\frac{\pi n}{l}x,$$

$$f_n(t)=\frac{2}{\pi n}\left[\varsigma_1\left((-1)^{n+1}\overline{\mu}_2(t-\tau)+\overline{\mu}_1(t-\tau)\right)-\left((-1)^{n+1}\dot{\overline{\mu}}_2(t)+\dot{\overline{\mu}}_1(t)\right)\right],$$

$n=1,2,\ldots.$

2.3 General solution

Using all previous results, the solution of the first boundary value problem for the equation (7) can be written in the form of sum:

$$\begin{aligned}
\xi(x,t) &= \sum_{n=1}^{\infty}\Big\{\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau)+\\
&+\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\left[\Phi_n'(s)+\lambda_1\left(\frac{\pi n}{l}\right)^2\Phi_n(s)\right]ds+\\
&+\int_{0}^{t}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)} f_n(s)\,ds\Big\}\sin\frac{\pi n}{l}x+\\
&+\overline{\mu}_1(t)+\frac{x}{l}\left[\overline{\mu}_2(t)-\overline{\mu}_1(t)\right],
\end{aligned} \tag{29}$$

$$\Phi_n(t)=\frac{2}{l}\int_0^l\overline{\varphi}(s,t)\sin\frac{\pi n}{l}sds+\frac{2}{\pi n}\left[(-1)^n\bar{\mu}_2(t)-\bar{\mu}_1(t)\right], n=1,2,...,$$

$$f_n(t)=\frac{2}{\pi n}\left[\varsigma_1\left((-1)^{n+1}\overline{\mu}_2(t-\tau)+\overline{\mu}_1(t-\tau)\right)-\left((-1)^{n+1}\dot{\overline{\mu}}_2(t)+\dot{\overline{\mu}}_1(t)\right)\right].$$

Similarly, the second equation from (6) has a solution:

$$\begin{aligned}
\eta(x,t) &= \sum_{n=1}^{\infty}\Big\{\exp_\tau\left\{\varsigma_2 e^{\lambda_2\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Psi_n(-\tau)+\\
&+\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_2 e^{\lambda_2\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2(t-s)}\left[\Psi_n'(s)+\lambda_2\left(\frac{\pi n}{l}\right)^2\Psi_n(s)\right]ds+\\
&+\int_{0}^{t}\exp_\tau\left\{\varsigma_2 e^{\lambda_2\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2(t-s)} g_n(s)\,ds\Big\}\sin\frac{\pi n}{l}x+\\
&+\overline{\theta}_1(t)+\frac{x}{l}\left[\overline{\theta}_2(t)-\overline{\theta}_1(t)\right],
\end{aligned} \tag{30}$$

$$\Psi_n(t)=\frac{2}{l}\int_0^l\overline{\psi}(s,t)\sin\frac{\pi n}{l}sds+\frac{2}{\pi n}\left[(-1)^n\bar{\theta}_2(t)-\bar{\theta}_1(t)\right], n=1,2,...$$

$$g_n(t)=\frac{2}{\pi n}\left[\varsigma_2\left((-1)^{n+1}\overline{\theta}_2(t-\tau)+\overline{\theta}_1(t-\tau)\right)-\left((-1)^{n+1}\dot{\overline{\theta}}_2(t)+\dot{\overline{\theta}}_1(t)\right)\right].$$

Then solutions of the boundary value problem of the initial system (1) with the conditions (2) finally are

$$u(x,t)=s_{11}\xi(x,t)+s_{12}\eta(x,t),\ v(x,t)=s_{21}\xi(x,t)+s_{22}\eta(x,t), \tag{31}$$

where the solutions $\xi(x,t)$, $\eta(x,t)$ of the reduced system (6) are defined in (29), (30), s_{ij}, $i,j=\overline{1,2}$ are the coefficients of the matrix of transformation S.

3 Existence Conditions for Solutions

The solution of the first boundary value problem of the equations (6) is presented in the form of formal series (29), (30). We will show that when certain conditions are satisfied the series converge and the representations are really the solutions of system of delay partial differential equations.

We will consider the first equations (7).

Theorem 3.1 *Let the functions $\Phi_n(t)$, $-\tau \leq t \leq 0$ and $f_n(t)$, $t \geq 0$, defined in (13), (22), satisfy the conditions*

$$\lim_{n\to+\infty} \max_{-\tau\leq t\leq T-\tau} |f_n(t)|\, e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} = 0,$$
$$\lim_{n\to+\infty} e^{-\lambda_2\left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} \max_{-\tau\leq t\leq 0} |\Phi_n(t)| = 0 \tag{32}$$

on an interval $(k-1)\tau \leq T < k\tau$. Then the expression (29) is a solution of the equation (7) for t: $0 \leq t \leq T$. And the function $\xi(x,t)$ has a continuous first-order derivative with respect to t and a second-order derivative with respect to x.

Proof. We will write the representation (29) as a sum of three terms:

$$\xi(x,t) = S_1(x,t) + S_2(x,t) + S_3(x,t) + \overline{\mu}_1(t) + \frac{x}{l}\left[\overline{\mu}_2(t) - \overline{\mu}_1(t)\right], \tag{33}$$

where

$$S_1(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{\pi n}{l}x,\ S_2(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{\pi n}{l}x,\ S_3(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{\pi n}{l}x,$$

$$A_n(t) = \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau),$$

$$B_n(t) = \int_{-\tau}^{0} \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)} \times$$
$$\times \left[\Phi_n'(s) + \lambda_1\left(\frac{\pi n}{l}\right)^2\Phi_n(s)\right] ds,$$

$$C_n(t) = \int_0^t \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)} f_n(s)\, ds.$$

1. Firstly we will consider coefficients $A_n(t)$, $n = 1, 2, ...$ of the first series $S_1(x,t)$. As follows from the definition of delay exponential function, formulated in (15), for any moment of time $T : (k-1)\tau \leq T < k\tau$, $k = 0, 1, 2, ...$ the following equality holds

$$A_n(T) = \exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, T\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau) = e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T+\tau)}\Phi_n(-\tau) \times$$

$$\times \left[1 + \varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!} + \varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!} + \varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!} + \right.$$

$$+ ... + \varsigma_1^k e^{k\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau} \frac{\left[T-(k-1)\tau\right]^k}{k!} \Bigg].$$

Hence

$$S_1(x,T) = \sum_{n=1}^{\infty} A_n(T) \sin \frac{\pi n}{l} x =$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T+\tau)} \Phi_n(-\tau) \Bigg[1 + \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau} \frac{T}{1!} + \varsigma_1^2 e^{2\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau} \frac{\left[T-\tau\right]}{2!} +$$

$$+ \varsigma_1^3 e^{3\left(\frac{\pi n}{l}\right)^2 \tau} \frac{\left[T-2\tau\right]^3}{3!} + \ldots + \varsigma_1^k e^{k\left(\frac{\pi n}{l}\right)^2 \tau} \frac{\left[T-(k-1)\tau\right]^k}{k!} \Bigg] \sin \frac{\pi n}{l} x =$$

$$= \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T+\tau)} \Phi_n(-\tau) \sin \frac{\pi n}{l} x + \varsigma_1 \frac{T}{1!} \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 T} \Phi_n(-\tau) \sin \frac{\pi n}{l} x +$$

$$+ \varsigma_1^2 \frac{\left[T-\tau\right]}{2!} \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T-\tau)} \Phi_n(-\tau) \sin \frac{\pi n}{l} x + \ldots +$$

$$+ \varsigma_1^k \frac{\left[T-(k-1)\tau\right]^k}{k!} \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} \Phi_n(-\tau) \sin \frac{\pi n}{l} x.$$

And if coefficients $\Phi_n(-\tau)$ are such that the following condition is satisfied

$$\lim_{n\to\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T-(k-1)\tau)} \left|\Phi_n(-\tau)\right| = 0,$$

then the series $S_1(x,t)$ converges absolutely and uniformly.

2. We will consider coefficients $B_n(t)$, $n = 1, 2, 3, ...$ of the second series $S_2(x,t)$. We will divide the integral into two and calculate the second integral by parts:

$$B_n(t) = \lambda_1 \left(\frac{\pi n}{l}\right)^2 \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \Phi_n(s)\, ds +$$

$$+ \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \Phi_n'(s)\, ds =$$

$$= \lambda_1 \left(\frac{\pi n}{l}\right)^2 \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \Phi_n(s)\, ds +$$

$$+ \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 t} \Phi_n(0) -$$

$$- \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t+\tau)} \Phi_n(-\tau) +$$

$$+ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau} \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-2\tau-s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \Phi_n(s)\, ds -$$

$$- \lambda_1 \left(\frac{\pi n}{l}\right)^2 \int_{-\tau}^{0} \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau-s \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (t-s)} \Phi_n(s)\, ds =$$

$$= \exp_\tau \left\{ \varsigma_1 e^{\lambda_1 \left(\frac{\pi n}{l}\right)^2 \tau}, t-\tau \right\} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 t} \Phi_n(0) -$$

$$-\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau)+$$

$$+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-2\tau-s\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds=$$

$$=B_{n1}(t)-B_{n2}(t)+B_{n3}(t).$$

Now we will consider the first series

$$B_{n1}(t)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t-\tau\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n(0).$$

By analogy with the previous case, for any moment of time $T:(k-2)\tau\le T<(k-1)\tau$ the following holds:

$$B_{n1}(T)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, T\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 t}\Phi_n(0)=e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n(0)\times$$

$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!}+\right.$$

$$\left.+...+\varsigma_1^{(k-1)}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-(k-2)\tau]^{(k-1)}}{(k-1)!}\right].$$

Hence

$$\sum_{n=1}^{\infty}B_{n1}(T)\sin\frac{\pi n}{l}x=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n(0)\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{T}{1!}+\right.$$

$$+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-2\tau]^3}{3!}+$$

$$\left.+...+\varsigma_1^{(k-1)}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[T-(k-2)\tau]^{(k-1)}}{(k-1)!}\right]\sin\frac{\pi n}{l}x=$$

$$=\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2 T}\Phi_n(0)\sin\frac{\pi n}{l}x+\varsigma_1\frac{T}{1!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-\tau)}\Phi_n(0)\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\frac{[T-\tau]}{2!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-2\tau)}\Phi_n(0)\sin\frac{\pi n}{l}x+\ldots+$$

$$+\varsigma_1^{(k-1)}\frac{[T-(k-2)\tau]^{(k-1)}}{(k-1)!}\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\Phi_n(0)\sin\frac{\pi n}{l}x.$$

And if coefficients $\Phi_n(0)$ are such that the following condition is satisfied

$$\lim_{n\to\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\left|\Phi_n(0)\right|=0,$$

then the series $\sum_{n=1}^{\infty}B_{n1}(T)\sin\frac{\pi n}{l}x$ converges absolutely and uniformly.

We will consider the second series

$$B_{n2}(t)=\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}, t\right\} e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t+\tau)}\Phi_n(-\tau)=A_n(t).$$

For any moment of time $T:(k-1)\tau\le T<k\tau$ the series $\sum_{n=1}^{\infty}B_{n2}(T)\sin\frac{\pi n}{l}x$ converges absolutely and uniformly if, as follows from the previous case, coefficients $\Phi_n(-\tau)$ are such that the following condition is satisfied

$$\lim_{n\to\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(T-(k-1)\tau)}\left|\Phi_n(-\tau)\right|=0,$$

Finally, for coefficients

$$B_{n3}(t)=\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},t-2\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(t-s)}\Phi_n(s)\,ds$$

at the moment of time $T:(k-1)\tau\le T<k\tau$, we make a substitution $T-2\tau-s=\omega$ and obtain:

$$B_{n3}(T)=\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{T-2\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\Phi_n(T-2\tau-\omega)\,d\omega.$$

Dividing the integral into two we have:

$$B_{n3}(T)=\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{T-2\tau}^{(k-2)\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\times$$
$$\times\Phi_n(T-2\tau-\omega)\,d\omega+$$
$$+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\int_{(k-2)\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+2\tau)}\Phi_n(T-2\tau-\omega)\,d\omega.$$

Therefore, owing to the mean value theorem, there exist values $\omega_1:T-2\tau\le\omega_1\le(k-2)\tau$, $\omega_2:(k-2)\tau\le\omega_2\le T-\tau$ for which the following holds:

$$B_{n3}(T)=\varsigma_1(k\tau-T)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega_1\right\}\Phi_n(T-2\tau-\omega_1)+$$
$$+\varsigma_1(T-(k-1)\tau)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega_2\right\}\Phi_n(T-2\tau-\omega_2).$$

Hence

$$B_{n3}(T)=\varsigma_1(k\tau-T)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}\Phi_n(T-2\tau-\omega_1)\times$$
$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_1}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-2\tau]^3}{3!}+\right.$$
$$\left.+...+\varsigma_1^{k-2}e^{(k-2)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_1-(k-3)\tau]^{k-2}}{(k-2)!}\right]+$$
$$+\varsigma_1(T-(k-1)\tau)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}\Phi_n(T-2\tau-\omega_2)\times$$
$$\times\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_2}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-\tau]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-2\tau]^3}{3!}+\right.$$
$$\left.+...+\varsigma_1^{k-1}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{[\omega_2-(k-2)\tau]^{k-1}}{(k-1)!}\right].$$

And

$$\sum_{n=1}^{\infty} B_{n3}\left(T\right)\sin\frac{\pi n}{l}x = \varsigma_1 \sum_{n=1}^{\infty}\Big\{\left(k\tau - T\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_1+\tau\right)}\Phi_n\left(T-2\tau-\omega_1\right)+$$

$$+\left(T-\left(k-1\right)\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_2+\tau\right)}\Phi_n\left(T-2\tau-\omega_2\right)\Big\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\sum_{n=1}^{\infty}\left\{\frac{\omega_1}{1!}\left(k\tau-T\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_1}\Phi_n\left(T-2\tau-\omega_1\right)+\right.$$

$$\left.+\frac{\omega_2}{1!}\left(T-\left(k-1\right)\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_2}\Phi_n\left(T-2\tau-\omega_2\right)\right\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^3\sum_{n=1}^{\infty}\left\{\frac{\left[\omega_1-\tau\right]}{2!}\left(k\tau-T\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_1-\tau\right)}\Phi_n\left(T-2\tau-\omega_1\right)+\right.$$

$$\left.+\frac{\left[\omega_2-\tau\right]}{2!}\left(T-\left(k-1\right)\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_2-\tau\right)}\Phi_n\left(T-2\tau-\omega_2\right)\right\}\sin\frac{\pi n}{l}x+\ldots+$$

$$+\varsigma_1^{k-1}\sum_{n=1}^{\infty}\left\{\frac{\left[\omega_1-\left(k-3\right)\tau\right]^{k-2}}{\left(k-2\right)!}\left(k\tau-T\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_1-\left(k-3\right)\tau\right)}\Phi_n\left(T-2\tau-\omega_1\right)+\right.$$

$$\left.+\frac{\left[\omega_2-\left(k-3\right)\tau\right]^{k-2}}{\left(k-2\right)!}\left(T-\left(k-1\right)\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_2-\left(k-3\right)\tau\right)}\Phi_n\left(T-2\tau-\omega_2\right)\right\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^k\frac{\left[\omega_1-\left(k-2\right)\tau\right]^{k-1}}{\left(k-1\right)!}\left(T-\left(k-1\right)\tau\right)\times$$

$$\times\sum_{n=1}^{\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega_2-\left(k-2\right)\tau\right)}\Phi_n\left(T-2\tau-\omega_2\right)\sin\frac{\pi n}{l}x.$$

If coefficients $\Phi_n\left(t\right)$ are such that the following condition is satisfied

$$\lim_{n\to+\infty}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(T-\left(k-1\right)\tau\right)}\max_{-\tau\le t\le 0}\left|\Phi_n(t)\right|=0,$$

the series $\sum_{n=1}^{\infty}B_{n3}\left(T\right)\sin\frac{\pi n}{l}x$ converges.

From the convergence of series $\sum_{n=1}^{\infty}B_{n1}\left(T\right)\sin\frac{\pi n}{l}x$, $\sum_{n=1}^{\infty}B_{n2}\left(T\right)\sin\frac{\pi n}{l}x$, $\sum_{n=1}^{\infty}B_{n3}\left(T\right)\sin\frac{\pi n}{l}x$ follows the convergence of series $S_2\left(x,t\right)$.

3. Now we will consider coefficients $C_n\left(t\right)$, $n=1,2,\ldots$ of the third series $S_3\left(x,t\right)$. For the fixed moment of time T: $(k-1)\tau\le T<k\tau$ we will make a substitution and write:

$$C_n\left(T\right)=\int_0^T\exp_\tau\left\{\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},T-\tau-s\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(T-s\right)}f_n\left(s\right)ds=$$

$$=\int_{-\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega+\tau\right)}f_n\left(T-\tau-\omega\right)d\omega=$$

$$=\int_{-\tau}^{0}\exp_\tau\left\{\varsigma_1e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\left(\omega+\tau\right)}f_n\left(T-\tau-\omega\right)d\omega+$$

$$+\int_0^{\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega+$$

$$+\int_{\tau}^{2\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega+$$

$$+\int_{(k-2)\tau}^{T-\tau}\exp_\tau\left\{\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau},\omega\right\}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega+\tau)}f_n\left(T-\tau-\omega\right)d\omega.$$

As follows from the mean value theorem, for each of integrals there are time moments

$$-\tau\le\omega_1\le 0,\ 0\le\omega_2\le\tau,\ \dots\ (k-2)\,\tau\le\omega_k\le T-\tau,$$

for which the following holds:

$$C_n\left(T\right)=\tau e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_1+\tau)}f_n\left(T-\tau-\omega_1\right)+$$

$$+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_2}{1!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_2+\tau)}f_n\left(T-\tau-\omega_2\right)+$$

$$+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_3}{1!}+\varsigma_1^2 e^{2\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_3-\tau\right]}{2!}+\varsigma_1^3 e^{3\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_3-2\tau\right]^3}{3!}\right]\times$$

$$\times e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_3+\tau)}f_n\left(T-\tau-\omega_3\right)+...+\tau\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_{k-1}}{1!}+...+\right.$$

$$\left.+\varsigma_1^{k-2}e^{(k-2)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_{k-1}-(k-3)\,\tau\right]^{k-2}}{(k-2)!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_{k-1}+\tau)}f_n\left(T-\tau-\omega_{k-1}\right)+$$

$$+\left[T-(k-1)\,\tau\right]\left[1+\varsigma_1 e^{\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\omega_k}{1!}+...+\right.$$

$$\left.+\varsigma_1^{k-1}e^{(k-1)\lambda_1\left(\frac{\pi n}{l}\right)^2\tau}\frac{\left[\omega_k-(k-2)\,\tau\right]^{k-1}}{(k-1)!}\right]e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k+\tau)}f_n\left(T-\tau-\omega_k\right).$$

Hence, we obtain that

$$S_3\left(x,T\right)=\sum_{n=1}^{\infty}C_n\left(T\right)\sin\frac{\pi n}{l}x=\sum_{n=1}^{\infty}\left\{\tau\sum_{i=1}^{k-1}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_i+\tau)}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k+\tau)}f_n\left(T-\tau-\omega_k\right)\right\}\times$$

$$\times\sin\frac{\pi n}{l}x-\varsigma_1\sum_{n=1}^{\infty}\left\{\tau\sum_{i=2}^{k-1}\frac{\omega_i}{1!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_i}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)\frac{\omega_k}{1!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2\omega_k}f_n\left(T-\tau-\omega_k\right)\right\}\sin\frac{\pi n}{l}x+$$

$$+\varsigma_1^2\sum_{n=1}^{\infty}\left\{\tau\sum_{i=3}^{k-1}\frac{\left[\omega_i-\tau\right]^2}{2!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_i-\tau)}f_n\left(T-\tau-\omega_i\right)+\right.$$

$$\left.+\left(T-(k-1)\,\tau\right)\frac{\left[\omega_k-\tau\right]^2}{2!}e^{-\lambda_1\left(\frac{\pi n}{l}\right)^2(\omega_k-\tau)}f_n\left(T-\tau-\omega_k\right)\right\}\sin\frac{\pi n}{l}x-$$

$$+ ... + \varsigma_1^{k-2} \sum_{n=1}^{\infty} \left\{ \tau \frac{[\omega_{k-1} - (k-3)\,\tau]^{k-2}}{(k-2)!} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (\omega_i - (k-3)\tau)} f_n\,(T - \tau - \omega_{k-1}) + \right.$$

$$\left. + [T - (k-1)\,\tau]\, \frac{[\omega_k - (k-3)\,\tau]^{k-2}}{(k-2)!} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (\omega_k - (k-3)\tau)} f_n\,(T - \tau - \omega_k) \right\} \sin \frac{\pi n}{l} x +$$

$$+ \varsigma_1^{k-1}\, [T - (k-1)\,\tau]\, \frac{[\omega_k - (k-2)\,\tau]^{k-1}}{(k-1)!} \times$$

$$\times \sum_{n=1}^{\infty} e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (\omega_k - (k-2)\tau)} f_n\,(T - \tau - \omega_k) \sin \frac{\pi n}{l} x.$$

And, if coefficients $f_n\,(t)$ satisfy the following condition

$$\lim_{n \to +\infty} \max_{-\tau \le t \le T - \tau} |f_n(t)|\, e^{-\lambda_1 \left(\frac{\pi n}{l}\right)^2 (T - (k-1)\tau)} = 0,$$

then the series $S_3(x,t)$ converges absolutely and uniformly.

Thus it was shown that for absolute and uniform convergence of the series $S_1(x,t)$, $S_2(x,t)$, $S_3(x,t)$ "fast reduction" on an index n of coefficients $\Phi_n\,(t)$, $-\tau \le t \le 0$ and $f_n\,(t)$, $0 \le t \le T$ is required.

Convergence of derivatives ξ'_t and ξ''_{xx} follows from the differentiability property of delay exponential function (Lemma 2.1). □

Proof of convergence of the series which represents the solution $\eta\,(x,t)$ is similar.

Corollary 3.1 *As the solutions $u\,(x,t)$, $v\,(x,t)$ are linear combinations of the functions $\xi\,(x,t)$, $\eta\,(x,t)$, they converge absolutely and uniformly, and their representations (31) are the solution of the boundary value problem of the initial system (1), (2).*

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Partial Control Design for Nonlinear Control Systems

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Abstract: This paper presents a general approach to design a partially stabilizing controller for nonlinear systems. In this approach, the nonlinear control system is divided into two subsystems, which are called the *first* and the *second* subsystems. This division is done based on the required stability properties of system's states. Furthermore, it is shown that partial control makes the possibility of converting the control problem into a simpler one by reducing the number of control input variables. The reduced input vector (the vector that includes components of input vector appearing in the *first* subsystem) is designed based on the new introduced control Lyapunov function called *partial control Lyapunov function* (PCLF) to asymptotically stabilize the first subsystem.

Keywords: *partial stability; partial control; partial control Lyapunov function (PCLF).*

Mathematics Subject Classification (2010): 34D20, 37N35, 70K99, 74H55, 93C10, 93D15.

1 Introduction

The problem of partial stability, that is stability with respect to a part of system's states, finds applications in many of engineering problems. In particular, partial stability arises in the study of inertial navigation systems, spacecraft stabilization via gimbaled gyroscopes or flywheels, electromagnetic, adaptive stabilization, guidance, etc. [1]– [14]. In the mentioned applications, although the plant may be unstable (in the standard concept), it might be *partially* asymptotically stable, i.e., some states may have convergent behavior. It is in contrast to many other engineering problems where Lyapunov stability (in its standard concept) is required [17]– [20]. For example, consider the equation of motion for the reaction wheel pendulum depicted in Figure 1 [15]:

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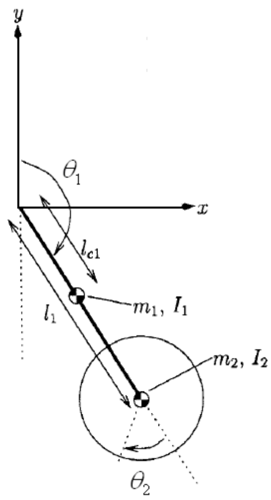

Figure 1: Coordinate convections for the reaction wheel pendulum [15].

$$\begin{aligned} & d_{11}\ddot{\theta}_1 + d_{12}\ddot{\theta}_2 + \phi\left(\theta_1\right) = 0, \\ & d_{21}\ddot{\theta}_1 + d_{22}\ddot{\theta}_2 = u, \end{aligned} \tag{1}$$

where θ_1 is the pendulum angle, θ_2 is the disk angle, u is the motor torque input and

$$\begin{aligned} & d_{11} = m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2, \\ & d_{12} = d_{21} = d_{22} = I_2, \\ & \phi(\theta_1) = -\bar{m} g \sin(\theta_1), \\ & \bar{m} = m_1 l_{c1} + m_2 l_1, \end{aligned} \tag{2}$$

where l_1 is the length of pendulum; l_{c1} is the position of the center of mass of the pendulum; m_1 is the mass of the pendulum; m_2 is the mass of disk; I_1, I_2 are the inertia of the pendulum and the disk around their center of masses. The reaction wheel pendulum is a physical pendulum with a symmetric disk attached to the end. The disk is free to spin about an axis which is parallel to the axis of rotation of the pendulum. Also, the disk is controlled by a DC-motor and the coupling torque generated by the angular acceleration of the disk can be used to actively control the system [15]. Suppose that a feedback control law should be designed so that $\dot{\theta}_1 \to 0$ and $\dot{\theta}_2$ be constant; that is, $\dot{\theta}_2\left(t\right) \to \Omega$ as $t \to \infty$ where $\Omega > 0$. This implies that $\theta_2\left(t\right) = \Omega t \to \infty$ as $t \to \infty$. Consequently, it is obvious that the reaction wheel pendulum is unstable in the standard concept; however, it is partially asymptotically stabilizable with respect to θ_1, $\dot{\theta}_1$ and $\dot{\theta}_2$.

Although partial stability has applications in many of engineering fields, there are a few papers regarding the design of control laws which stabilize only part of system's states [2]– [12] and advantages of partial control technique are not fully recognized. Among the existing papers in the field of partial control, most of them only consider a case study and try to design control laws for partial stability of their specific applications. Applications are Euler dynamical system [3], permanent rotations of a rigid body, relative equilibrium of a satellite, stationary motions of a gimbaled gyroscope [2] and chaos synchronization [7]. The references [2], [4], [9]– [11] focus on designing partial control and have given some way of designing. However, it is worth noting that the control schemes posed in these references are uneasy to realize and are usable only for systems with some special structures. In [12], a new class of nonlinear systems which is called "partially passive system" was introduced and some theorems for partial stabilization were developed.

In this paper, some new partial stabilization theorems for nonlinear dynamical systems are posed. It is shown that partial control makes the possibility of converting the control problem into a simpler one having fewer control input variables; which is one of the main contributions of this paper. In all of the existing papers in the field of partial control, the input vector is *wholly* designed; but in this paper by designing the reduced input vector, the advantage of partial control in simplifying the problem by reducing the control variables is recognized. The system's state is separated into two parts and accordingly the nonlinear dynamical system is divided into two subsystems. The subsystems, hereafter, are referred to as the "*first*" and the "*second*" subsystems. The reduced control input vector (the vector that includes components of input vector which appear in the *first* subsystem) is designed in such a way to guarantee asymptotic stability of the nonlinear system with respect to the first part of state vector. The design procedure is based on selection of a proper control Lyapunov function which is called partial control Lyapunov function. It's name is because that in this function only the first part of states is appeared.

The remainder of this paper is arranged as follows. First, the preliminaries on partial stability/control are given in Section 2. In Section 3, the theorems for partial control design are presented and explained in detail. Finally, conclusions are made in Section 4.

2 Preliminaries

In this section, the definitions and notations of partial stability are introduced. Consider a nonlinear system in the form;

$$\dot{x} = f(x), \qquad x(t_0) = x_0, \tag{3}$$

where $x \in R^n$ is the state vector. Let vectors x_1 and x_2 denote the partitions of the state vector, respectively. Therefore, $x = (x_1^T, x_2^T)^T$ where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$ and $n_1 + n_2 = n$. As a result, the nonlinear system (3) can be divided into two parts (the *first* and the *second* subsystems) as follows

$$\begin{aligned} \dot{x}_1(t) &= F_1(x_1(t), x_2(t)), \qquad x_1(t_0) = x_{10}, \\ \dot{x}_2(t) &= F_2(x_1(t), x_2(t)), \qquad x_2(t_0) = x_{20}, \end{aligned} \tag{4}$$

where $x_1 \in D \subseteq R^{n_1}$, D is an open set including the origin, $x_2 \in R^{n_2}$ and $F_1 : D \times R^{n_2} \to R^{n_1}$ is such that for every $x_2 \in R^{n_2}$, $F_1(0, x_2) = 0$ and $F_1(., x_2)$ is locally Lipschitz in x_1. Also, $F_2 : D \times R^{n_2} \to R^{n_2}$ is such that for every $x_1 \in D$, $F_2(x_1, .)$ is locally Lipschitz in x_2, and $I_{x_0} = [0,\ \tau_{x_0})$, $0 < \tau_{x_0} \leq \infty$ is the maximal interval of existence of solution $(x_1(t), x_2(t))$ of (4) $\forall t \in I_{\mathrm{x}_0}$. Under these conditions, the existence and uniqueness of solution is ensured. Now, stability of the dynamical system (4) with respect to x_1 can be defined as follows [5]:

Definition 2.1 1. The nonlinear system (4) is Lyapunov stable with respect to x_1 if for every $\varepsilon > 0$ and $x_{20} \in R^{n_2}$, there exists $\delta(\varepsilon, x_{20}) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$ for all $t \geq 0$.

2. The nonlinear system (4) is asymptotically stable with respect to x_1, if it is Lyapunov stable with respect to x_1 and for every $x_{20} \in R^{n_2}$, there exists $\delta = \delta(x_{20}) > 0$ such that $\|x_{10}\| < \delta$ implies $\lim_{t\to\infty} x_1(t) = 0$.

It is important to note that this partial stability definition (which is given in [5]) is different from past definitions of partial stability [1,4]. In past definitions, it is required that $F_1(0,0) = 0$ and $F_2(0,0) = 0$. Also, the initial condition of the whole system should be in a neighborhood of the origin which is not required in Definition 2.1. The main advantage of considering the condition $F_1(0, x_2) = 0$ for every x_2, is that it makes the possibility of investigating the partial stability even if a part of system's states goes to infinity. Using this fact, authors of [5] present the unification of partial stability theory for autonomous systems and stability theory for nonlinear time-varying systems. This unification allows the stability theory of time-varying systems to be presented as a special case of autonomous partial stability theory.

In order to analyze partial stability, the following theorem and its corollary are taken from [5]. Note that in the following theorem, $\dot V(x_1, x_2) = V'(x_1, x_2)F(x_1, x_2)$ where the row vector of $\partial V(x)/\partial x$ is shown by $V'(x)$ and $F(x_1, x_2) = \begin{bmatrix} F_1^T(x_1, x_2) & F_2^T(x_1, x_2) \end{bmatrix}^T$.

Theorem 2.1 *Consider the nonlinear dynamical system (4). If there exist a continuously differentiable function $V : D \times R^{n_2} \to R$ and class K functions $\alpha(.)$ and $\gamma(.)$ such that*

$$V(0, x_2) = 0, \qquad x_2 \in R^{n_2}, \tag{5}$$

$$\alpha(\|x_1\|) \le V(x_1, x_2), \qquad (x_1, x_2) \in D \times R^{n_2}, \tag{6}$$

$$\dot V(x_1, x_2) \le -\gamma(\|x_1\|), \qquad (x_1, x_2) \in D \times R^{n_2}, \tag{7}$$

then, the nonlinear dynamical system (4) is asymptotically stable with respect to x_1.

Proof. See [5]. □

Corollary 2.1 *Consider the nonlinear dynamical system (4). If there exist a positive definite continuously differentiable function $V : D \to R$, and a class K function $\gamma(.)$ such that*

$$V'(x_1)F_1(x_1, x_2) \le -\gamma\left(\|x_1\|\right), \qquad (x_1, x_2) \in D \times R^{n_2}, \tag{8}$$

then, the equilibrium point of the nonlinear dynamical system (4) is asymptotically stable with respect to x_1.

Now, consider the following autonomous nonlinear control system:

$$\begin{aligned} \dot x_1(t) &= F_1(x_1, x_2, \mathrm{u}(x_1, x_2)), \qquad x_1(t_0) = x_{10},\\ \dot x_2(t) &= F_2(x_1, x_2, \mathrm{u}(x_1, x_2)), \qquad x_2(t_0) = x_{20}, \end{aligned} \tag{9}$$

where $\mathrm{u} \in R^m$ and $F_1 : D \times R^{n_2} \times R^m \to R^{n_1}$ is such that for every $x_2 \in R^{n_2}$, $F_1(., x_2, .)$ is locally Lipschitz in x_1 and u. Also, $F_2 : D \times R^{n_2} \times R^m \to R^{n_2}$ is such that for every $x_1 \in D$, $F_2(x_1, ., .)$ is locally Lipschitz in x_2 and u. These assumptions guarantee the local existence and uniqueness of the solution of the differential equations (9).

Definition 2.2 The nonlinear control system (9) is said to be asymptotically stabilizable with respect to x_1, if there exists some admissible feedback control law $\mathrm{u} = k(x_1, x_2)$, which makes system (9) asymptotically stable with respect to x_1.

3 An Approach for Partial Control Design

This section presents a feasible design algorithm for partial stabilization of nonlinear systems. Suppose the $\dot{x}_1$-subsystem in Eq. (9) is affine with respect to the control input (the $\dot{x}_2$-equation may have a general dynamical form). Therefore,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + \textstyle\sum_{i=1}^{m} g_{1i}(x_1, x_2) u_i, \\ \dot{x}_2(t) &= F_2(x_1, x_2, \mathrm{u}), \end{aligned} \tag{10}$$

where u_i is the i^{th} component of input vector u. Also, $g_{1i} \in R^{n_1}$, for i=1,2,...,m. Let define r = number of $(g_{1i} \neq 0)_{i=1,\ldots,m}$. Hence, r is the number of components of input vector which appear in $\dot{x}_1$-subsystem. Thus $0 \leq \mathrm{r} \leq m$. Now, with respect to the value of r, two cases may be considered.

3.1 Case 1: $\mathrm{r} \neq 0$

By augmenting the r nonzero vectors g_{1i} in a matrix, the nonlinear system (10) can be rewritten as follows;

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + G_1(x_1, x_2)\mathrm{u}_1, \\ \dot{x}_2 &= F_2(x_1, x_2, \mathrm{u}), \end{aligned} \tag{11}$$

where $\mathrm{u}_1 \in R^{\mathrm{r}}$ is the reduced version of input vector u, that contains r control variables appearing in $\dot{x}_1$-subsystem, $G_1(x_1, x_2)$ is an $n_1 \times \mathrm{r}$ matrix where its columns are the r nonzero vectors g_{1i}. In this case, the task is to find an appropriate u_1, which guarantees partial stabilization of nonlinear system (11) with respect to x_1.

Theorem 3.1 *Consider the nonlinear dynamical system (11). Suppose $V(x_1) : D \to R$ is a positive definite continuously differentiable function (which is called partial control Lyapunov function) with the property that no solution x_1 of the unforced system* (11) *can stay identically in the set $\{V'(x_1) = 0\}$ other than the trivial solution $x_1(t) \equiv 0$. Also, suppose $\gamma(.)$ is class K function. Then, the system may be asymptotically stabilizable with respect to x_1 through the following reduced input vector*

$$\mathrm{u}_1 = k_1(x_1, x_2) = \begin{cases} \frac{b^T\{-V'(x_1)f_1 - \gamma(\|x_1\|)\}}{bb^T}, & where \;\; bb^T \neq 0, \\ 0, & where \;\; bb^T = 0, \end{cases} \tag{12}$$

where $b = V'(x_1)G_1(x_1, x_2)$. It is stressed that only in the points of state space $x_1 - x_2$ where $bb^T = 0$, the following condition should be satisfied:

$$V'(x_1)f_1(x_1, x_2) = -\gamma\left(\|x_1\|\right) \qquad \forall (x_1, x_2), \;\; \text{where } bb^T = 0. \tag{13}$$

Proof. By use of the control law (12), the time derivative of $V(x_1)$ in the line of system's trajectory is

$$\begin{aligned} \dot{V}(x_1) &= V'(x_1)\dot{x}_1 \\ &= V'(x_1)f_1 + V'(x_1)G_1 \left[\frac{(V'(x_1)G_1)^T\{-V'(x_1)f_1 - \gamma(\|x_1\|)\}}{(V'(x_1)G_1)(V'(x_1)G_1)^T} \right] \\ &= V'(x_1)f_1 + \{-V'(x_1)f_1 - \gamma\left(\|x_1\|\right)\} \\ &= -\gamma\left(\|x_1\|\right). \end{aligned} \tag{14}$$

Therefore, according to Corollary 2.1, the nonlinear system (11) is asymptotically stable with respect to x_1. For the case where $bb^T = 0$, if condition (13) is satisfied, then by taking $\mathrm{u}_1 = 0$, partial stability will be achieved. □

Note: When $V'(x_1) = 0$, then $bb^T = 0$. In the points where $bb^T = 0$, condition (13) should be satisfied, which results in $\gamma(\|x_1\|) = 0$. Since, $\gamma(.)$ is a class K function, thus $\gamma(\|x_1\|) = 0 \Rightarrow x_1 = 0$. Therefore, as mentioned in Theorem 3.1, $V(x_1)$ should be chosen in a way that $V'(x_1) = 0 \Rightarrow x_1 \equiv 0$.

3.2 Case 2: r = 0

This situation means that there is no component of input vector in $\dot{x}_1$-subsystem. Suppose that $\dot{x}_2$-subsystem is affine with respect to input. Therefore,

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2)\mathrm{u}. \end{aligned} \tag{15}$$

This system may be viewed as a cascade connection of two subsystems where x_2 is to be viewed as an input for *first* subsystem. The form (15) is usually referred to as the regular form. Assume that x_2 and u both belong to R^m (in other words, $n_2 = m$), and $G_2(x_1, x_2)$ is an m by m nonsingular matrix. This assumption is not so restrictive and many design methods, which are based on regular forms, e.g., backstepping or sliding mode techniques use such an assumption [16]. In this case, the task is to find an appropriate u; which guarantees partial stabilization of the closed-loop system.

Theorem 3.2 *Consider the nonlinear dynamical system (15). Suppose $V(x_1) : D \to R$ is a partial control Lyapunov function, $\gamma(.)$ is a class K function and $\varphi(x_1)$ is a smooth function. The design of the function $\varphi(x_1)$ is such that*

$$V'(x_1)(f_1(x_1, \varphi(x_1))) \le -\gamma(\|x_1\|). \tag{16}$$

Therefore, the nonlinear system (15) may be asymptotically stabilized with respect to x_1 by the following input vector

$$\mathrm{u} = G_2^{-1}[\varphi'(x_1) f_1 - f_2]. \tag{17}$$

Proof. Substitution of (17) in $\dot{x}_2$-subsystem (15) yields,

$$\begin{aligned} \dot{x}_2 &= f_2 + G_2\mathrm{u} \\ &= f_2 + G_2 G_2^{-1}[\varphi'(x_1) f_1 - f_2] \\ &= \varphi'(x_1) f_1 \end{aligned} \tag{18}$$

which results in $x_2 = \varphi(x_1)$. Since the condition (16) means that the *first* subsystem ($\dot{x}_1$-subsystem) may be asymptotically stabilized by a virtual input in the form $x_2 = \varphi(x_1)$ (according to Corollary 2.1). Therefore, the control law (17) partially stabilized the nonlinear system (15) with respect to x_1. □

3.3 Example. Partial stabilization of reaction wheel pendulum

The reaction wheel pendulum was described in Introduction. We define the states $z_1 = \theta_1$, $z_2 = \dot{\theta}_1$, $z_3 = \theta_2$ and $z_4 = \dot{\theta}_2$, The system's equations (1) can be written as follows

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\tfrac{d_{22}}{\det D}\phi(z_1) - \tfrac{d_{12}}{\det D}u, \\ \dot{z}_3 &= z_4, \\ \dot{z}_4 &= \tfrac{d_{21}}{\det D}\phi(z_1) + \tfrac{d_{11}}{\det D}u, \end{aligned} \tag{19}$$

where $\det D = d_{11}d_{22} - d_{12}d_{21} > 0$. The problem is to stabilize the downward position of the pendulum, that is $z_1 = 0$, $z_2 = 0$, while stability of the rest of states is not of interest. Therefore, the state vector $x = [z_1, z_2, z_3, z_4]^T$ can be divided into $x_1 = [z_1, z_2]^T$ and $x_2 = [z_3, z_4]^T$. By separating the states into x_1 and x_2, one has: r = 1 and $\mathrm{u}_1 = u$. The task is to design u according to Theorem 3.1 to achieve asymptotic stability with respect to x_1. Consider that for $\dot{x}_1$-subsystem $f_1 = \left[\begin{array}{cc} z_2 & -\frac{d_{22}}{\det D}\phi(z_1) \end{array}\right]^T$ and $G_1 = \left[\begin{array}{cc} 0 & -\frac{d_{12}}{\det D} \end{array}\right]^T$. By taking the partial control Lyapunov function $V(x_1) = 0.5\left(z_1^2 + z_1 z_2 + z_2^2\right)$ then $b = V'(x_1)G_1 = -\frac{d_{12}}{\det D}\left(z_2 + 0.5z_1\right)$. Therefore, the points $bb^T = 0$ are equal to the points $z_2 = -0.5z_1$. First of all, the condition (13) should be checked. The left side of condition (13) is:

$$V'(x_1)f_1|_{bb^T=0} = V'(x_1)f_1|_{z_2=-0.5z_1} = -\frac{3}{8}z_1^2. \tag{20}$$

By choosing $\gamma\left(\|x_1\|\right) = \alpha z_1^2 + \beta z_2^2$; $\alpha,\beta > 0$, the positive constants α and β may be chosen such that $V'(x_1)f_1|_{z_2=-0.5z_1} = -\,\gamma\left(\|x_1\|\right)|_{z_2=-0.5z_1} = -\frac{3}{8}z_1^2$. This condition is satisfied for example for $\alpha = 0.25$ and $\beta = 0.5$. Now, according to Theorem 3.1, u is:

$$u = \begin{cases} -\frac{\det D}{d_{12}} \frac{-(z_1+0.5z_2)z_2 + \frac{d_{22}}{\det D}(z_2+0.5z_1)\phi(z_1) - 0.25z_1^2 - 0.5z_2^2}{z_2+0.5z_1} & for\ z_2 \neq -0.5z_1, \\ 0 & for\ z_2 = -0.5z_1. \end{cases} \tag{21}$$

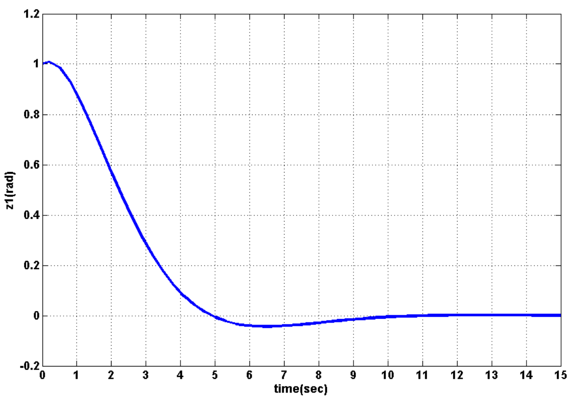

Figure 2: Time response of z_1 (the pendulum angle).

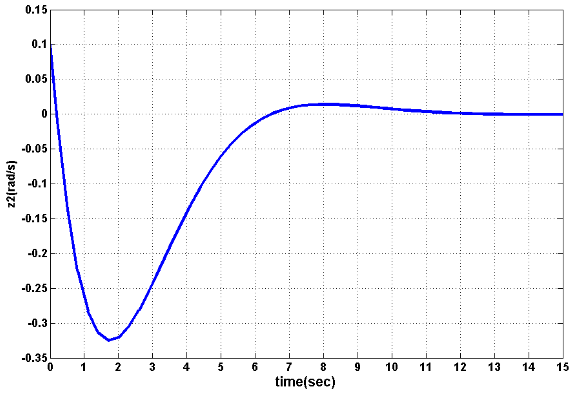

Fig gle).

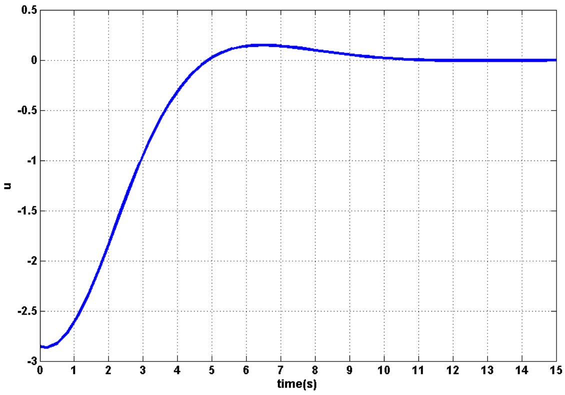

Figure 4: Time response of u (the motor torque input).

To check theoretical results, the closed loop system with controller (21) was simulated. The parameters of the system were chosen as $d_{11} = 0.004571$, $d_{22} = d_{12} = d_{21} = 2.495 \times 10^{-5}$, $\bar{m} = 0.35841$ that are physical parameters of the system located at the Automatic Control Dept., Lund Institute of Technology [15]. The initial conditions are

$$z_1(0) = 1, \quad z_2(0) = 0.1, \quad z_3(0) = z_4(0) = 0.$$

Figures 2 and 3 show the time responses of z_1 and z_2 in the closed loop system, respectively. As seen, the closed loop system shows quite fast convergence of z_1 and z_2 to zero. Also, the time response of controller (21) is shown in Figure 4.

4 Conclusion

In this paper, the problem of partial stabilization which has various applications in many of dynamic systems was considered and a general approach for stabilization of a nonlinear system with respect to a part of system's states was proposed. It was shown that in partial stabilization, the control input vector can be simplified by reducing its control variables. The reduced input vector was designed based on partial control Lyapunov function in a way that the asymptotic stabilization of a part of system's states was achieved. The proposed method was used in designing the partial controller for reaction wheel pendulum.

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Wavelet Neural Network Based Adaptive Tracking Control for a Class of Uncertain Nonlinear Systems Using Reinforcement Learning

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Abstract: In this paper an adaptive critic based wavelet neural network (WNN) based tracking control strategy for a class of uncertain systems in continous time is proposed. The adaptive critic WNN controller comprises two WNNs: critic WNN and action WNN. The critic WNN is approximating the strategic utility function, whereas the action WNN is minimizing both the strategic utility function and the unknown nonlinear dynamic estimation errors. Adaptation laws are developed for the online tuning of wavelets parameters. The uniformly ultimate boundedness of the closed-loop tracking error is verified even in the presence of WNN approximation errors and bounded unknown disturbances, using the Lyapunov approach and with novel weight updating rules. Finally some simulations are performed to verify the effectiveness and performance of the theoretical development.

Keywords: *wavelet neural networks; optimal control; adaptive control; reinforcement learning; Lyapunov functional.*

1 Introduction

Typical control strategies are based on a mathematical model that captures as much information as possible about the plant to be controlled. The ultimate objective is not to design the best controller for the plant model, but for the real time plant. This objective is addressed by robust control theory by including in the model a set of uncertainties. Robust control techniques are applied to the plant model, augmented with uncertainties and candidate controllers, to analyze the stability of the actual system. This is a powerful

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tool for practical controller design, but designing a controller that remains stable in the presence of uncertainties limits the aggressiveness of the resulting controller, and can result in suboptimal control performance [1, 2].

In this paper, the robust control techniques are combined with a reinforcement learning algorithm to improve the performance of robust controller while maintaining the stability of the system. Reinforcement learning is a class of algorithms for solving multi-step, sequential decision problems by finding a policy for choosing sequences of actions that optimize the sum of some performance criterion over time [3]– [7].

In recent years, learning-based control methodology using Neural networks (NNs) has become an alternative to adaptive control since NNs are considered as general tools for modeling nonlinear systems [17]. Work on adaptive NN control using the universal NN approximation property is now pursued by several groups of researchers. By using neural network (NN) as an approximation tool, the assumptions on linear parameterized nonlinearities in adaptive controller designing aspects have greatly been relaxed. It also broadens the class of the uncertain nonlinear systems which can be effectively dealt by adaptive controllers. However there are some difficulties associated with NN based controller. The basis functions are generally not orthogonal or redundant; i.e., the network representation is not unique and is probably not the most efficient one and the convergence of neural networks may not be guaranteed. Also the training procedure for NN may be trapped in some local minima depending on the initial settings. Wavelet neural networks are the modified form of the NN having the properties of space and frequency localization properties leading to a superior learning capabilities and fast convergence. Thus WNN based control systems can achieve better control performance than NN based control systems [8]– [10].

Adaptive actor-critic WNN-based control has emerged as a promising WNN approach due to its potential to find approximate solutions to dynamic programming [11]– [14]. In the actor-critic WNN based control a long-term system performance measure can be optimized, in contrast to the short-term performance measure used in classical adaptive and WNN control. While the role of the actor is to select actions, the role of the critic is to evaluate the performance of the actor. This evaluation is used to provide the actor with a signal that allows it to improve its performance, typically by updating its parameters along an estimate of the gradient of some measure of performance, with respect to the actor's parameters. The critic WNN approximates a certain strategics utility function that is similar to a standard Bellman equation, which is taken as the long-term performance measure of the system. The weights of action WNN are tuned online by both the critic WNN signal and the filtered tracking error. It minimizes the strategic utility function and uncertain system dynamic estimation errors so that the optimal control signal can be generated [3].

This paper deals with the designing of reinforcement learning WNN based adaptive tracking controller for a class of uncertain nonlinear systems. WNN are used for approximating the system uncertainty as well as to optimize the performance of the control strategy.

The paper is organized as follows. Section 2 deals with the system preliminaries, system description is given in Section 3. WNN based controller designing aspects are discussed in Section 4. Section 5 describes the tuning algorithm for actor-critic wavelets. The stability analysis of the proposed control scheme is given in Section 6. Effectiveness of the proposed strategy is illustrated through an example in Section 7 while Section 8 concludes the paper.

2 System Preliminaries

2.1 Wavelet neural network

Wavelet network is a type of building block for function approximation. The building block is obtained by translating and dilating the mother wavelet function. Corresponding to certain countable family ofa_m and $\mathrm{b_n}$, wavelet function can be expressed as

$$\left\{ a_m^{-d/2} \psi \left(\frac{x - b_n}{a_m} \right) : m \in Z, n \in Z^d \right\}. \tag{1}$$

Consider

$$a_m = a_0^m, b_n = n a_0^{-m} b_0, m \in Z, n \in Z^d. \tag{2}$$

The wavelet in (1) can be expressed as

$$\psi_{mn} = \left\{ a_0^{-md/2} \psi \left(a_0^{-m} x - n b_0 \right) : m \in Z, n \in Z^d \right\}, \tag{3}$$

where the scalar parameters a_0 and $\mathrm{b_0}$ define the step size of dilation and translation discretizations (typically a_0=2 and $\mathrm{b_0}$=1) and $x = [x_1, x_2, ..., x_n]^T \in R^n$ is the input vector.

Output of an n dimensional WNN with m wavelet nodes is

$$f = \sum_{m \in Z} \sum_{n \in Z^d} \alpha_{mn} \psi_{mn}. \tag{4}$$

3 System Description

Consider a nonlinear system of the form

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 = x_3, \\ &\vdots \\ &\dot{x}_n = f(x) + u, \\ &y = x_1, \end{aligned} \tag{5}$$

where $x = [x_1, x_2, ..., x_n]^T, u, y$ are state variable, control input and output respectively. $f = [f_1, f_2, ..., f_n]^T : \Re^{n+1} \to \Re^n$ are smooth unknown, nonlinear functions of state variables.

Rewriting the system (5) as

$$\begin{aligned} &\dot{x} = Ax + B(f(x) + u(t)), \\ &y = Cx, \end{aligned} \tag{6}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

The objective is to formulate a state feedback control law to achieve the desired tracking performance. The control law is formulated using the transformed system (6). Let $\bar{y}_d = [y_d, \dot{y}_d, \ldots, \overset{n-1}{y_d}]^T$ be the vector of desired tracking trajectory. The following assumptions are taken for the systems under consideration.

Assumption 3.1 Desired trajectory $y_d\,(t)$ is assumed to be smooth, continuous C^n and available for measurement.

4 Basic Controller Design Using Filtered Tracking Error

Define the state tracking error vector $\hat{e}(t)$ as

$$\hat{e}(t) = \hat{x}(t) - \bar{y}_d(t). \tag{7}$$

The filter tracking error is defined as

$$\hat{r} = K\hat{e},$$

where $K = [k_1, k_2, \ldots k_{n-1}, 1]$ is an appropriately chosen coefficient vector such that $\hat{e} \to 0$ exponentially as $\Re \to 0$. Differentiating it along the trajectory of the systems, we get

$$\dot{\hat{r}} = K_e\hat{e} + f(x) + u - \overset{n}{y_d}\,. \tag{8}$$

Applying the feedback linearization method, the control law is defined as

$$u = (\overset{n}{y_d} - \hat{f}(x) - K_e\hat{e} - \hat{r}), \tag{9}$$

where $K_e = [0, k_1, k_2, \ldots, k_{n-1}]$. Substituting (9) in (8),

$$\dot{\hat{r}} = -\hat{r} + \tilde{f}(x). \tag{10}$$

Stability of the system (6) with the proposed control strategy will be analyzed in the subsequent section.

5 Adaptive WNN Controller Design

A novel strategic utility function is defined as the long-term performance measure for the system. It is approximated by the WNN critic signal. The action WNN signal is constructed to minimize this strategic utility function by using a quadratic optimization function. The critic WNN and action WNN weight tuning laws are derived. Stability analysis using the Lyapunov direct method is carried out for the closed-loop system (6) with novel weight tuning updates.

5.1 Strategic utility function

The utility function $p(k) = [p_i(k)]_{i=1}^m \in \Re^m$ is defined on the basis of the filtered tracking error $\hat{r}$ and is given by [3]:

$$\begin{aligned} p_i(k) &= 0, \quad \text{if } \hat{r}_i^2 \le \eta, \\ \mathrm{p_i(k)} &= 1, \quad \text{if } \hat{r}_i^2 > \eta, \end{aligned} \tag{11}$$

where $p_i(k) \in \Re, i = 1, 2, \ldots, m$ and $\eta \in \Re$ is the predefined threshold, $p(k)$ can be considered as the current performance index. The long term system performance measure $Q'(k) \in \Re^m$ can be defined using the binary utility function as

$$Q'(k) = \alpha^N p(k+1) + \alpha^{N-1} p(k+2) + \ldots + \alpha^{k+1} p(N) + \ldots, \tag{12}$$

where $\alpha \in \Re$ and $0 < \alpha < 1$ and N is the horizon. Above equation may be rewritten as

$$Q(k) = \min_{u(k)} \{\alpha Q(k-1) - \alpha^{N+1} p(k)\}.$$

This measure is similar to standard Bellman's equation [15].

5.1.1 Critic WNN

$Q'(k)$is approximated by the critic WNN by defining the prediction error as

$$e_c(k) = \hat{Q}(k) - \alpha(\hat{Q}(k-1) - \alpha^N p(k)), \tag{13}$$

where $\hat{Q}(k) = \hat{w}_1^T(k)\phi_1(v_1^T x(k)) = \hat{w}_1^T(k)\phi_1(k)$, $e_c(k) \in \Re^m$, $\hat{Q}(k) \in \Re^m$ is the critic signal, $w_1(k) \in \Re^{n_1 \times m}$ and $v_1 \in \Re^{nm \times n_1}$ represent the weight estimates, $\phi_1(k) \in \Re^{n1}$is the wavelet activation function and n_1 is the number of nodes in the wavelet layer. The objective function to be minimized by the critic NN is defined as:

$$E_c(k) = \frac{1}{2} e_c^T(k) e_c(k). \tag{14}$$

The weight update rule for the critic NN is a gradient-based adaptation, which is given by [3]

$$\hat{w}_1(k+1) = \hat{w}_1(k) + \Delta\hat{w}_1(k),$$

where

$$\Delta\hat{w}_1(k) = \alpha_1 \left[-\frac{\partial E_c(k)}{\partial \hat{w}_1(k)} \right] \tag{15}$$

or

$$\hat{w}_1(k+1) = \hat{w}_1(k) - \alpha_1 \phi_1(k) \times (\hat{w}_1^T(k)\phi_1(k) + \alpha^{N+1} p(k) - \alpha \hat{w}_1^T(k-1)\phi_1(k-1))^T, \tag{16}$$

where $\alpha_1 \in \Re$ is the WNN adaptation gain. The critic WNN weights are tuned by the reinforcement learning signal and discounted values of critic WNN past outputs.

5.1.2 Action WNN

The output of the action NN is to approximate the unknown nonlinear function $f(x(k))$ and to provide an optimal control signal to be the part of the overall input $u(k)$ as

$$\hat{f}(k) = \hat{w}_2^T(k)\phi_2(v_2^T x(k)) = \hat{w}_2^T(k)\phi_2(k), \tag{17}$$

where $\hat{w}_2(k) \in \Re^{n_2 \times m}$ and $v_2 \in \Re^{nm \times n_2}$ represent the matrix of weight estimate, $\phi_2(k) \in \Re^{n_2}$ is the activation function, n_2 is the number of nodes in the hidden layer. Suppose that the unknown target output-layer weight for the action WNN is w_2 then we have

$$f(k) = w_2^T(k)\phi_2(v_2^T x(k))\varepsilon_2(x(k)) = w_2^T(k)\phi_2(k)\varepsilon_2(x(k)), \tag{18}$$

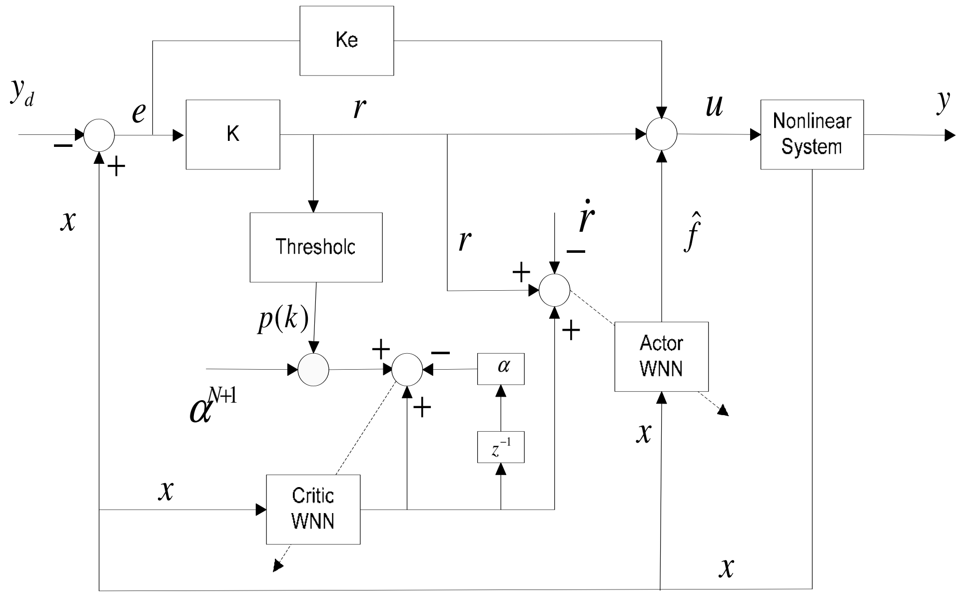

Figure 1: Block diagram of the closed loop system.

where $\varepsilon_2(x(k)) \in \Re^m$ is the WNN approximation error. Combining (17) and (18),

$$\tilde{f}(k) = \hat{f}(k) - f(k) = (\hat{w}_2(k) - w_2)^T \phi_2(k) - \varepsilon_2(x(k)), \tag{19}$$

where $\tilde{f}(k) \in \Re^m$ is the functional estimation error. The action WNN weights are tuned by using the functional estimation error$\tilde{f}(k)$and the error between the desired strategic utility function$Q_d(k) \in \Re^m$ and the critic signal $\hat{Q}(k)$ as shown in figure 2. Define

$$e_a(k) = \tilde{f}(k) + (\hat{Q}(k) - Q_d(k)). \tag{20}$$

The objective is to make the utility function $Q_d(k)$ zero at every step. Thus (20) becomes

$$e_a(k) = \tilde{f}(k) + \hat{Q}(k). \tag{21}$$

The objective function to be minimized by the action NN is given by

$$E_a(k) = \frac{1}{2} e_a^T(k) e_a(k). \tag{22}$$

The weight update rule for the action NN is also a gradient based adaptation, which is defined as

$$\hat{w}_2(k+1) = \hat{w}_2(k) + \Delta\hat{w}_2(k),$$

where

$$\Delta\hat{w}_2(k) = \alpha_2 \left[-\frac{\partial E_a(k)}{\partial \hat{w}_2(k)} \right] \tag{23}$$

or

$$\hat{w}_2(k+1) = \hat{w}_2(k) - \alpha_2 \phi_2(k)(\hat{Q}(k) + \tilde{f}(k))^T, \tag{24}$$

where $\alpha_2 \in \Re$ is the WNN adaptation gain.
The WNN weight updating rule in (24) cannot be implemented in practice since the nonlinear function $f(x(k))$ is unknown. However, using (10), the functional estimation error is given by

$$\tilde{f}(k) = \dot{\hat{r}} + \hat{r}. \tag{25}$$

Substituting (25) into (24), $\hat{w}_2(k+1) = \hat{w}_2(k) - \alpha_2 \phi_2(k)(\hat{Q}(k) + \dot{\hat{r}} - \hat{r})^T$.
Here the weight update for the action WNN is tuned by the critic WNN output, current filtered tracking error, and a conventional outer-loop signal as shown in Figure 2.

6 Stability Analysis

Consider a Lyapunov functional of the form

$$V = \frac{1}{2}\hat{r}^2. \tag{26}$$

Differentiating it along the trajectories of the system, we have

$$\dot{V} = \hat{r}(K_e \hat{e} + K(\hat{f}(\hat{x}) + u(t) - v_r - \overset{n}{\underset{d}{y}}).$$

By the substitution of control law $u(t)$ in the above equation,

$$\dot{V} = \hat{r}(-K\hat{r} + \tilde{f}(\hat{x}) - v_r).$$

$$\dot{V} \leq -K\hat{r}^2 + |\hat{r}| \left|\tilde{f}(\hat{x})\right| - \hat{r}v_r).$$

Substituting the robust control term $v_r = -\frac{(\rho^2+1)\hat{r}}{2\rho^2}$ in the above equation, we get

$$\dot{V} \leq -s_1\hat{r}^2 + s_2(|\hat{r}| \left|\tilde{f}(\hat{x})\right|)^2,$$

where $s_1 = (K + \frac{K}{2})$ and $s_2 = \frac{K\rho^2}{2}$. The system is stable as long as

$$s_1\hat{r}^2 \geq s_2(|\hat{r}| \left|\tilde{f}(\hat{x})\right|)^2. \tag{27}$$

7 Simulation Results

Simulation is performed to verify the effectiveness of proposed reinforcement learning WNN based control strategy. Consider a system of the form

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 = 0.01x_1 \sin x_2 + u, \\ &y = x_1. \end{aligned} \tag{28}$$

System belongs to the class of uncertain nonlinear systems defined by (5) with $n = 2$. The proposed controller strategy is applied to this system with an objective to solve the tracking problem of system. The desired trajectory is taken as $y_d = 0.5 \sin t +$

$0.1\cos 0.5t + 0.3$. Initial conditions are taken as $x(0) = [0.5, 0]^T$. Attenuation levels for robust controller are taken as 0.01. Controller gain vector is taken as $k = [35, 5]$. Wavelet networks with discrete Shannon's wavelet as the mother wavelet is used for approximating the unknown system dynamics. Wavelet parameters for these wavelet networks are tuned online using the proposed adaptation laws. Initial conditions for all the wavelet parameters are set to zero. Simulation results are shown in Figure 1. As observed from the figures, system response tracks the desired trajectory rapidly.

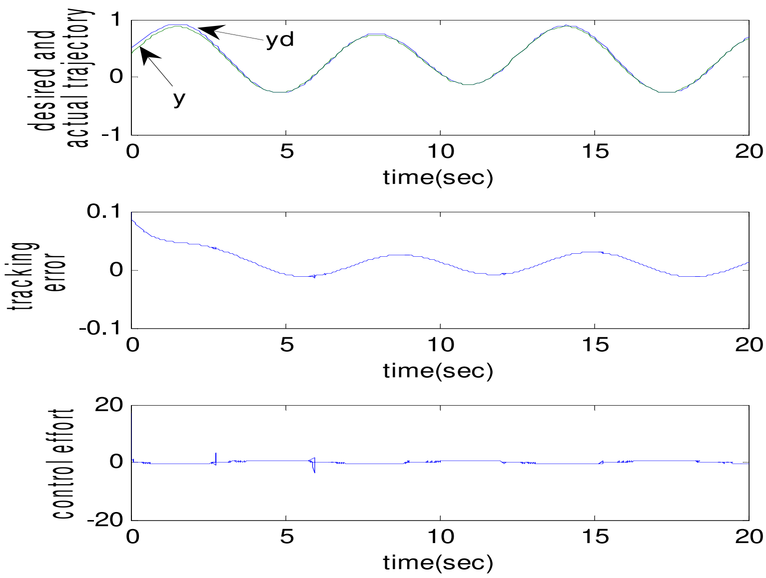

Figure 2: System output and tracking error.

8 Conclusion

A reinforcement learning WNN based adaptive tracking control strategy is proposed for a class of systems with unknown system dynamics. Adaptive wavelet networks are used for approximating the unknown system dynamics. Adaptation laws are developed for online tuning of the wavelet parameters. The stability of the overall system is guaranteed by using the Lyapunov functional. The theoretical analysis is validated by the simulation results.

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Sum of Linear Ratios Multiobjective Programming Problem: A Fuzzy Goal Programming Approach

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Abstract: Sum of ratios optimization is an interesting field of research. This paper presents a solution method for sum of linear ratios multiobjective programming (SOLR – MOP) problem using the fuzzy goal programming technique. Each membership function of fuzzy objectives is approximated into linear function by using first order Taylor's theorem about the vertex of the feasible region where the objective function has maximum value. Then the resulted approximated linearized membership functions may be used for the formulation of fuzzy goal programming. So the problem is solved using fuzzy goal programming technique. The efficiency of the method is measured by numerical examples.

Keywords: *multiobjective programming; fractional programming; fuzzy multiobjective fractional programming; sum of ratio fractional program; fuzzy goal programming.*

Mathematics Subject Classification (2010): 90C29, 90C32.

1 Introduction

Ratio criteria are used to measure the efficiency of a system in any different fields of engineering and management sciences. The ratio optimization problem is called the fractional programming. These may be applied to different disciplines such as financial sector, inventory management, production planning, banking sector and others. Basically it is used for modeling real life problems with one or more objectives such as debt/equity, profit/cost, inventory/sales, actual cost/standard cost, output/employees, nurses/patients ratios etc. with respect to some constraints.

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The ratio optimization problem with linear functions and linear constraints is called linear fractional programming (LFP) problem. If these problems have more than one objective then the problem is known as multiobjective linear fractional programming (MOLFP) problem.

If the ratio optimization problem has sum of linear ratios (ratios of affine functions), then the fractional programming problem (LFP) is known as sum of linear ratios programming (SOLR-P) problem.

A general sum of linear ratios programming (SOLR-P) problem is defined in the following way:

$$\begin{aligned}
&\text{Max} \quad F(x) = \text{Max}\left\{\sum_{j}^{p} \frac{f_j(x)}{m_j(x)}\right\} = \text{Max}\sum_{j}^{p} \frac{c_{0j}^T x + \alpha_{0j}}{d_{0j}^T x + \beta_{0j}} \\
&\text{subject to} \\
&x \in S, \;\; x \geq 0, \qquad\qquad (1) \\
&\text{where} \;\; p \geq 2, x, c_{0j},\, d_{0j} \in R^n, \;\; \alpha_{0j}\,, \beta_{0j} \;\in R.
\end{aligned}$$

The feasible region S is a nonempty, compact, convex set in R^n. The function $f_j(x) = c_{0j}^T x + \alpha_{0j}$, and $m_j(x) = d_{0j}^T x + \beta_{0j}$ are positive for all $x \in S$. Note that under these assumptions, the global maximum for problem (1) is attained by at least one point in S.

If we take more than one objectives in problem (1), then the problem is known as sum of linear ratios multiobjective programming (SOLR-MOP) problem, mathematically it can be written as:

$$\begin{aligned}
&\text{Max}\, F(x) = [F_1(x),\, F_2(x),\, \ldots F_k(x)], \text{where} \\
&F_i(x) = \sum_{j}^{p} \frac{f_{ij}(x)}{m_{ij}(x)}, \qquad\qquad (2) \\
&x \in S, \;\; x \geq 0, \;\; p \geq 2, \;\; x, \;\; c_{ij}, \;\; d_{ij} \in R^n, \;\; \alpha_{ij}\,, \beta_{ij} \;\in R.
\end{aligned}$$

and $f_{ij}(x) = c_{ij}^T x + \alpha_{ij}$, $m_{ij}(x) = d_{ij}^T x + \beta_{ij}$ are positive for all $x \in S$, where $S = \{x \,:\, Ax\,(\leq, =, \geq)\, b,\, x \geq 0,\, x \in R^n,\, b \in R^m,\, A \in R^{m \times n}\}$, $(i = 1, 2, \ldots, k,\; j = 1, 2, \ldots, p)\, \forall\, x \in S$. Here, S is assumed to be non-empty compact convex set in R^n and all $F_i(x)$ having continuous partial derivative in the feasible region S.

Sum of ratios fractional program was one of the least researched fractional program until about 1990. During last decade, interest in these programs has become especially strong. This is because, from a practical point of view, the sum of ratios fractional programs have numerous applications in the fields as discussed above but still multiobjective sum of ratios problem has least attention.

Various solution approaches have been proposed in the literature for sum of ratios fractional program. In [6], Cambini et. al. proposed a simplex type finite algorithm for the case $p = 2$ in problem (1) and find the global optimal solution. Later, Konno et. al.[13] proposed a finite parametric simplex type algorithm for the solution of linear sum of fractional programs. They give the minimization of the sum of two ratios.

In [4], Benson presented a branch- and - bound algorithm for globally solving the nonlinear sum of ratios problem. The algorithm has reduced the computational difficulty by conducting branch - and - bound search in R^p space rather than R^n space and the algorithm is applied in numerical examples for verification. Benson [3] proposed a branch - and - bound algorithm using the concave envelopes for the same problem.

In the algorithm, upper bounds are computed by maximizing concave envelopes of a sum of ratios function over intersection of the feasible region of the equivalent problem with rectangular sets systematically subdivided as branch and bound search procedure. The convergence of the algorithm is also presented and computational advantage is also highlighted. Other algorithms are also presented by Benson in [2, 23, 25].

In [8], Shen et. al. solved the sum of convex - convex ratios problem with non-convex feasible region. They used a branch bound scheme where the Lagrange duality theory is used to obtain lower bounds and the convergence of the algorithm is also proved. Shen and Wang [5] proposed also a branch bound algorithm for globally solving the sum of ratios with coefficients. They reduced the problem in equivalent sequence of linear programming problem by utilizing linearization technique.

In [10], Dür et. al. gave a branch bound solution algorithm for sum of ratios problem using rectangular partitions in Euclidean space of dimension p. For the bounding procedures, they used dual constructions and the calculation of efficient points of a corresponding multiobjective optimization problem.

Jaberipour and Khorram [11] proposed a harmony search algorithm for solving a sum - of - ratios problem. They also presented the numerical examples for demonstration, effectiveness and robustness of the proposed method and they claimed that all the solution obtained by their method are superior to those obtained by other methods.

In [16], Kuno developed a branch- and- bound algorithm for maximizing a sum of $p \geq 2$ linear ratios on a polytope. They embedded the problem in $2p$-dimensional space and constructed the bounding operations. The operations are carried out in p-dimensional space and rectangular branch bound method is used to find the solution. They also discussed the convergence criteria and also reviewed some computational results.

Konno and Yamashita [15] proposed a method to minimize the sums and products of linear fractional functions. They developed efficient deterministic algorithms for globally minimizing the sum and the product of several linear fractional functions over a polytope using outer approximation algorithm in given problem. They showed that the Charnes Cooper transformation plays an essential role in solving these problems. Also a simple bounding technique using linear multiplicative programming techniques has remarkable effects on structured problems.

In [14], Konno and Fukaisi presented a practical algorithm for solving low rank linear multiobjective programming problems and minimize the sum of product of two linear functions and also solved low rank linear fractional programming problems as minimization of sum of linear fractional functions over a polytope. Recently Gao et. al. [22] gave the extension of branch bound algorithm as maximization of sum of nonlinear ratios problem. They also presented the complexity of the problem and discussed some numerical experiments on the extended algorithm.

In [26], Gao and Shi presented a comprehensive review on branch - and bound algorithms for solving sum of ratios problem and they made a comparison between two branch-and bound approaches for solving the sum-of ratios problem. They also modify the algorithm for nonlinear sum-of ratios problem.

Multiobjective programming problems have been extensively studied for several decades and the research is based on the theoretical background. As a matter of fact many ideas and approaches have their foundation in the theory of fractional programming. Multiobjective linear fractional programming problems using fuzzy set theory has been studied in [19, 21, 27, 28]. Luhandjula [28] has given a solution method for MOLFP using linguistic approach. Dutta, Rao and Tewari [27] modified linguistic approach of

Luhandjula [28] to solve MOLFP using fuzzy set theoretic approach. Recently, Güzel and Sivri [20] have given Taylor series approach to solve MOLFP and in [16], they developed another approach. Toksari [21], developed an algorithm to solve FMOLFP by Taylor series approach and he linearized the membership functions instead of objection functions.

Fuzzy set theory becomes the efficient tool for solving various types of non-linear systems [30, 31, 32].

Our objective in this paper is to propose a simple method to the solution of sum of linear ratio multiobjective programming (SOLR-MOP)(2) problem using fuzzy goal programming approach. In this approach, each membership function associated with each objective of SOLR – MOP is approximated into linear function and then it is solved by fuzzy goal programming method. In the proposed article, we have attempted to handle multiobjective case for sum of linear ratios using fuzzy goal programming approach which is not attempted in the literature. The proposed algorithm is applied to three numerical examples.

2 Sum of Linear Ratios Fuzzy multiobjective Programming Problem (SOLR-FMOP)

If an uncertain aspiration level is introduced to each of the objectives of SOLR-MOP, then these fuzzy objectives are called fuzzy goals. The sum of linear ratios fuzzy multiobjective programming (SOR-FMOP) problem can be defined as

$$
\begin{aligned}
&\text{Find } X(x_1,\, x_2, ...x_n) \quad \text{such that}\\
&F_i(x) \lessapprox g_i \;\text{ or }\; F_i(x) \;\gtrapprox\; g_i\, \forall \quad (i = 1, 2, \dots, k,\; j = 1, 2, \dots, p) \qquad (3)\\
&\text{subject to}\\
&x \in S = \{x \in R^n,\, Ax(\leq, =, \geq)b, x \geq 0 \,\text{with}\; b \in R^m,\, A \in R^{m\times n}\},\\
&F_i(x) = \sum_{j}^{p} \frac{c_{ij}^T x + \alpha_{ij}}{d_{ij}^T x + \beta_{ij}},
\end{aligned}
$$

where g_i is the aspiration level of the i^{th} objective F_i and $\lessapprox, \gtrapprox$ indicate fuzziness of the aspiration level. The membership function $\mu_i(x)$ must be described for each fuzzy goal. A membership function can be explained as given below.
If $F_i(x) \lessapprox g_i$, then

$$
\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \leq g_i, \\ \dfrac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i}, & if \quad g_i \leq F_i(x) \leq \overline{t_i}, \\ 0 & if \quad F_i(x) \geq \overline{t_i}. \end{cases} \tag{4}
$$

If $F_i(x) \geq g_i$, then

$$
\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \gtrapprox g_i, \\ \dfrac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}}, & if \quad \underline{t_i} \leq F_i(x) \leq g_i, \\ 0 & if \quad F_i(x) \leq \underline{t_i}, \end{cases} \tag{5}
$$

and $\overline{t_i}$ and $\underline{t_i}$ are the upper tolerance limit and lower tolerance limit, respectively, for the i^{th} fuzzy goal. Then the problem (3) is called sum of linear ratios fuzzy multiobjective programming problem (SOLR-FMOP).

3 Goal Programming

The concept of goal programming (GP) was first introduced by Charnes and Cooper in 1961 [7] as a tool to resolve infeasible linear programming problems. Thereafter, significant methodological development of GP was made by Ignizio [18] and others. The overall purpose of GP is to minimize the deviations between the achievement of goals and their aspiration levels. A typical GP is expressed as follows

$$\begin{aligned}
&\text{Minimize} \sum_{i=1}^{k} |F_i(x) - g_i| \\
&\text{subject to} \\
&x \in X = \{x \in R^n;\ Ax \le b,\ x \ge 0\},
\end{aligned} \tag{6}$$

where F_{ij} is the linear function of the i^{th} goal and g_i is the aspiration level of the i^{th} goal.
Let $F_i(x) - g_i = d_i^+ - d_i^-, \qquad d_i^-, d_i^+ \ge 0$. Problem (6) can be formulated as follows

$$\begin{aligned}
&\text{Minimize} \sum_{i=1}^{k} (d_i^+ + d_i^-) \\
&\text{subject to} \\
&F_i(x) - d_i^+ + d_i^- - g_i = 0, \qquad i = 1, 2, \ldots k, \\
&d_i^+, d_i^- \ge 0,\ \ x \in X = \{x \in R^n;\ Ax \le b,\ x \ge 0\},
\end{aligned} \tag{7}$$

where $d_i^- \ge 0$, $d_i^+ \ge 0$ are, respectively under - and over - deviations of the i^th goal. Problem (7) has been applied to solve many real world problems.

3.1 Fuzzy goal programming

In fuzzy goal programming approaches, the highest degree of membership function is 1. So, for the defined membership function in (4) and (5), the flexible membership goals with aspiration levels 1 can be expressed as

$$\frac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}} + d_i^- - d_i^+ = 1 \qquad \text{or} \qquad \frac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i} + d_i^- - d_i^+ = 1, \tag{8}$$

where $d_i^- \ge 0$, $d_i^+ \ge 0$ with $d_i^+ . d_i^- = 0$ are, respectively, under - and over -deviations from the aspiration levels.

In conventional GP, the under- and over-deviational variables are included in the achievement function or minimized and that depends upon the type of the objective functions to be optimized.

In this approach, only the under - deviational variable d_i^- is required to achieve the aspired levels of the fuzzy goals. It may be noted that any over - deviation from fuzzy goal indicates the full achievement of the membership value. Recently, B. B. Pal. et.al [19] proposed an efficient goal programming (GP) method for solving fuzzy multiobjective linear fractional programming problems.

4 Mathematical Modeling of Problem

We consider the sum of linear ratios multiobjective programming (SOLR-MOP) problem of the form:

$$\begin{aligned}
&\text{Max}\, F(x) = \{\, F_1(x),\, F_2(x),\, \ldots, F_k(x)\,\}\\
&\quad F_i(x) = \sum_{j}^{p} \frac{c_{ij}^T x + \alpha_{ij}}{d_{ij}^T x + \beta_{ij}}, \qquad (9)\\
&\text{where} \quad d_{ij}^T x + \beta_{ij} > 0,\, \forall \quad (i = 1,\, 2,\, \ldots, k,\; j = 1, 2, \ldots p)\\
&\text{subject to}\\
&x \in S = \{Ax \le\, b,\, x \ge\, 0,\, x,\, c_{ij}^T,\, d_{ij}^T, \in R^n,\, b \in R^m,\\
&A = (m \times n)\text{ matrix},\, \alpha_{ij},\, \beta_{ij},\, \in R\}.
\end{aligned}$$

Assume fuzzy aspiration level g_i and tolerance limit $(\overline{t_i},\, \underline{t_i})$ for each objective function $F_i(x)$. We construct the membership function for each objective function using Zimmermann max-min approach [29]. Then the problem (9) becomes

$$\begin{aligned}
&\text{Find} \quad X(x_1,\, x_2, \ldots, x_n)\\
&\text{so as to satisfy}\\
&F_i(x) \lessapprox g_i\\
&\text{or} \qquad (10)\\
&F_i(x) \gtrapprox g_i\\
&\text{subject to} \qquad x \in S = \{x \in R^n,\, Ax \le b, x \ge 0 \text{ with } b \in R^m,\, A \in R^{m\times n}\}\\
&\text{and} \qquad F_i(x) = \sum_{j}^{p} \frac{c_{ij}^T x + \alpha_{ij}}{d_{ij}^T x + \beta_{ij}}, \text{ where} \quad d_{ij}^T x + \beta_{ij} > 0, \forall\, i \text{ and } j,
\end{aligned}$$

where g_i is the aspiration level of the i^{th} objective function $F_i(x)$. The membership function $\mu_i(x)$, described for each fuzzy goal, is given by equation (4) and equation (5). Suppose that all $F_i(x)$ and all of their partial derivatives of order less than or equal to $n+1$ are continuous on the feasible region S. So the membership functions $\mu_i(x)$ of each $F_i(x)$ are having same property in the feasible region.

The proposed algorithm can be explained in three steps and linear approximation of membership functions is motivated by Toksari [21].

Step 1: Determine the vertex of the feasible region, $x_q^* = \{\, x_{q1}^*,\, x_{q2}^*,\, \ldots, x_{qn}^*\}$ for which the i^{th} membership function is maximized associated with the i^{th} objective $F_i(x), \forall\; i = 1,\, 2,\, \ldots, k$ and $j = 1, 2, \ldots, p,$ where n is the number of variable and q is finite.

Step 2: Transform each fractional membership function into linear membership function by using first order Taylor's theorem

$$\mu_i(x) = \widetilde{\mu_i}(x) \cong \mu_i(x_q^*) + [(x_1 - x_{q1}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_1} + (x_2 - x_{q2}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_2} + \dots \quad (11)$$

$$+ (x_n - x_{qn}^*)\frac{\partial \mu_i(x_{q^*})}{\partial x_n}] + O(h^2),$$

$$\mu_i(x) = \widetilde{\mu_i}(x) \cong \mu_i(x_q^*) + \sum_{j=1}^{n}[(x_j - x_{qj}^*)\frac{\partial \mu_i(x_q^*)}{\partial x_j}] + O(h^2), \quad (12)$$

where, if $F_i(x) \lessapprox g_i$, then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \leq g_i, \\ \dfrac{\overline{t_i} - F_i(x)}{\overline{t_i} - g_i}, & if \quad g_i \leq F_i(x) \leq \overline{t_i}, \\ 0 & if \quad F_i(x) \geq \overline{t_i}. \end{cases} \quad (13)$$

If $F_i(x) \geq g_i$, then

$$\mu_i(x) = \begin{cases} 1, & if \quad F_i(x) \geq g_i, \\ \dfrac{F_i(x) - \underline{t_i}}{g_i - \underline{t_i}}, & if \quad \underline{t_i} \leq F_i(x) \leq g_i, \\ 0 & if \quad F_i(x) \leq \underline{t_i}. \end{cases}$$

subject to

$x \in X = \{Ax \leq b, x \geq 0, x, c_{ij}^T, d_{ij}^T, \in R^n, b \in R^m,$
$A = (a_{ij})_{m\times n}, \alpha_{ij}, \beta_{ij}, \in R\}$.

Now in (12), these are linearized approximated membership function of fuzzy objectives. Then the problem can be solved by assuming fuzzy goals.

Step 3: Find $x^* = \{x_1^*, x_2^*, \dots, x_n^*\}$ using fuzzy goal formulation. Apply fuzzy goal programming approach for the linearized membership functions $\widetilde{\mu_i}(x)$ in (12) of F_i. The flexible membership goals with aspiration levels 1 can be expressed as

$$\widetilde{\mu_i}(x) + d_i^- - d_i^+ = 1, \quad (14)$$

where $d_i^-, d_i^+ \geq 0$, with $d_i^+ d_i^+ = 0$ are respectively under- and over- deviations from the aspiration levels.

Now the fuzzy goal programming formulation can be obtained as

$$\begin{aligned} &\text{Minimize} \sum_{i=1}^{k} d_i^- \\ &\text{subject to} \\ &\widetilde{\mu_i}(x) - d_i^+ + d_i^- = 1, \qquad i = 1, 2, \dots k \\ &d_i^+, d_i^- \geq 0 \\ &x \in S = \{x \in R^n; Ax \leq b, x \geq 0\} \text{ with } d_i^+ . d_i^+ = 0. \end{aligned} \quad (15)$$

In the problem (15), S is a non empty convex bounded set having feasible points . The LPP (15) can be solved easily, which gives the efficient solution of (SOLR-MOP)

(3). The values of membership functions at the optimal point gives the satisfaction level (degree) of objective function to the solution.

5 Numerical Examples

Example 1: Consider a SOLR-MOP with two objective functions:

$$\begin{aligned}
&\text{Max}\left\{\frac{x_1+2x_2}{2x_1+x_2+5}+\frac{9x_1+2x_2}{7x_1+3x_2+1},\quad \frac{2x_1+3x_2+5}{x_1+1}+\frac{5x_1+4x_2}{x_1+x_2}\right\}\\
&\text{subject to}\\
&\qquad x_1-x_2\geq 2,\\
&\qquad 4x_1+5x_2\leq 25,\\
&\qquad x_1+9x_2\geq 9,\\
&\qquad x_1\geq 5,\\
&\qquad x_1,\, x_2\geq 0.
\end{aligned} \tag{16}$$

It is observed that $f_{ij} \geq 0$, $m_{ij} \geq 0, (i = 1, 2$ and $j = 1, 2)$ for each x in the feasible region.

If the fuzzy aspiration levels of the two objectives are 1.806, and 7.83, then find x in order to satisfy the following fuzzy goals:

$$\left(\frac{x_1+2x_2}{2x_1+x_2+5}+\frac{9x_1+2x_2}{7x_1+3x_2+1}\right)\gtrapprox 1.806,\quad \left(\frac{2x_1+3x_2+5}{x_1+1}+\frac{5x_1+4x_2}{x_1+x_2}\right)\gtrapprox 7.83.$$

The tolerance limits for the two fuzzy goals are (1.620, 7.05) respectively. The membership functions for the two fuzzy goals are

$$\mu_1(x)=\begin{cases}1, & if \quad F_1(x)\geq g_i,\\ \dfrac{F_i(x)-\underline{t_i}}{g_i-\underline{t_i}}, & if \quad \underline{t_i}\leq F_i(x)\leq g_i,\\ 0, & if \quad F_i(x)\leq \underline{t_i}.\end{cases}$$

i.e.

$$\mu_1(x)=\begin{cases}1, & if \quad F_1(x)\geq 1.806,\\ \dfrac{\left(\dfrac{x_1+2x_2}{2x_1+x_2+5}+\dfrac{9x_1+2x_2}{7x_1+3x_2+1}\right)-1.620}{0.19}, & if\ 1.620\leq F_1(x)\leq 1.806,\\ 0, & if \quad F_1(x)\leq 1.620.\end{cases} \tag{17}$$

$$\mu_2(x)=\begin{cases}1, & if \quad F_2(x)\geq 7.83,\\ \dfrac{\left(\frac{2x_1+3x_2+5}{x_1+1}+\dfrac{5x_1+4x_2}{7x_1+3x_2+1}\right)-7.05}{0.78}, & if\ 7.05\leq F_2(x)\leq 7.83,\\ 0, & if \quad F_2(x)\leq 7.05.\end{cases} \tag{18}$$

Expand the membership functions $\mu_1(x)$ about point (5, 0.44) and $\mu_2(x)$ about point (5, 1)

$$\mu_1(x) \cong \widetilde{\mu_1}(x) = \mu_1(5,\, 0.44) + (x_1 - 5)\frac{\partial \mu_1(5,\, 0.44)}{\partial x_1} + (x_2 - 0.44)\frac{\partial \mu_1(5,\, 0.44)}{\partial x_2},$$
$$\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.14x_1 + 0.54x_2 + 0.06, \tag{19}$$
$$\mu_2(x) \cong \widetilde{\mu_2}(x) = \mu_2(5,\, 1) + (x_1 - 5)\frac{\partial \mu_2(5,\, 1)}{\partial x_1} + (x_2 - 1),\frac{\partial \mu_2(5,\, 1)}{\partial x_2},$$
$$\mu_2(x) \cong \widetilde{\mu_2}(x) = -0.18x_1 + 0.50x_2 + 1.4. \tag{20}$$

Now apply the fuzzy goal programming technique:

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^-)\\
&\ \text{subject to}\\
&\widetilde{\mu_1}(x) - d_1^+ + d_1^- = 1,\\
&\widetilde{\mu_2}(x) - d_2^+ + d_2^- = 1,\\
&d_1^-, d_1^+, d_2^-, d_2^+ \ge 0,\\
&x \in S = \{x \in R^n;\ Ax \le b,\ x \ge 0\} \text{ with } d_1^+.d_1^+ = 0 \,\text{and}\, d_2^+.d_2^+ = 0.
\end{aligned} \tag{21}$$

Thus new LPP is obtained

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^-)\\
&\ \text{subject to}\\
&0.14x_1 + 0.54x_2 - d_1^+ + d_1^- = 0.94,\\
&-0.18x_1 - 0.50x_2 - d_2^+ + d_2^- = -0.4,\\
&\quad x_1 - x_2 \ge 2,\\
&\quad 4x_1 + 5x_2 \le 25,\\
&\quad x_1 + 9x_2 \ge 9,\\
&\quad x_1 \ge 5,\\
&\quad x_1,\, x_2 \ge 0, \quad \text{with} \quad d_1^+.d_1^+ = 0 \text{ and } d_2^+.d_2^+ = 0.
\end{aligned} \tag{22}$$

The optimal solution of the above problem is given by $x_1 = 5$, $x_2 = 1$, $d_1^- = 0$, $d_1^+ = 0.30$, $d_2^- = 0$, $d_2^+ = 0$ and the membership values are $\mu_1 = 0.12$, $\mu_2 = 1$. The optimal solution of the problem (22) is at the point $(5, 1)$ and minimum value is 0. The point (5, 1) is the efficient solution of the given original problem in the feasible region with optimal values of the functions $F_1 = 1.643, F_2 = 7.83$. The membership function values at (5, 1) indicate that goals F_1 and F_2 are satisfied 12% and 100% respectively, for the obtained solution.

Example 2: Let us consider a SOLR - MOP with three objective functions

$$\text{Max} \quad \{F_1(x) = \frac{x_1 + 4x_2}{2x_1 + x_2 + 1} + \frac{9x_1 + 2x_2}{x_1 + 3x_2 + 1} + \frac{x_1 + 3x_2}{x_2 + 1}, \tag{23}$$
$$F_2 = \frac{3x_1 + 8x_2}{x_1 + x_2 + 3}, + \frac{x_1 + 2x_2}{2x_1 + 3x_2 + 2} + \frac{x_1 + 2x_2}{3x_1 + x_2 + 2}\}$$

subject to

$$\begin{aligned} & x_1 - x_2 \geq 5, \\ & 4x_1 + 5x_2 \leq 25, \\ & x_1 \geq 5, \\ & x_1, \, x_2 \geq 0. \end{aligned} \tag{24}$$

If the fuzzy aspiration levels of two objectives are (9.08, 2.76) respectively, then find x in order to satisfy the following goals:

$$F_1(x) \gtrapprox 9.08, \quad F_2(x) \gtrapprox 2.76. \tag{25}$$

The tolerance limits for the three fuzzy goals are $(8.79, 2.51)$ respectively. The membership functions for the two fuzzy goals are given by

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \quad \geq 9.08, \\ \dfrac{\frac{x_1+4x_2}{2x_1+x_2+1} + \dfrac{9x_1 + 2x_2}{x_1 + 3x_2 + 1} + \dfrac{x_1 + 3x_2}{x_2 + 1} - 8.79}{0.29}, & if \; 8.79 \leq F_1(x) \leq 9.08, \\ 0 & if \quad F_1(x) \leq 8.79. \end{cases} \tag{26}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \quad \geq 2.76, \\ \dfrac{\frac{3x_1+8x_2}{x_1+x_2+3}, + \dfrac{x_1 + 2x_2}{2x_1 + 3x_2 + 2} + \dfrac{x_1 + 2x_2}{3x_1 + x_2 + 2} - 2.51}{0.25}, & if 2.51 \leq F_2(x) \leq 2.76, \\ 0 & if \quad F_2(x) \leq 2.51. \end{cases} \tag{27}$$

Both membership functions are expanded by using first order Taylor's theorem about the point (6.25, 0) in the feasible region. The linearized forms of membership functions are obtained

$$\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.68x_1 + 4.47x_2 - 3.25, \tag{28}$$
$$\mu_2(x) \cong \widetilde{\mu_2}(x) = 0.48x_1 + 3.09x_2 - 2. \tag{29}$$

Now apply the fuzzy goal programming technique and the new LPP is obtained

$$\begin{aligned} & \text{Minimize } (d_1^- + d_2^-) \\ & \text{ subject to} \\ & 0.68x_1 + 4.47x_2 - d_1^+ + d_1^- = 4.25, \\ & 0.48x_1 + 3.09x_2 - d_2^+ + d_2^- = 3, \\ & \quad x_1 - x_2 \geq 5, \\ & \quad 4x_1 + 5x_2 \leq 25, \\ & \quad x_1 \geq 5, \\ & \quad x_1, \, x_2 \geq 0. \quad \text{with} \quad d_1^+ . d_1^+ = 0 \text{ and } d_2^+ . d_2^+ = 0. \end{aligned} \tag{30}$$

The alternate optimal solution is obtained but the best minimum value is 0 at $x_1 = 5.56$, $x_2 = 0.56$, $d_1^- = 0$, $d_1^+ = 0.51$, $d_2^- = 0.01$, $d_2^+ = 0$ and the membership values are $\mu_1 = 0$, $\mu_2 = 1$. So the optimal solution of problem (31) is at $(5.56, 0.56)$. The point (5.56, 0.56) is the efficient solution of the given original problem in the feasible region with optimal values of the functions $F_1 = 7.93, F_2 = 3.12$. The membership function values at (5.56, 0.56) indicate that goals F_1 and F_2 are satisfied 0% and 100% respectively, for the obtained solution.

Example 3: Let us consider a SOLR - MOP with three objective functions

$$\begin{aligned}
\text{Max} \quad \{ & F_1(x) = \frac{x_1}{x_2+1} + \frac{x_2}{2x_1+3}, \\
& F_2(x) = \frac{x_2+4}{x_1+2x_2+1} + \frac{x_1+2}{3x_1+x_2+2}, \\
& F_3(x) = \frac{x_1+2x_2}{x_1+3x_2+2} + \frac{5x_1+x_2}{2x_1+5x_2+3} \} \\
\text{subject to} & \\
& x_1 \le 6, \\
& x_2 \le 6, \\
& 2x_1 + x_2 \le 9, \\
& -2x_1 + x_2 \le 5, \\
& x_1 - x_2 \le 5, \\
& x_1, x_2 \ge 0.
\end{aligned} \tag{31}$$

If the fuzzy aspiration levels of the three objectives are (4.5, 5, 2.57) respectively, then

$$F_1(x) \gtrapprox 4.5, \quad F_2(x) \gtrapprox 5, \quad F_3(x) \gtrapprox 2.57. \tag{32}$$

The tolerance limits for the two fuzzy goals are 0, 0.86, 0 respectively. The membership functions for the three fuzzy goals are

$$\mu_1(x) = \begin{cases} 1, & if \quad F_1(x) \ge 4.5, \\ \dfrac{\dfrac{x_1}{x_2+1} + \dfrac{x_2}{2x_1+3} - 0}{4.5}, & if \quad 0 \le F_1(x) \le 4.5, \\ 0, & if \quad F_1(x) \le 0. \end{cases} \tag{33}$$

$$\mu_2(x) = \begin{cases} 1, & if \quad F_2(x) \ge 5, \\ \dfrac{\dfrac{x_2+4}{x_1+2x_2+1} + \dfrac{x_1+2}{3x_1+x_2+2} - 0.86}{4.14}, & if \quad 0.86 \le F_2(x) \le 5, \\ 0 & if \quad F_2(x) \le 0.86. \end{cases} \tag{34}$$

$$\mu_3(x) = \begin{cases} 1, & if \quad F_3(x) \ge 2.57, \\ \dfrac{\dfrac{x_1+2x_2}{x_1+3x_2+2} + \dfrac{5x_1+x_2}{2x_1+5x_2+3} - 0}{2.57}, & if \quad 0 \le F_3(x) \le 2.57, \\ 0, & if \quad F_3(x) \le 0. \end{cases} \tag{35}$$

By expanding the first order Taylor's theorem for membership functions μ_1, μ_2 and μ_3 about points (4.5, 0), (0, 0) and (4.5, 0) respectively in the feasible region:

$$\mu_1(x) \cong \widetilde{\mu_1}(x) = 0.22x_1 - 4.22x_2 + 0.01, \tag{36}$$

$$\mu_2(x) \cong \widetilde{\mu_2}(x) = 1.33x_1 - 1.69x_2 + 1, \tag{37}$$

$$\mu_3(x) \cong \widetilde{\mu_3}(x) = 0.059x_1 - 0.71x_2 + 0.74. \tag{38}$$

Apply fuzzy goal programming technique, the new LPP is obtained

$$\begin{aligned}
&\text{Minimize } (d_1^- + d_2^- + d_3^-) \\
&\ \text{subject to} \\
&0.22x_1 - 4.22x_2 - d_1^+ + d_1^- = 0.99, \\
&1.33x_1 - 1.69x_2 - d_2^+ + d_2^- = 0, \\
&0.059x_1 - 0.71x_2 - d_3^+ + d_3^- = 0.26, \\
&\quad x_1 \le 6, \\
&\quad x_2 \le 6, \\
&\quad 2x_1 + x_2 \le 9, \\
&\quad -2x_1 + x_2 \le 5, \\
&\quad x_1 - x_2 \le 5, \\
&\quad x_1,\, x_2 \ge 0, \quad \text{with} \quad d_1^+.d_1^+ = 0,\, d_2^+.d_2^+ = 0 \text{ and } d_3^-.d_3^+ = 0.
\end{aligned} \tag{39}$$

Optimal solution of the problem (40) is at the point $x_1 = 4.5$, $x_2 = 0$, $d_1^- = 0$, $d_2^- = 0$, $d_3^- = 0, d_1^+ = 0$, $d_2^+ = 5.99\, d_3^+ = 0.01$ and the minimum value is 0. The efficient solution of the given problem is $x_1 = 4.5, x_2 = 0$, $F_1 = 4.5$, $F_2 = 0.86$, $F_3 = 2.57$ and the membership values are $\mu_1 = 1,\ \mu_2 = 0,\ \mu_3 = 1$. The membership function values at (4.5, 0) indicate that goals F_1, F_2 and F_3 are satisfied 100%, 0% and 100% respectively, for the obtained solution.

6 Conclusion

In this paper, a new algorithm has been proposed to optimize sum of linear ratios multiobjective programming (SOLR-MOP)problem using fuzzy set theory and goal programming method. Most of the reported work is based on the single objective optimization. So, the proposed algorithm is a simple procedure to optimize sum of linear ratios in multiobjective case. This reduces computational complexity as compared to the previous reported work.

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Approximate Controllability of Nonlocal Semilinear Time-varying Delay Control Systems

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Abstract: In this work the controllability problem for a class of semilinear control system with nonlocal initial conditions is considered. Under some simple conditions the relation between the reachable set of semilinear system and that of its corresponding linear system is established. In particular, approximate controllability of semilinear abstract control system is proved. Examples are presented to explain the application of the proposed result.

Keywords: *infinite-dimensional spaces; semilinear time-varying delay systems; approximate controllability; nonlocal conditions.*

Mathematics Subject Classification (2010): 93B05.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{C}_t = C([-\tau, t]; X), \tau > 0, 0 \leq t \leq T < \infty$, be a Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the norm $||\phi||_{\mathcal{C}_t} = \sup_{-\tau \leq \eta \leq t} ||\phi(\eta)||$. Now, consider the following nonlocal semilinear delay control system

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) + f(t, x(t), x_{b(t)}) \text{ on } (0, T], \\ h(x) &= \phi \text{ on } [-\tau, 0], \end{aligned} \tag{1}$$

where the state variable $x(\cdot)$ takes values in Banach space X and the control function $u(\cdot)$ belongs to $Y = L^2([0, T]; U)$, the Banach space of admissible control functions with a Banach space U. Standing assumptions on system operators are as follows:

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(H1) $A : X \supset D(A) \to X$ is a linear operator such that it generates a C_0-semigroup on X, denoted by $S(t) : t \geq 0$. Let $M \geq 1$ and $\omega \geq 0$ be such that $||S(t)|| \leq Me^{\omega t}$; $t \geq 0$.

(H2) $b : [0, T] \to [0, T]$ is a map such that it satisfies the property $b(t) \leq t$, $\forall t \in [0, T]$. For a continuous function $x \in \mathcal{C}_T$ and $t \in [0, T]$, $x_{b(t)} \in \mathcal{C}_0$ and is defined by $x_{b(t)}(\theta) = x(b(t) + \theta)$; $\theta \in [-\tau, 0]$.

(H3) $h : \mathcal{C}_0 \to \mathcal{C}_0$, and there exists a function $\chi \in \mathcal{C}_0$ such that $h(\chi) = \phi$.

(H4) Nonlinear map $f : [0, T] \times X \times \mathcal{C}_0 \to X$ is continuous in first variable and satisfies the Lipschitz-like condition in second and third argument, that is, there exists some constant $l > 0$ such that $\|f(t, x(t), y_{b(t)}) - f(t, v(t), w_{b(t)})\| \leq l(\|x - v\|_{\mathcal{C}_T} + \|y - w\|_{\mathcal{C}_T})$ for all $x, y, v, w \in \mathcal{C}_T$ and $t \in [0, T]$.

(H5) $B : U \to X$ is a bounded linear operator.

Semilinear differential equation (1) can be seen as an abstract formulation for many control systems described by partial or functional differential equations. Here, nonlocal condition is generally more practical for the physical measurements as compared to the classical condition. The importance of nonlocal conditions has been discussed in the pioneering work by Byszewski and Lakshmikantham [6,7]. Nonlocal conditions were used by Deng in [10] to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube. It is a well known fact that the problem of controllability of semilinear systems in infinite-dimensional spaces can be converted into solvability problem of a functional operator equation in appropriate Banach spaces, and fixed-point theory has been widely used in the literature to establish this solvability; [2,9,14,15]. These concepts has been extended to infinite-dimensional semilinear delay control systems with local or nonlocal initial conditions, among others, we refer to the papers [5, 17, 19, 21, 23, 24, 26] for local conditions and papers [3, 4, 13, 16] for nonlocal conditions.

The purpose of this paper is to compare the trajectory reachable set of nonlinear system (1) to the trajectory reachable set of its corresponding linear system [$f = 0$ in (1)] and this is motivated by the paper of Naito and Park [19] and Ryu, Park, and Kwun [21]. In particular, approximate controllability of system (1) is shown provided the corresponding linear system is controllable. In the proof of the main controllability result in the next section, we do not require any inequality condition, compactness of $S(t)$, and uniform-boundedness of f. In this respect, this paper relaxes some restrictions made by earlier authors if an another simple condition is satisfied by the system operators. In the last section, theory is illustrated with some examples.

2 The Main Results

Let us first consider the following functional delay differential system:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), x_{b(t)}), & t \in (0, T], \\ h(x) = \phi, & \text{on } [-\tau, 0]. \end{cases} \tag{2}$$

Definition 2.1 A solution function $x \in \mathcal{C}_T$ of the integral equation

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)f(s, x(s), x_{b(s)})\mathrm{d}s, & t \in [0, T], \end{cases} \tag{3}$$

is called a mild solution of problem (2).

The existence and uniqueness of the mild solution of (2) is discussed in the following theorem, and the proof is motivated by the work of Bahuguna and his coworkers, see [1,11].

Theorem 2.1 *If assumptions (H1)-(H4) are satisfied, then there exists a mild solution of (2) on* $[0,\, T]$ *for some* $T > 0$*. Moreover, the mild solution is unique if and only if* χ *is unique.*

Proof. We choose a $T > 0$ such that $2lTMe^{\omega T} < 1$. Define a map $\mathcal{F}$ from $\mathcal{C}_T$ into itself by

$$(\mathcal{F}x)(t) = \begin{cases} \chi(t), & t \in [-\tau,\, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)f(s, x(s), x_{b(s)})\mathrm{d}s, & t \in [0,\, T]. \end{cases} \tag{4}$$

It is clear that $\mathcal{F}$ is well defined and assumption $(H3)$ ensures a fixed point of $\mathcal{F}$ on $t \in [-\tau,\, 0]$. Now we show that $\mathcal{F}$ is a contraction for the case when $t \in [0,\, T]$. For this purpose, consider any $x, y \in \mathcal{C}_T$, then we have

$$\begin{aligned} ||(\mathcal{F}x)(t) - (\mathcal{F}y)(t)||_X &\leq ||\int_0^t S(t-s)(f(s, x(s), x_{b(s)}) - f(s, y(s), y_{b(s)}))\mathrm{d}s||_X \\ &\leq 2lTMe^{\omega T}||x-y||_{\mathcal{C}_T}. \end{aligned} \tag{5}$$

Since $2lTMe^{\omega T} < 1$, $\mathcal{F}$ is a contraction on $\mathcal{C}_T$ and hence by Banach Contraction Principle $\mathcal{F}$ has a unique fixed point. Obviously, the uniqueness of χ in $(H3)$ reveals the uniqueness of the mild solution. □

From the above result, a mild solution of the control system (1) can be written as follows

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau,\, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)[Bu(s) + f(s,\, x(s),\, x_{b(s)})]\mathrm{d}s, & t \in [0,\, T]. \end{cases} \tag{6}$$

Note that, mild solution (6) depends on control functions $u(\cdot)$. The solution of (6) under a control $u(\cdot)$, denoted by $x(\cdot; u)$, is called the trajectory (state) function of (1) under $u(\cdot)$. The set of all possible trajectories, denoted by

$$K_\alpha(f) := \{x(\cdot; u) \in C([\alpha,\, T];\, X) :\, u \in L^2([0,\, T]; U),\, 0 < \alpha \leq T\} \tag{7}$$

is called the trajectory reachable set of system (1). In particular, the set of all possible terminal states, denoted by

$$K_T(f) := \{x(T; u) \in X :\, u \in L^2([0,\, T]; U)\} \tag{8}$$

is called the reachable set of system (1) at terminal time T.

Definition 2.2 System (1) is said to be approximate controllable on $[0,\, T]$ if $\overline{K_T(f)} = X$, where $\overline{K_T(f)}$ stands for the closure of $K_T(f)$ in X.

Now, we define two functions $F : \mathcal{C}_T \to L^2([0,T]; X)$ and $B_1 : Y \to L^2([0,T]; X]$ as $(Fx)(t) = f(t,\, x(t),\, x_{b(t)})$, $\quad (B_1u)(t) = Bu(t)$.

Theorem 2.2 *Under assumptions (H1)-(H5) and* $R(F) \subseteq \overline{R(B_1)}$*, we have* $\overline{K_\alpha(f)} \supseteq K_\alpha(0)$.

Proof. Let $x(\cdot) \in K_\alpha(0)$, there exists a control $u \in Y$ such that

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)Bu(s)\mathrm{d}s, & t \in [0, T]. \end{cases} \tag{9}$$

Due to range condition, for a given $\epsilon > 0 \ \exists \ w \in Y$ such that

$$\|Fx - B_1 w\|_{L^2([0,T];X)} \le \epsilon. \tag{10}$$

Now, let $y(\cdot)$ be mild solution of (1) corresponding to control $u - w$. Then

$$\begin{aligned} x(t) - y(t) &= \int_0^t S(t-s)Bw(s)\mathrm{d}s - \int_0^t S(t-s)f(s, y(s), y_{b(t)})\mathrm{d}s \\ &= \int_0^t S(t-s)(B_1 w - Fx)(s)\mathrm{d}s + \int_0^t S(t-s)(Fx - Fy)(s)\mathrm{d}s. \end{aligned} \tag{11}$$

Using (H4) and (10) we have

$$\begin{aligned} \|x(t) - y(t)\| &\le Me^{\omega T}\int_0^t \|(B_1 w - Fx)(s)\|\mathrm{d}s + Me^{\omega T}\int_0^t \|(Fx - Fy)(s)\|\mathrm{d}s \\ &\le Me^{\omega T}\sqrt{T}\epsilon + 2Mle^{\omega T}\int_0^t ||x - y||_{\mathcal{C}_T}\mathrm{d}s. \end{aligned} \tag{12}$$

This implies

$$||x - y||_{\mathcal{C}_T} \le Me^{\omega T}\sqrt{T}\epsilon + 2Mle^{\omega T}\int_0^t ||x - y||_{\mathcal{C}_T}\mathrm{d}s. \tag{13}$$

Now, using Gronwall's inequality it can be shown that

$$||x - y||_{\mathcal{C}_T} \le Me^{\omega T}\sqrt{T}\epsilon \exp(2lTMe^{\omega T}). \tag{14}$$

From the above inequality it is clear that $||x - y||_{\mathcal{C}_T}$ can be made arbitrary small by choosing suitable w. Hence the theorem is proved. □

Corollary 2.1 *Under assumptions of the above theorem, system (1) is approximate controllable if its corresponding linear system is approximate or exact controllable.*

Proof. The proof is a particular case of Theorem 2.2 at $\alpha = T$. □

Remark 2.1 Fixed-point theory arguments make it necessary to assume uniform boundedness of nonlinear term f with certain inequality condition involving various system parameters, and/or compactness of semigroup $T(t)$. But, these conditions (specially inequality conditions) are not easy to verify in many situations. In this paper these conditions are replaced with a range condition $R(F) \subset \overline{R(B_1)}$. Note that this range condition is satisfied trivially for the system (1) if B is the identity operator. Obviously, Theorem 2.2 gives the controllability of system (1) when $b(t) = t$ in the case of constant delay, and this case is explained in Example 3.1.

3 Application

Example 3.1 Consider the following mathematical model

$$\begin{aligned}
\frac{\partial}{\partial t}y(t,x) &= \frac{\partial^2}{\partial x^2}y(t,x) + u(t,x) + (\int_0^1 y(t,x)dx)y(t,x) \\
&+ (\int_0^1 y(t-\tau,x)dx)y(t-\tau,x), 0 \le x \le 1, t \in [0,T], \quad (15) \\
y(t,0) &= y(t,1) = 0, t \in [0,T], \\
\frac{1}{\tau}\int_{-\tau}^0 e^{2s}y(s,x)ds &= y_0(x), 0 \le x \le 1,
\end{aligned}$$

where $y(t,.)$, $u(t,.)$, $y_0 \in L^2(0,1)$. If we take

(1) $X = L^2(0,1)$ as the state space and $y(t,\cdot) = \{y(t,x) : 0 \le x \le 1\}$ as the state.

(2) input trajectory $u(t,.)$ as the control and $U = L^2(0,1)$ as the control space. Note that, here $X = U$.

(3) $A : D(A) \subset X \to X$ defined by $A(z) = \frac{d^2z}{dx^2}$ with domain $D(A) = H^2(0,1)\bigcap H_0^1(0,1)$. Then A is an infinitesimal generator of a C_0-semigroup of bounded linear operators; see [8].

(4) $B = I$.

(5) $b(t) = t$, and $y_{b(t)}(\theta) \equiv y(t-\tau,\cdot)$ (so this is a constant time-delay case).

(6) $f : [0,T] \times X \times \mathcal{C}_0 \to X$, $T > 0$ defined by

$$f(t, y(t,\cdot), y_{b(t)}) = (\int_0^1 y(t,x)dx)y(t,\cdot) + (\int_0^1 y(t-\tau,x)dx)y(t-\tau,\cdot),$$

where $0 \le x \le 1, t \in [0,T]$. It is not hard to see that f satisfies (H4).

(7) $h(z)(\theta) = g(z)$ for $z \in \mathcal{C}_0$, $\theta \in [-\tau,0]$; $\phi(\theta) = y_0$. Here, $g : \mathcal{C}_0 \to X$ is such that $g(z)(x) = \frac{1}{\tau}\int_{-\tau}^0 e^{2s}z(s,x)ds$. For this definition of h, we can find a function $\chi \in \mathcal{C}_0$, given by $\chi(\theta) = \frac{1}{k}y_0$ on $[-\tau,\, 0]$ with $k = \int_0^\tau \frac{1}{\tau}e^{-2s}ds \neq 0$, such that

$$h(\chi)(\theta) = \frac{1}{\tau}\int_{-\tau}^0 e^{2s}(\frac{1}{k}y_0)ds = y_0 = \phi(\theta), \text{ that is, } h(\chi) = \phi.$$

Then (15) resembles control system (1) and has a mild solution (6) on $[-\tau,T]$. Now take $Y := L^2([0,\,T];L^2(0,1))$, $B_1 = I : Y \to Y$, $F : \mathcal{C}_T \to Y$ as $(Fz)(t) = f(t,\, z(t),\, z_{b(t)})$. Then it is clear that $R(F) \subset \overline{R(B_1)}$. Since the corresponding linear system is approximate controllable; [8], system (15) is approximate controllable due to Theorem 2.2. Mathematical model (15) may be seen as the population dynamics, see [12], where $y(t,.)$ represents the population density at time t and the term $\frac{\partial^2}{\partial x^2}y(t,x)$ describes the internal migration. Moreover, the continuous functions $B, D : [0,T] \to \mathbb{R}_+$ given by $B(t) = \int_0^1 y(t-\tau,x)dx$ and $D(t) = \int_0^1 y(t,x)dx$, represent average birth and death rates, respectively, τ is the delay due to pregnancy, and source term $u(t,x)$ represents a control.

Example 3.2 Consider the control system governed by the following semilinear heat equation

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x)\\ &+ f(t,y,y_{b(t)});\ 0<t<T, 0<x<\pi,\ -\tau\le\theta\le 0,\\ y(t,0) &= y(t,\pi)=0, t\in[0,T],\end{aligned} \tag{16}$$

with the same initial condition as in the above example, where $y(t,.)$, $y_0 \in L^2(0,\pi)$. Then (16) can be converted into (1), if we take:

(1) $X = L^2(0,\pi)$ as the state space and $y(t,.) = \{y(t,x) : 0\le x\le\pi\}$ as the state.

(2) input trajectory $u(t,.)$ as the control.

(3) $A : D(A) \subset X \to X$ defined by $A(z) = \frac{d^2z}{dx^2}$ with domain $D(A) = H^2(0,\pi)\bigcap H_0^1(0,\pi)$. Then, $\overline{D(A)} = X$ and A is an infinitesimal generator of a C_0-semigroup of bounded linear operators; see [8]. Further, if we take $\{\phi_n(x) = (2/\pi)^{1/2}\sin(nx); 0\leqslant x\leqslant\pi; n\in\mathbb{N}\}$, then $\{\phi_n\}$ is an orthonormal basis of X and ϕ_n is an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of operator A. Then the C_0-semigroup generated by A has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions.

(4) $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty\}$, with norm $|u|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$ as the control space. B is a continuous linear map from U to X defined as

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \text{ for } u = \sum_{n=2}^{\infty} u_n\phi_n \in U.$$

(5) $b(t) = k|\sin t|$, $k\in(0,1)$ or $b(t) = \frac{t^2}{1+t^2}$.

(6) h and χ are the same as in Example 3.1.

It shows that (16) has a mild solution (6) on $[-\tau,T]$ provided f is Lipschitz continuous. Although, the same example has been discussed in [9, 18, 27] (with or without delay and under local conditions), but approximate controllability was proved under restrictions such as the uniform boundedness on f or some inequality constraints. This paper shows that the approximate controllability also follows for non-uniform bounded function f without having to satisfy any inequality constraint and without using the compactness of C_0-semigroup. For example, consider the function f given by $f(t,z,z_{b(t)}) = \alpha(\|z\|_{\mathcal{C}_T} + \|z_{b(t)}\|_{\mathcal{C}_0})(\phi_3(x)+\phi_4(x))$, where α is a positive constant. Here f is Lipschitz and $R(F)\subseteq R(B_1)$. Moreover, this example shows that time-varying aftereffect and generalized nonlocal conditions can also be handled by the theorem proved in the previous section. In the above example $b(t) = k|\sin t|$, $k\in(0,1)$ or $b(t) = \frac{t^2}{1+t^2}$ is a theoretical construction but many physical and biological processes include time-varying aftereffect phenomena in their inner dynamics, see [20].

Example 3.3 Consider the system of infinite ordinary differential equations:

$$\frac{dx(t)}{dt} = Ax(t) + u(t) + f(t,x(t),x_{b(t)}),\quad \sum_{i=1}^{l} c_i x(\theta_i) = x_0, \tag{17}$$

where $x(t) = (x_1(t), x_2(t), \ldots) \in l^2$. Then (17) resembles control system (1), if we take

(1) $X = l^2$ as the state space and $x(t)$ as the state.

(2) input $u(t) = (u_1(t), u_2(t), \ldots)$ as the control and $U = l^2$ as the control space. Note that, here $X = U$.

(3) A is a self-adjoint operator on X defined by $Ae_i = \lambda_i e_i$ where $\{e_i\}$ is an orthonormal basis of X and $\{\lambda_i\}$is a decreasing sequence of positive numbers such that $\lim\limits_{i\to\infty} \lambda_i = \lambda_0 > 0$. Then A is an infinitesimal generator of a C_0-semigroup of bounded linear operators defined by $T(t)x = \Big(e^{\lambda_1 t}x_1, e^{\lambda_2 t}x_2, \ldots\Big)$.

(4) $B = I$ and b is the same as in Example 3.2.

(5) f is defined by $f(t, x(t), x_{b(t)}) = (f_1(t, x(t), x_{b(t)}), f_2(t, x(t), x_{b(t)}), \ldots)$, $\quad 0 \le t \le T$.

(6) $h(z)(\theta) = g(z)$ for $z \in \mathcal{C}_0$, $\theta \in [-\tau, 0]$; $\phi(\theta) = x_0$. Here, $g : \mathcal{C}_0 \to X$ is such that $g(z) = \sum_{i=1}^{l} c_i z(\theta_i); -\tau \le \theta_1 < \theta_2 < \cdots < \theta_l \le 0$. For this definition of h, we can find a function $\chi \in \mathcal{C}_0$, given by $\chi(\theta) = \frac{1}{k}x_0$ on $[-\tau,\ 0]$ with $k = \sum_{i=1}^{l} c_i$.

The approximate controllability of the linear system corresponding to (17) has been proved by Triggiani [25]. In [22], the approximate controllability of (17) (without delay and with local Cauchy condition) has been shown via the solvability of some operator equations under the following conditions:
(i) The linear system is approximate controllable,
(ii) A generates a compact semigroup $T(t)$,
(iii) The nonlinear operator $f(t, x)$ satisfies the Lipschitz condition,
(iv) The operator f satisfies the growth condition $\|f(x(t))\|_X \le a\|x(t)\|_X + b$,
(v) System constants satisfy the constraint $\frac{e^{\lambda_1 T}\sqrt{T}}{2} \cdot \sqrt{2MbT(e^{2MbT} - 1)} < \frac{e^{2T\lambda_0} - 1}{2e^{\lambda_1 T}\sqrt{T}\lambda_0}$, where $\|T(t)\| \le e^{\lambda_1 \tau} = M$ for $0 \le t \le T$.

But due to Theorem 2.2, it follows that the system (17) is approximate controllable only under the above conditions (i) and (iii) for nonlinear operators those satisfy the range condition, *e.g.* f is defined as $f_1(t, x(t), x_{b(t)}) = a\|x\| + b\|x_{b(t)}\| + c$; a, b, and c are positive constants and $f_i(t, x(t), x_{b(t)}) = 0$ for all $i = 2, 3, \ldots$. This shows that the inequalities such as (v) above, assumed by earlier author are not required to be considered.

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Existence of Positive Solutions of a Nonlinear Third-Order M-Point Boundary Value Problem for p-Laplacian Dynamic Equations on Time Scales

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Abstract: In this paper, by using fixed-point theorems in cones, we study the existence of at least one, two and three positive solution of a nonlinear third-order m-point p-Laplacian boundary value problem on time scale.

Keywords: *time scales; nontrivial solution; fixed-point theorems.*

Mathematics Subject Classification (2010): 39A10, 34B15, 34B16.

1 Introduction

We study the third-order m-point boundary value problems (MPBVP) on time scales with p-Laplacian,

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla}(t) + p(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathrm{T_k}\cap T^{k^2}}, \tag{1}$$

$$u^{\triangle\nabla}(\rho(0)) = 0, \ u^{\triangle}(T) = 0, \ u(\rho(0)) = B(\sum_{1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \tag{2}$$

where Φ_p is p-Laplacian operator, i.e. $\Phi_p(s) = |s|^{p-2}s$, $p > 1$ and $(\Phi_p)^{-1} = \Phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Here $\rho(0) < \xi_1 < \xi_2 < ... < \xi_{m-2} < \sigma(T)$.

(H1) $\alpha_i \in [0,\infty)$, $i = 1,2,3...$ and $f : [0,T] \times [0,\infty) \to [0,\infty)$ is left-dense continuous function,

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(H2) $p : [0, T] \to [0, \infty)$ is left-dense continuous function,

(H3) $B : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the existence of $B_0 \geq B_1 > 0$ such that $B_0 s \leq B(s) \leq B_1 s$, for $s \in [0, \infty)$.

A time scale $\mathbf{T}$ is a nonempty closed subset of $\mathbf{R}$. We make the blanket assumption 0, T are points in $\mathbf{T}$. By an interval $[0, T]$, we always mean the intersection of the real interval $[0, T]$ with the given time scale; that is $[0, T] \cap \mathbf{T}$. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively, $\sigma(t) = \inf\{\tau \in \mathbf{T} | \tau > t\} \in \mathbf{T}$, $\rho(r) = \sup\{\tau \in \mathbf{T} | \tau < r\}$ for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T_k} = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T_k} = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T^k} = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T^k} = \mathbf{T}$. Some basic definitions and theorems on time scales can be found in the books [4, 5].

p-Laplacian problems with two point, three point and multi point boundary conditions for ordinary differential equations and difference equations have been studied by several authors (see [6, 10, 16] and the references therein). Recently, there has been much attention paid to the existence of positive solution for second-order and third-order nonlinear boundary value problems on time scales [1, 2, 9, 11, 12, 15, 17, 18]. However, to the best of our knowledge, there are not many results concerning third-order p-Laplacian dynamic equations on time scales.

In [8], Yanging Guo, Changlang Yu, Jufang Wang considered the existence of three positive solutions for the following m-point boundary value problems on infinite intervals

$$(\varphi_p(x'(t)))' + \phi(t) f(t, x(t), x'(t)) = 0, \ 0 < t < \infty, \tag{3}$$

$$x(0) = \sum_{1}^{m-2} a_i x'(\eta_i), \ \lim_{t \to \infty} x'(t) = 0. \tag{4}$$

They used Avery–Henderson fixed-point theorem on a cone to prove the existence of three positive solutions to the $(3) - (4)$ nonlinear problems.

In [15], Sihua Liang, Jihui Zhang, Zhiyong Wang prove the existence of three positive solutions for the following second order m-point boundary value problems

$$(\Phi(p(t) u^{\triangle}(t)))^{\nabla} + a(t) f(u(t)) = 0, \ t \in [0, T]_{\mathbf{T^k} \cap \mathbf{T_k}}, \tag{5}$$

$$u(0) - B_0\big(\sum_{1}^{m-2} a_i u^{\triangle}(\xi_i)\big) = 0, \ u^{\triangle}(T) = 0. \tag{6}$$

for some dynamic equations on time scales using Legget–Williams fixed-point theorem.

In [11], Zhimin He obtained the existence of at least double positive solutions of the following three-point boundary value problems

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla} + a(t) f(u(t)) = 0, \ t \in [0, T], \tag{7}$$

$$u(0) - B_0(u^{\triangle}(\eta)) = 0, \ u^{\triangle}(T) = 0, \tag{8}$$

or

$$u^{\triangle}(0) = 0, \ u(T) + B_1(u^{\triangle}(\eta) = 0, \tag{9}$$

by using double fixed-point theorem.

In [9], Wei Hang, Maoxing Liu considered the third-order nonlinear problem such that

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla} + a(t)f(u(t)) = 0,\ t \in [0,T], \tag{10}$$

$$\alpha u(0) - \beta u^{\triangle}(0) = 0,\ u(T) = \sum_{1}^{m-2} a_i u(\xi_i),\ u^{\triangle\nabla}(0) = 0. \tag{11}$$

They used the fixed-point theorem which is given by V.Lakshmikantham in [7] to prove the existence of at least one nontrivial solution to the nonlinear problem $(10)-(11)$.

Motivated by the results [15], in this paper, we will study the existence of multiple positive solutions of third-order p-Laplacian MPBVP $(1)-(2)$.

The aim of this paper is to establish some simple criteria for the existence of positive solutions of the p-Laplacian MPBVP $(1)-(2)$. This paper is organized as follows: In Section 2 we first present some properties of the solution of the linear p-Laplacian MPBVP corresponding to $(1)-(2)$. In Section 3, we state the fixed-point theorems in order to prove main results and we get the existence of at least one, two and three positive solutions for nonlinear p-Laplacian MPBVP $(1)-(2)$.

2 Preliminaries and Lemmas

To prove main results, we will give several lemmas and the following lemmas are based on the linear p-Laplacian MPBVP

$$(\Phi_p(u^{\triangle\nabla})^{\nabla}(t) + h(t) = 0,\quad t \in [0,T]_{\mathrm{T_k}\cap T^{k^2}}, \tag{12}$$

$$u^{\triangle\nabla}(\rho(0)) = 0,\ u(\rho(0)) = B(\sum_{1}^{m-2} a_i u^{\triangle}(\xi_i)),\ u^{\triangle}(T) = 0. \tag{13}$$

Lemma 2.1 *For $h \in \mathbf{C}_{ld}([0,T]\times\mathbf{R})$, the problems* (12) *and* (13) *have the unique solution*

$$u(t) = B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s) + \int_{\rho(0)}^{t}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r. \tag{14}$$

Proof. From the equation (12) we can easily obtain

$$u^{\triangle\nabla}(s) = -\Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau),\quad u^{\triangle}(t) = \int_{t}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s$$

Therefore, we have

$$u(t) = u(\rho(0)) + \int_{\rho(0)}^{t}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r.$$

Applying the boundary conditions (2.13) we have

$$u(t) = B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s) + \int_{\rho(0)}^{t}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r.$$

It is easy to see that the p-Laplacian MPBVP $(\Phi_p(u^{\triangle\nabla}(t))^{\nabla}=0, u^{\triangle\nabla}(\rho(0))=0$, $u(\rho(0))=B(\sum_1^{m-2} a_i u^{\triangle}(\xi_i))=0,\ u^{\triangle}(T)=0$ has only the trival solution. Thus u is the unique solution of (12) and (13). The proof is complete. $\square$

Let X denote Banach space $\mathbf{C}_{ld}([\rho(0),T],[0,\infty))$ with the norm $\|u\|=\sup|u(t)|$, $t\in[\rho(0),T]$. Define the cone $P\subset X$ by

$$P=\{u\in X: u(t)>0,\ u^{\triangle}(t)>0,\ t\in[\rho(0),T],\ u\ is\ concave\}. \tag{15}$$

For $u\in P$ define the operator L by

$$\begin{aligned} Lu(t)=B(\sum_1^{m-2} a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\ +\ \int_{\rho(0)}^{t}(\int_r^T\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r. \end{aligned} \tag{16}$$

Obviously, from the definition of L we have $Lu(t)\geq 0$ and for $t\in[\rho(0),T]$ we get

$$(Lu)^{\triangle}(t)=\int_t^T\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s\geq 0.$$

As

$$(Lu)^{\triangle\nabla}(t)=-\Phi_q(\int_{\rho(0)}^{t}p(\tau)f(\tau,u(\tau))\nabla\tau)\leq 0,$$

then Lu is concave. Therefore $L:P\to P$ and $\|Lu\|=\sup|Lu(t)|=Lu(T)$ for $t\in[\rho(0),T]$.

Also it is easy to check that L is a completely continuous operator by a standard application of the Arzela-Ascoli theorem.

Lemma 2.2 *If $u\in P$ and $\|u\|=\sup|u(t)|,\ t\in[\rho(0),T]$, then*

$$u(t)\geq\frac{t-\rho(0)}{T-\rho(0)}\|u\|. \tag{17}$$

Proof. It can be easily shown by the similar way as in Lemma 3.1 in the reference [14].

3 Existence of Positive Solutions

In this section we will prove the existence of multiple positive solutions of our problem. We will need also the following Krasnoselkii's fixed-point theorem to prove the existence of at least one positive solution of p-Laplacian MPBVP (1)–(2).

Theorem 3.1 [13] *Let X be a Banach space and $P\subset X$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of P with $0\in P$, $\overline{\Omega}_1\subset\Omega_2$, and let $L:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator such that either*

(i) $\|Lu\|\leq\|u\|$ *for $u\in P\cap\partial\Omega_1$, $\|Lu\|\geq\|u\|$ for $u\in P\cap\partial\Omega_2$;*

or

(ii) $\|Lu\|\geq\|u\|$ *for $u\in P\cap\partial\Omega_1$, $\|Lu\|\leq\|u\|$ for $P\cap\partial\Omega_2$ hold.*

Then L has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Theorem 3.2 *Assume conditions* $(H_1) - (H_3)$ *are satisfied. In addition, suppose there exist numbers* $0 < r < R < \infty$ *such that*

(i) $f(\tau, u(\tau)) \le \Phi_p(\frac{u}{k_1}),\ if\ 0 \le u \le r,$

and

(ii) $f(\tau, u(\tau)) \ge \Phi_p(\frac{u}{k_2}),\ if\ R \le u \le \infty,$

where

$$k_1 = B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)\nabla\tau)\nabla s + \int_{\rho(0}^T (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)\nabla\tau)\nabla s)\triangle r,$$

$$k_2 = \int_{\rho(0)}^T (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)\Phi_p(\frac{\tau}{T})\nabla\tau)\nabla s)\triangle r.$$

Then the p-Laplacian MPBVP (1) − (2) *has at least one positive solution.*

Proof. Define the cone P as in (15). It is also easy to check that $L : P \to P$ is completely continuous and $LP \subset P$. If $u \in P$ with $\|u\| = r$ then we get

$$\begin{aligned}
\|Lu\| &\le B(\sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)|f(\tau, u(\tau))|\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^T (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)|f(\tau, u(\tau))|\nabla\tau)\nabla s)\triangle r \\
&\le B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)\Phi_p(\frac{u}{k_1})\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^T (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)\Phi_p(\frac{u}{k_1})\nabla\tau)\nabla s)\triangle r \\
&= \frac{u}{k_1}[B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^T (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)\nabla\tau)\nabla s)\triangle r] \\
&= \|u\|.
\end{aligned}$$

So if we set

$$\Omega_1 = \{u \in \mathbf{C}_{ld}([\rho(0), T], [0, \infty)) : \ \|u\| < r\},$$

then $\|Lu\| \le \|u\|$ for $u \in P \cap \partial\Omega_1$.

Let us now set

$$\Omega_2 = \{u \in \mathbf{C}_{ld}([\rho(0), T], [0, \infty)) : \ \|u\| < R\},$$

then for $u \in P$ with $\|u\| = R$, we have

$$\begin{aligned}
\|Lu\| &= |B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{T} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r| \\
&\geq \int_{\rho(0)}^{T} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) |f(\tau, u(\tau))|\nabla\tau)\nabla s)\triangle r \\
&\geq \int_{\rho(0)}^{T} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{u}{k_2})\nabla\tau)\nabla s)\triangle r \\
&\geq \int_{\rho(0)}^{T} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{\tau}{T}\frac{\|u\|}{k_2})\nabla\tau)\nabla s)\triangle r \\
&= \frac{\|u\|}{k_2}[\int_{\rho(0)}^{T} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{\tau}{T})\nabla\tau)\nabla s)\triangle r \\
&= \|u\|.
\end{aligned}$$

Hence $\|Lu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Thus by the first part of Theorem 3.1, L has a fixed point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Therefore the p-Laplacian MPBVP $(1)-(2)$ has at least one positive solution. □

Applying the following Avery–Henderson fixed point theorem, we will prove the existence of at least two positive solutions to the p-Laplacian MPBVP $(1)-(2)$.

Theorem 3.3 [3] *Let P be a cone in a real Banach space X. Set*
$P(\psi, z) = \{u \in P : \psi(u) < z\}$
If η and ψ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants z and γ
$\psi(u) \leq \theta(u) \leq \eta(u)$ *and* $\|u\| \leq \gamma\psi(u)$
for all $u \in P(\overline{\psi, z})$. Suppose that there exist positive numbers $x < y < z$ such that
$\theta(\lambda u) \leq \lambda\theta(u)$ *for all $0 < \lambda < 1$ and $u \in \partial P(\theta, y)$.*
If $L : P(\overline{\psi, z}) \to P$ is completely continuous operator satisfying
(i) $\psi(Lu) > z$ *for all $u \in \partial P(\psi, z)$*
(ii) $\theta(Lu) < y$ *for all $u \in \partial P(\theta, y)$*
(iii) $P(\eta, x) \neq \emptyset$ *and $\eta(Lu) > x$ for all $u \in \partial P(\eta, x)$. Then L has at least two fixed points u_1 and u_2 such that*

$$x < \eta(u_1) \text{ with } \theta(u_1) < y \text{ and } y < \theta(u_2) \text{ with } \psi(u_2) < z.$$

Theorem 3.4 *Assume $(H_1)-(H_3)$ hold. Suppose there exist positive numbers $x < \frac{F}{E}y < \frac{(\xi_1-\rho(0))F}{(T-\rho(0))E}z$ such that the function f satisfies the following conditions:*
(i) $f(s, u) > \Phi_p(\frac{z}{D})$ *for* $s \in [\xi_1, T]$ *and* $u \in [z, \frac{T-\rho(0)}{\xi_1-\rho(0)}z]$,
(ii) $f(s, u) < \Phi_p(\frac{y}{E})$ *for* $s \in [\rho(0), T]$ *and* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}y]$,
(iii) $f(s, u) > \Phi_p(\frac{x}{F})$ *for* $s \in [\rho(0), \xi_{m-2}]$ *and* $u \in [0, \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}x]$.
for some positive constants D, E and F. Then p-Laplacian MPBVP $(1)-(2)$ has at least two positive solutions u_1 and u_2 such that

$$u_1(\xi_1) < y \text{ and } u_1(\xi_{m-2}) > x, u_2(\xi_1) > y \text{ and } u_2(\xi_1) < z.$$

Let us define the positive constants D, E and F such that

$$D = B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau\Big)\nabla s + \int_{\rho(0)}^{\xi_1}\Big(\int_{\xi_1}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau\Big)\nabla s\Big)\triangle r,$$
$$E = B_1 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{T} p(\tau)\nabla\tau\Big)\nabla s + \int_{\rho(0)}^{\xi_1}\Big(\int_{\rho(0)}^{T} \Phi_q\Big(\int_{\rho(0)}^{T} p(\tau)\nabla\tau\Big)\nabla s\Big)\triangle r,$$
$$F = B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau\Big)\nabla s + \int_{\rho(0)}^{\xi_{m-2}}\Big(\int_{\xi_{m-2}}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau\Big)\nabla s\Big)\triangle r.$$

Proof. Define the cone P as in (15). We know L is completely continuous and $LP \subset P$. Let the nonnegative increasing continuous functionals ψ, θ and η be defined on the cone by
$\psi(u) = \min u(t) = u(\xi_1),\ t \in [\xi_1, \xi_{m-2}]$,
$\theta(u) = \max u(t) = u(\xi_1),\ t \in [\rho(0), \xi_1]$,
$\eta(u) = \max u(t) = u(\xi_{m-2}),\ t \in [\rho(0), \xi_{m-2}]$.

For each $u \in P,\ \psi(u) = \theta(u) \leq \eta(u)$. In addition for each $u \in P$

$$\psi(u) = u(\xi_1) \geq \frac{\xi_1 - \rho(0)}{T - \rho(0)} \|u\|. \tag{18}$$

Also $\theta(0) = 0$ and we have $\theta(\lambda u) = \lambda\theta(u)$ and for $u \in P$ and $\lambda \in [0, 1]$.

We now verify that all conditions of Theorem 3.3 are satisfied.
If $u \in \partial P(\psi, z)$ then $\psi(u) = min_{t\in[\xi_1,\xi_{m-2}]} u(t) = u(\xi_1) = z$. So we have $u(t) \geq z$, for $t \in [\xi_1, T]$, and from (18) $z \leq u(t) \leq \|u\| \leq \frac{T-\rho(0)}{\xi_1-\rho(0)} z$ for $t \in [\xi_1, T]$. Then assumption (i) implies $f(s, u) > \Phi_p(\frac{z}{D})$ for $s \in [\xi_1, T]$.

Since $Lu \in P$ we get

$$\begin{aligned}
\psi(Lu) &= Lu(\xi_1) \\
&= B\Big(\sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s\Big) \\
&+ \int_{\rho(0)}^{\xi_1}\Big(\int_{r}^{T} \Phi_q\Big(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s\Big)\triangle r \\
&> B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}\Big(\int_{\xi_1}^{T} \Phi_q\Big(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s\Big)\triangle r \\
&> B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}\Big(\int_{\xi_1}^{T} \Phi_q\Big(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau\Big)\nabla s\Big)\triangle r
\end{aligned}$$

$$
\begin{aligned}
&> \frac{z}{D}\{B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= z.
\end{aligned}
$$

Hence condition (i) of Theorem 3.3 is satisfied.

Secondly, we show that (ii) of Theorem 3.3 is fulfilled. For this, we select $u \in \partial P(\theta, y)$. Then

$$\theta(u) = \max_{t\in[\rho(0),\xi_1]} u(t) = u(\xi_1) = y.$$

We know from (2.17)

$$0 \le u(t) \le \frac{T-\rho(0)}{\xi_1-\rho(0)} y,$$

for $t \in [\rho(0), T]$. Then assumption (ii) implies

$$f(s,u) < \Phi_p(\frac{y}{E}),$$

for $s \in [\rho(0), T]$. Therefore

$$
\begin{aligned}
\theta(Lu) &= Lu(\xi_1) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{y}{E}\{B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= y.
\end{aligned}
$$

Then condition (ii) of Theorem 3.3 holds.

Finally, we verify that (iii) of Theorem 3.3 is also satisfied.
Since $0 \in P$ and $x > 0$, $P(\eta, x) \neq \emptyset$, that $\eta(0) = 0 < x$. Now let $u \in \partial P(\eta, x)$. Then

$$\eta(u) = \max_{t\in[\rho(0),\xi_{m-2}]} u(t) = u(\xi_{m-2}) = x.$$

We know from (2.17)

$$0 \le u(t) \le \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)} x,$$

for $t \in [\rho(0), \xi_{m-2}]$. Then assumption (iii) implies $f(s,u) > \Phi_p(\frac{x}{F})$ for $s \in [\rho(0), \xi_{m-2}]$. As before, we get

$$\begin{aligned}
\eta(Lu) &= Lu(\xi_{m-2}) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\xi_{m-2}}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> \frac{x}{F}\{B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\xi_{m-2}}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= x.
\end{aligned}$$

Since all conditions of Theorem 3.3 are satisfied, the p-Laplacian MPBVP (1) – (2) has at least two positive solutions u_1 and u_2 such that

$$x < \eta(u_1),\ \theta(u_1) < y \text{ and } y < \theta(u_2),\ \psi(u_2) < z. \ \square$$

We will use the following Legget-Williams fixed point theorem to prove the existence of at least three positive solutions to the p-Laplacian MPBVP (1) – (2).

Theorem 3.5 [14]*Let P be a cone in a Banach space X. Set*
$P(\gamma, c) = \{u \in P : \gamma(u) < c\}$.
Let α, β and γ be three increasing nonnegative and continuous functionals on P, satisfying for some $c > 0$ and $A > 0$ such that
$\gamma(u) \le \beta(u) \le \alpha(u),\ \|u\| \le A\gamma(u),$
for all $u \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $L : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that
(i) $\gamma(Lu) < c$ *for all* $u \in \partial P(\gamma, c)$;
(ii) $\beta(Lu) > b$ *for all* $u \in \partial P(\beta, b)$;
(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Lu) < a$ *for all* $u \in \partial P(\alpha, a)$.
Then L has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that
$0 \le \alpha(u_1) < a < \alpha(u_2),\ \ \beta(u_2) < b < \beta(u_3),\ \ \gamma(u_3) < c.$

Theorem 3.6 *Assume that conditions $(H_1) - (H_3)$ are satisfied. Suppose there exist positive numbers $a < b < c$ such that function f satisfies the following conditions:*
(i) $f(s,u) < \Phi_p(\frac{c}{E})$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}c]$,
(ii) $f(s,u) > \Phi_p($
$fracbD)$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}b]$,

(iii) $f(s,u) < \Phi_p(\frac{a}{G})$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}a]$.
Then there exist at least three positive solutions u_1, u_2, u_3 *of p-Laplacian MPBVP* $(1.1)-(1.2)$ *such that*

$$0 \leq \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

For notational convenience, we denote G *by*

$$G = B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s)\triangle r$$

and also we will take the constants D *and* E *as in Theorem* 3.4.

Proof. We define completely continuous operator L by (2.16). Let $u \in \partial P(\gamma, c)$ then $Lu(t) \geq 0$ for $t \in [0,T]$. We know that $L : \overline{P(\gamma, c)} \to P$. Let the nonnegative increasing continuous functionals γ, β and α be defined on the cone by
$\gamma(u) = \max u(t) = u(\xi_1), \quad t \in [\rho(0), \xi_1]$,
$\beta(u) = \min u(t) = u(\xi_1), \quad t \in [\xi_1, \xi_{m-2}]$,
$\alpha(u) = \max u(t) = u(\xi_{m-2}), \quad t \in [\rho(0), \xi_{m-2}]$.
For each $u \in P$ we have

$$\gamma(u) = \beta(u) \leq \alpha(u), \quad \gamma(u) = u(\xi_1) \geq \frac{\xi_1 - \rho(0)}{T-\rho(0)}\|u\|.$$

We now show that all the conditions of Theorem 3.5 are satisfied. To make use of property (i) of Theorem 3.5, we choose $u \in \partial P(\gamma, c)$. Then $\gamma(u) = \max_{t\in[\rho(0),\xi_1]} u(t) = u(\xi_1) = c$. If we recall that $\|u\| \leq \frac{T-\rho(0)}{\xi_1-\rho(0)}\gamma(u) = \frac{T-\rho(0)}{\xi_1-\rho(0)}c$, we have for all $t \in [\rho(0), T]$

$$0 \leq u(t) \leq \frac{T-\rho(0)}{\xi_1 - \rho(0)}c.$$

Then assumption (i) of Theorem 3.6 implies $f(s,u) < \Phi_p(\frac{c}{E})$ for all $s \in [\rho(0), T]$,

$$\begin{aligned}
\gamma(Lu) &= Lu(\xi_1) \\
&= B(\sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)f(\tau, u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1} (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{c}{E}\{B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= c.
\end{aligned}$$

Hence condition (i) of Theorem 3.5 is satisfied.

Secondly we show that (ii) of Theorem 3.5 is fulfilled. For this, we select $u \in \partial P(\beta, b)$. Then $\beta(u) = \min_{t\in[\xi_1,\xi_{m-2}]} u(t) = u(\xi_1) = b$. This means $u(t) > b \ \ t \in [\xi_1, T]$ and since $u \in P$, we have $b \le u(t) \le \|u\| \le \frac{T-\rho(0)}{\xi_1-\rho(0)} b$ for all $u \in P$. So we have

$$b \le u(t) \le \frac{T-\rho(0)}{\xi_1-\rho(0)} b,$$

for all $t \in [\xi_1, T]$. Then assumption (ii) of Theorem 3.6 implies $f(s,u) > \Phi_p(\frac{b}{D})$ for all $s \in [\xi_1, T]$,

$$\begin{aligned}
\beta(Lu) &= Lu(\xi_1) \\
&= B(\sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1} (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> B_0 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^T \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> \frac{b}{D}[B_0 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^T \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s)\triangle r] \\
&= b.
\end{aligned}$$

Then condition (ii) of Theorem 3.5 holds.

Finally we verify that (iii) of Theorem 3.5 is also satisfied. We note that $u(t) \equiv \frac{a}{2}$ is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{2} < a$ for $t \in [\rho(0), T]$. So $P(\alpha, a) \neq \emptyset$. Now let $u \in \partial P(\alpha, a)$, then $\alpha(u) = a$. This implies that $0 \le u(t) \le a$ for $t \in [\rho(0), \xi_{m-2}]$. Note that $\|u\| \le \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)} \alpha(u) = \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)} a$ for all $t \in [\rho(0), \xi_{m-2}]$. So

$$0 \le u(t) \le \frac{T-\rho(0)}{\xi_1-\rho(0)} a,$$

for all $s \in [\rho(0), \xi_{m-2}]$. As before, we get

$$\begin{aligned}
\alpha(Lu) &= Lu(\xi_{m-2}) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{a}{G}\{B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= a.
\end{aligned}$$

The condition (iii) of Theorem 3.5 is satisfied. Therefore Theorem 3.5 implies that L has at least three fixed points which are positive solutions $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

The proof of Theorem 3.6 is complete. $\square$

We can illustrate our result which is given in Theorem 3.4 in the following example.

Example 3.1 Let $\mathbf{T}= [0,1] \cup [2,3]$. We consider the following p-Laplacian dynamic equation:

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla}(t) + p(t)f(t,u(t)) = 0,\ t \in [0,3]_{\mathbf{T_k} \cap \mathbf{T^{k^2}}} \tag{19}$$

satisfying the boundary conditions

$$u^{\triangle\nabla}(0) = 0, u^{\triangle}(3) = 0, u(0) = \sum_{1}^{2} \alpha_i u^{\triangle}(\xi_i), \tag{20}$$

where $p = q = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $m = 4$, $p(t) \equiv 1$, $B_0 = B_1 = 1$ and

$$f(t,u) = f(u) = \begin{cases} \frac{u^2}{10^4} + \frac{6}{10}\ , & 0 \leq u \leq 10^3, \\ 100.6 + 2(u - 10^3)\ , & u > 10^3. \end{cases}$$

Taking $x = 1$, $y = 10$, $z = 10^4$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{5}{2}$; it is easy to see that $D = \frac{15}{8}$, $E = 12$, $F = 10$, $x < \frac{F}{E}y < \frac{F}{6E}z$ and then f(u) satisfies

$$\begin{aligned}
f(u) &> \Phi_2(\frac{z}{D}) = 5334 \quad u \in [10^4, 6 \times 10^4], \\
f(u) &< \Phi_2(\frac{y}{E}) = 0.84 \quad u \in [0, 60], \\
f(u) &> \Phi_2(\frac{x}{F}) = 0.1 \quad u \in [0, \frac{6}{5}].
\end{aligned}$$

The use of Theorem 3.4 implies four point BVP (19) − (20) has at least two positive solutions u_1, u_2 satisfying

$$u_1(\tfrac{1}{2}) < 10 \text{ and } u_1(\tfrac{5}{2}) > 1,\ u_2(\tfrac{1}{2}) > 10 \text{ and } u_2(\tfrac{5}{2}) < 10^4. \ \square$$

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Weakly Connected Nonlinear Systems: Boundedness and Stability of Motion

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This self-contained book provides systematic instructive analysis of boundedness and stability of weakly connected nonlinear systems.

There are 5 chapters. In Chapter 1, mathematical foundation on theory of integral and differential inequalities, comparison techniques, direct Lyapunov method, and stability definitions for systems with small parameter are discussed in details. These results are very handy for researchers who wish to find them in their research and/or to follow the reading of the book.

In Chapter 2, it contains recently developed approaches, based on the direct Lyapunov method and the comparison technique, to the investigation of boundedness of motion of weakly connected nonlinear systems with two different measures.

In Chapter 3, the subject matter is on the stability of solutions of a weakly connected system of differential equations based on the direct Lyapunov method and the comparison technique. In particular, sufficient conditions for asymptotic and uniform stability are obtained by using the auxiliary vector functions. It also contains a formulation of polystability of motion of a nonlinear system with small parameter.

In Chapter 4, a general approach, based on the generalization of the direct Lyapunov method combined with the asymptotic method of nonlinear mechanics, is applied to the study of stability of solutions for nonlinear systems with small perturbing forces.

In Chapter 5, fundamental results on the boundedness and stability of hybrid systems with weakly connected subsystems are presented. They are obtained based on the generalization of the direct Lyapunov method, where both vector ad matrix-valued auxiliary functions are used.

This is an excellent book for researchers and graduate students working in the areas of systems with a small parameter, which include nonlinear systems of weakly connected equations. It contains many fundamental results that are needed for carrying out research in the areas. This book is certainly a very welcome addition. It has made it much easier for graduate students or new researchers in the areas to gain up-to-date necessary background knowledge for starting to do research. It is also an excellent reference book for established researches.

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