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Existence of Positive Solutions of a Nonlinear Third-Order M-Point Boundary Value Problem for p-Laplacian Dynamic Equations on Time Scales

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Abstract: In this paper, by using fixed-point theorems in cones, we study the existence of at least one, two and three positive solution of a nonlinear third-order m-point p-Laplacian boundary value problem on time scale.

Keywords: *time scales; nontrivial solution; fixed-point theorems.*

Mathematics Subject Classification (2010): 39A10, 34B15, 34B16.

1 Introduction

We study the third-order m-point boundary value problems (MPBVP) on time scales with p-Laplacian,

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla}(t) + p(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathrm{T_k}\cap T^{k^2}}, \tag{1}$$

$$u^{\triangle\nabla}(\rho(0)) = 0, \; u^{\triangle}(T) = 0, \; u(\rho(0)) = B(\sum_{1}^{m-2} \alpha_i u^{\triangle}(\xi_i)), \tag{2}$$

where Φ_p is p-Laplacian operator, i.e. $\Phi_p(s) = |s|^{p-2}s$, $p > 1$ and $(\Phi_p)^{-1} = \Phi_q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Here $\rho(0) < \xi_1 < \xi_2 < ... < \xi_{m-2} < \sigma(T)$.

(H1) $\alpha_i \in [0,\infty)$, $i = 1,2,3...$ and $f : [0,T] \times [0,\infty) \to [0,\infty)$ is left-dense continuous function,

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(H2) $p : [0, T] \to [0, \infty)$ is left-dense continuous function,

(H3) $B : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies the existence of $B_0 \geq B_1 > 0$ such that $B_0 s \leq B(s) \leq B_1 s$, for $s \in [0, \infty)$.

A time scale $\mathbf{T}$ is a nonempty closed subset of $\mathbf{R}$. We make the blanket assumption 0, T are points in $\mathbf{T}$. By an interval $[0, T]$, we always mean the intersection of the real interval $[0, T]$ with the given time scale; that is $[0, T] \cap \mathbf{T}$. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ, respectively, $\sigma(t) = \inf\{\tau \in \mathbf{T} | \tau > t\} \in \mathbf{T}$, $\rho(r) = \sup\{\tau \in \mathbf{T} | \tau < r\}$ for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T_k} = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T_k} = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T^k} = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T^k} = \mathbf{T}$. Some basic definitions and theorems on time scales can be found in the books [4, 5].

p-Laplacian problems with two point, three point and multi point boundary conditions for ordinary differential equations and difference equations have been studied by several authors (see [6, 10, 16] and the references therein). Recently, there has been much attention paid to the existence of positive solution for second-order and third-order nonlinear boundary value problems on time scales $[1, 2, 9, 11, 12, 15, 17, 18]$. However, to the best of our knowledge, there are not many results concerning third-order p-Laplacian dynamic equations on time scales.

In [8], Yanging Guo, Changlang Yu, Jufang Wang considered the existence of three positive solutions for the following m-point boundary value problems on infinite intervals

$$(\varphi_p(x'(t)))' + \phi(t) f(t, x(t), x'(t)) = 0, \ 0 < t < \infty, \tag{3}$$

$$x(0) = \sum_{1}^{m-2} a_i x'(\eta_i), \ \lim_{t \to \infty} x'(t) = 0. \tag{4}$$

They used Avery–Henderson fixed-point theorem on a cone to prove the existence of three positive solutions to the $(3) - (4)$ nonlinear problems.

In [15], Sihua Liang, Jihui Zhang, Zhiyong Wang prove the existence of three positive solutions for the following second order m-point boundary value problems

$$(\Phi(p(t) u^{\triangle}(t)))^{\nabla} + a(t) f(u(t)) = 0, \ t \in [0, T]_{\mathbf{T^k} \cap \mathbf{T_k}}, \tag{5}$$

$$u(0) - B_0\big(\sum_{1}^{m-2} a_i u^{\triangle}(\xi_i)\big) = 0, \ u^{\triangle}(T) = 0. \tag{6}$$

for some dynamic equations on time scales using Legget–Williams fixed-point theorem.

In [11], Zhimin He obtained the existence of at least double positive solutions of the following three-point boundary value problems

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla} + a(t) f(u(t)) = 0, \ t \in [0, T], \tag{7}$$

$$u(0) - B_0(u^{\triangle}(\eta)) = 0, \ u^{\triangle}(T) = 0, \tag{8}$$

or

$$u^{\triangle}(0) = 0, \ u(T) + B_1(u^{\triangle}(\eta) = 0, \tag{9}$$

by using double fixed-point theorem.

In [9], Wei Hang, Maoxing Liu considered the third-order nonlinear problem such that

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla} + a(t)f(u(t)) = 0,\ t \in [0, T], \tag{10}$$

$$\alpha u(0) - \beta u^{\triangle}(0) = 0,\ u(T) = \sum_{1}^{m-2} a_i u(\xi_i),\ u^{\triangle\nabla}(0) = 0. \tag{11}$$

They used the fixed-point theorem which is given by V.Lakshmikantham in [7] to prove the existence of at least one nontrivial solution to the nonlinear problem $(10)-(11)$.

Motivated by the results [15], in this paper, we will study the existence of multiple positive solutions of third-order p-Laplacian MPBVP $(1) - (2)$.

The aim of this paper is to establish some simple criteria for the existence of positive solutions of the p-Laplacian MPBVP $(1) - (2)$. This paper is organized as follows: In Section 2 we first present some properties of the solution of the linear p-Laplacian MPBVP corresponding to $(1) - (2)$. In Section 3, we state the fixed-point theorems in order to prove main results and we get the existence of at least one, two and three positive solutions for nonlinear p-Laplacian MPBVP $(1) - (2)$.

2 Preliminaries and Lemmas

To prove main results, we will give several lemmas and the following lemmas are based on the linear p-Laplacian MPBVP

$$(\Phi_p(u^{\triangle\nabla})^{\nabla}(t) + h(t) = 0, \quad t \in [0, T]_{\mathrm{T_k} \cap T^{k^2}}, \tag{12}$$

$$u^{\triangle\nabla}(\rho(0)) = 0,\ u(\rho(0)) = B(\sum_{1}^{m-2} a_i u^{\triangle}(\xi_i)),\ u^{\triangle}(T) = 0. \tag{13}$$

Lemma 2.1 *For $h \in \mathbf{C}_{ld}([0, T] \times \mathbf{R})$, the problems* (12) *and* (13) *have the unique solution*

$$u(t) = B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s) + \int_{\rho(0)}^{t} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r. \tag{14}$$

Proof. From the equation (12) we can easily obtain

$$u^{\triangle\nabla}(s) = -\Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau), \quad u^{\triangle}(t) = \int_{t}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s$$

Therefore, we have

$$u(t) = u(\rho(0)) + \int_{\rho(0)}^{t} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r.$$

Applying the boundary conditions (2.13) we have

$$u(t) = B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s) + \int_{\rho(0)}^{t} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} h(\tau)\nabla\tau)\nabla s)\triangle r.$$

It is easy to see that the p-Laplacian MPBVP $(\Phi_p(u^{\triangle\nabla}(t))^{\nabla}=0, u^{\triangle\nabla}(\rho(0))=0,$ $u(\rho(0))=B(\sum_{1}^{m-2} a_i u^{\triangle}(\xi_i))=0,\ u^{\triangle}(T)=0$ has only the trival solution. Thus u is the unique solution of (12) and (13). The proof is complete. □

Let X denote Banach space $\mathbf{C}_{ld}([\rho(0),T],[0,\infty))$ with the norm $\|u\|=\sup|u(t)|$, $t\in[\rho(0),T]$. Define the cone $P\subset X$ by

$$P=\{u\in X: u(t)>0,\ u^{\triangle}(t)>0,\ t\in[\rho(0),T],\ u\ is\ concave\}. \tag{15}$$

For $u\in P$ define the operator L by

$$\begin{aligned} Lu(t)=B(&\sum_{1}^{m-2} a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\ &+\int_{\rho(0)}^{t}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r. \end{aligned} \tag{16}$$

Obviously, from the definition of L we have $Lu(t)\geq 0$ and for $t\in[\rho(0),T]$ we get

$$(Lu)^{\triangle}(t)=\int_{t}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s\geq 0.$$

As

$$(Lu)^{\triangle\nabla}(t)=-\Phi_q(\int_{\rho(0)}^{t}p(\tau)f(\tau,u(\tau))\nabla\tau)\leq 0,$$

then Lu is concave. Therefore $L: P\to P$ and $\|Lu\|=\sup|Lu(t)|=Lu(T)$ for $t\in[\rho(0),T]$.

Also it is easy to check that L is a completely continuous operator by a standard application of the Arzela-Ascoli theorem.

Lemma 2.2 *If $u\in P$ and $\|u\|=\sup|u(t)|,\ t\in[\rho(0),T]$, then*

$$u(t)\geq\frac{t-\rho(0)}{T-\rho(0)}\|u\|. \tag{17}$$

Proof. It can be easily shown by the similar way as in Lemma 3.1 in the reference [14].

3 Existence of Positive Solutions

In this section we will prove the existence of multiple positive solutions of our problem. We will need also the following Krasnoselkii's fixed-point theorem to prove the existence of at least one positive solution of p-Laplacian MPBVP (1)–(2).

Theorem 3.1 [13] *Let X be a Banach space and $P\subset X$ be a cone. Assume Ω_1 and Ω_2 are open bounded subsets of P with $0\in P$, $\overline{\Omega}_1\subset\Omega_2$, and let $L: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator such that either*

(i) $\|Lu\|\leq\|u\|$ *for $u\in P\cap\partial\Omega_1$,* $\|Lu\|\geq\|u\|$ *for $u\in P\cap\partial\Omega_2$;*

or

(ii) $\|Lu\|\geq\|u\|$ *for $u\in P\cap\partial\Omega_1$,* $\|Lu\|\leq\|u\|$ *for $P\cap\partial\Omega_2$ hold.*

Then L has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

Theorem 3.2 *Assume conditions* $(H_1)-(H_3)$ *are satisfied. In addition, suppose there exist numbers* $0<r<R<\infty$ *such that*

(i) $f(\tau,u(\tau))\leq \Phi_p(\frac{u}{k_1}),\ if\ 0\leq u\leq r,$

and

(ii) $f(\tau,u(\tau))\geq \Phi_p(\frac{u}{k_2}),\ if\ R\leq u\leq \infty,$

where

$$k_1=B_1\sum_1^{m-2}a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\nabla\tau)\nabla s+\int_{\rho(0}^{T}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\nabla\tau)\nabla s)\triangle r,$$

$$k_2=\int_{\rho(0)}^{T}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\Phi_p(\frac{\tau}{T})\nabla\tau)\nabla s)\triangle r.$$

Then the p-Laplacian MPBVP $(1)-(2)$ *has at least one positive solution.*

Proof. Define the cone P as in (15). It is also easy to check that $L:P\to P$ is completely continuous and $LP\subset P$. If $u\in P$ with $\|u\|=r$ then we get

$$\begin{aligned}\|Lu\| &\leq B(\sum_1^{m-2}a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)|f(\tau,u(\tau))|\nabla\tau)\nabla s)\\ &+\int_{\rho(0)}^{T}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)|f(\tau,u(\tau))|\nabla\tau)\nabla s)\triangle r\\ &\leq B_1\sum_1^{m-2}a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\Phi_p(\frac{u}{k_1})\nabla\tau)\nabla s\\ &+\int_{\rho(0)}^{T}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\Phi_p(\frac{u}{k_1})\nabla\tau)\nabla s)\triangle r\\ &=\frac{u}{k_1}[B_1\sum_1^{m-2}a_i\int_{\xi_i}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\nabla\tau)\nabla s\\ &+\int_{\rho(0)}^{T}(\int_{r}^{T}\Phi_q(\int_{\rho(0)}^{s}p(\tau)\nabla\tau)\nabla s)\triangle r]\\ &=\|u\|.\end{aligned}$$

So if we set

$$\Omega_1=\{u\in \mathbf{C}_{ld}([\rho(0),T],[0,\infty)):\ \|u\|<r\},$$

then $\|Lu\|\leq\|u\|$ for $u\in P\cap\partial\Omega_1$.

Let us now set

$$\Omega_2=\{u\in \mathbf{C}_{ld}([\rho(0),T],[0,\infty)):\ \|u\|<R\},$$

then for $u \in P$ with $\|u\| = R$, we have

$$\begin{aligned}
\|Lu\| &= |B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{T}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r| \\
&\geq \int_{\rho(0)}^{T}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)|f(\tau,u(\tau))|\nabla\tau)\nabla s)\triangle r \\
&\geq \int_{\rho(0)}^{T}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{u}{k_2})\nabla\tau)\nabla s)\triangle r \\
&\geq \int_{\rho(0)}^{T}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{\tau}{T}\frac{\|u\|}{k_2})\nabla\tau)\nabla s)\triangle r \\
&= \frac{\|u\|}{k_2}[\int_{\rho(0)}^{T}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)\Phi_p(\frac{\tau}{T})\nabla\tau)\nabla s)\triangle r \\
&= \|u\|.
\end{aligned}$$

Hence $\|Lu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Thus by the first part of Theorem 3.1, L has a fixed point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Therefore the p-Laplacian MPBVP $(1)-(2)$ has at least one positive solution. □

Applying the following Avery–Henderson fixed point theorem, we will prove the existence of at least two positive solutions to the p-Laplacian MPBVP $(1)-(2)$.

Theorem 3.3 [3] *Let P be a cone in a real Banach space X. Set*
$P(\psi, z) = \{u \in P : \psi(u) < z\}$
If η and ψ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants z and γ
$\psi(u) \leq \theta(u) \leq \eta(u)$ *and* $\|u\| \leq \gamma\psi(u)$
for all $u \in \overline{P(\psi, z)}$. Suppose that there exist positive numbers $x < y < z$ such that
$\theta(\lambda u) \leq \lambda\theta(u)$ *for all* $0 < \lambda < 1$ *and* $u \in \partial P(\theta, y)$.
If $L : \overline{P(\psi, z)} \to P$ is completely continuous operator satisfying
(i) $\psi(Lu) > z$ *for all* $u \in \partial P(\psi, z)$
(ii) $\theta(Lu) < y$ *for all* $u \in \partial P(\theta, y)$
(iii) $P(\eta, x) \neq \emptyset$ *and* $\eta(Lu) > x$ *for all* $u \in \partial P(\eta, x)$. *Then L has at least two fixed points u_1 and u_2 such that*

$$x < \eta(u_1) \text{ with } \theta(u_1) < y \text{ and } y < \theta(u_2) \text{ with } \psi(u_2) < z.$$

Theorem 3.4 *Assume $(H_1) - (H_3)$ hold. Suppose there exist positive numbers $x < \frac{F}{E}y < \frac{(\xi_1-\rho(0))F}{(T-\rho(0))E}z$ such that the function f satisfies the following conditions:*
(i) $f(s,u) > \Phi_p(\frac{z}{D})$ *for* $s \in [\xi_1, T]$ *and* $u \in [z, \frac{T-\rho(0)}{\xi_1-\rho(0)}z]$,
(ii) $f(s,u) < \Phi_p(\frac{y}{E})$ *for* $s \in [\rho(0), T]$ *and* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}y]$,
(iii) $f(s,u) > \Phi_p(\frac{x}{F})$ *for* $s \in [\rho(0), \xi_{m-2}]$ *and* $u \in [0, \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}x]$.
for some positive constants D, E and F. Then p-Laplacian MPBVP $(1)-(2)$ has at least two positive solutions u_1 and u_2 such that

$$u_1(\xi_1) < y \text{ and } u_1(\xi_{m-2}) > x, u_2(\xi_1) > y \text{ and } u_2(\xi_1) < z.$$

Let us define the positive constants D, E and F such that

$$D = B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s)\triangle r,$$
$$E = B_1 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s)\triangle r,$$
$$F = B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_{m-2}} (\int_{\xi_{m-2}}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau)\nabla\tau)\nabla s)\triangle r.$$

Proof. Define the cone P as in (15). We know L is completely continuous and $LP \subset P$. Let the nonnegative increasing continuous functionals ψ, θ and η be defined on the cone by
$\psi(u) = \min u(t) = u(\xi_1),\ t \in [\xi_1, \xi_{m-2}]$,
$\theta(u) = \max u(t) = u(\xi_1),\ t \in [\rho(0), \xi_1]$,
$\eta(u) = \max u(t) = u(\xi_{m-2}),\ t \in [\rho(0), \xi_{m-2}]$.

For each $u \in P,\ \psi(u) = \theta(u) \leq \eta(u)$. In addition for each $u \in P$

$$\psi(u) = u(\xi_1) \geq \frac{\xi_1 - \rho(0)}{T - \rho(0)} \|u\|. \tag{18}$$

Also $\theta(0) = 0$ and we have $\theta(\lambda u) = \lambda\theta(u)$ and for $u \in P$ and $\lambda \in [0, 1]$.

We now verify that all conditions of Theorem 3.3 are satisfied.
If $u \in \partial P(\psi, z)$ then $\psi(u) = min_{t\in[\xi_1,\xi_{m-2}]} u(t) = u(\xi_1) = z$. So we have $u(t) \geq z$, for $t \in [\xi_1, T]$, and from (18) $z \leq u(t) \leq \|u\| \leq \frac{T-\rho(0)}{\xi_1-\rho(0)} z$ for $t \in [\xi_1, T]$. Then assumption (i) implies $f(s, u) > \Phi_p(\frac{z}{D})$ for $s \in [\xi_1, T]$.

Since $Lu \in P$ we get

$$\begin{aligned}
\psi(Lu) &= Lu(\xi_1)\\
&= B(\sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\\
&+ \int_{\rho(0)}^{\xi_1} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r\\
&> B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s\\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r\\
&> B_0 \sum_1^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s\\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau) f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r
\end{aligned}$$

$$
\begin{aligned}
&> \frac{z}{D}\{B_0\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= z.
\end{aligned}
$$

Hence condition (i) of Theorem 3.3 is satisfied.

Secondly, we show that (ii) of Theorem 3.3 is fulfilled. For this, we select $u \in \partial P(\theta, y)$. Then

$$\theta(u) = \max_{t\in[\rho(0),\xi_1]} u(t) = u(\xi_1) = y.$$

We know from (2.17)

$$0 \leq u(t) \leq \frac{T-\rho(0)}{\xi_1-\rho(0)}y,$$

for $t \in [\rho(0), T]$. Then assumption (ii) implies

$$f(s,u) < \Phi_p(\frac{y}{E}),$$

for $s \in [\rho(0), T]$. Therefore

$$
\begin{aligned}
\theta(Lu) &= Lu(\xi_1) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{y}{E}\{B_1\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= y.
\end{aligned}
$$

Then condition (ii) of Theorem 3.3 holds.

Finally, we verify that (iii) of Theorem 3.3 is also satisfied.
Since $0 \in P$ and $x > 0$, $P(\eta, x) \neq \emptyset$, that $\eta(0) = 0 < x$. Now let $u \in \partial P(\eta, x)$. Then

$$\eta(u) = \max_{t\in[\rho(0),\xi_{m-2}]} u(t) = u(\xi_{m-2}) = x.$$

We know from (2.17)

$$0 \leq u(t) \leq \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}x,$$

for $t \in [\rho(0), \xi_{m-2}]$. Then assumption (iii) implies $f(s,u) > \Phi_p(\frac{x}{F})$ for $s \in [\rho(0), \xi_{m-2}]$. As before, we get

$$\begin{aligned}
\eta(Lu) &= Lu(\xi_{m-2}) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau)) \nabla\tau) \nabla s) \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau) f(\tau, u(\tau)) \nabla\tau) \nabla s) \triangle r \\
&> B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) f(\tau, u(\tau)) \nabla\tau) \nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\xi_{m-2}}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) f(\tau, u(\tau)) \nabla\tau) \nabla s) \triangle r \\
&> \frac{x}{F}\{B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) \nabla\tau) \nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\xi_{m-2}}^{T} \Phi_q(\int_{\rho(0)}^{\xi_{m-2}} p(\tau) \nabla\tau) \nabla s) \triangle r\} \\
&= x.
\end{aligned}$$

Since all conditions of Theorem 3.3 are satisfied, the p-Laplacian MPBVP (1) – (2) has at least two positive solutions u_1 and u_2 such that

$$x < \eta(u_1),\ \theta(u_1) < y \textit{ and } y < \theta(u_2),\ \psi(u_2) < z. \ \square$$

We will use the following Legget-Williams fixed point theorem to prove the existence of at least three positive solutions to the p-Laplacian MPBVP (1) – (2).

Theorem 3.5 [14]*Let P be a cone in a Banach space X. Set*
$P(\gamma, c) = \{u \in P : \gamma(u) < c\}$.
Let α, β and γ be three increasing nonnegative and continuous functionals on P, satisfying for some $c > 0$ and $A > 0$ such that
$\gamma(u) \le \beta(u) \le \alpha(u)$, $\|u\| \le A\gamma(u)$,
for all $u \in \overline{P(\gamma, c)}$. Suppose there exist a completely continuous operator $L : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that
(i) $\gamma(Lu) < c$ *for all* $u \in \partial P(\gamma, c)$;
(ii) $\beta(Lu) > b$ *for all* $u \in \partial P(\beta, b)$;
(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Lu) < a$ *for all* $u \in \partial P(\alpha, a)$.
Then L has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$ such that
$0 \le \alpha(u_1) < a < \alpha(u_2)$, $\beta(u_2) < b < \beta(u_3)$, $\gamma(u_3) < c$.

Theorem 3.6 *Assume that conditions $(H_1) - (H_3)$ are satisfied. Suppose there exist positive numbers $a < b < c$ such that function f satisfies the following conditions:*
(i) $f(s,u) < \Phi_p(\frac{c}{E})$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}c]$,
(ii) $f(s,u) > \Phi_p($
$fracbD)$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_1-\rho(0)}b]$,

(iii) $f(s,u) < \Phi_p(\frac{a}{G})$ *for all* $u \in [0, \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}a]$.

Then there exist at least three positive solutions u_1, u_2, u_3 *of p-Laplacian MPBVP* $(1.1)-(1.2)$ *such that*

$$0 \le \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

For notational convenience, we denote G *by*

$$G = B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s)\triangle r$$

and also we will take the constants D *and* E *as in Theorem* 3.4.

Proof. We define completely continuous operator L by (2.16). Let $u \in \partial P(\gamma, c)$ then $Lu(t) \ge 0$ for $t \in [0,T]$. We know that $L : \overline{P(\gamma,c)} \to P$. Let the nonnegative increasing continuous functionals γ, β and α be defined on the cone by

$\gamma(u) = \max u(t) = u(\xi_1), \quad t \in [\rho(0), \xi_1],$
$\beta(u) = \min u(t) = u(\xi_1), \quad t \in [\xi_1, \xi_{m-2}],$
$\alpha(u) = \max u(t) = u(\xi_{m-2}), \quad t \in [\rho(0), \xi_{m-2}].$

For each $u \in P$ we have

$$\gamma(u) = \beta(u) \le \alpha(u), \quad \gamma(u) = u(\xi_1) \ge \frac{\xi_1 - \rho(0)}{T - \rho(0)}\|u\|.$$

We now show that all the conditions of Theorem 3.5 are satisfied. To make use of property (i) of Theorem 3.5, we choose $u \in \partial P(\gamma, c)$. Then $\gamma(u) = \max_{t\in[\rho(0),\xi_1]} u(t) = u(\xi_1) = c$. If we recall that $\|u\| \le \frac{T-\rho(0)}{\xi_1-\rho(0)}\gamma(u) = \frac{T-\rho(0)}{\xi_1-\rho(0)}c$, we have for all $t \in [\rho(0), T]$

$$0 \le u(t) \le \frac{T-\rho(0)}{\xi_1 - \rho(0)}c.$$

Then assumption (i) of Theorem 3.6 implies $f(s,u) < \Phi_p(\frac{c}{E})$ for all $s \in [\rho(0), T]$,

$$\begin{aligned}
\gamma(Lu) &= Lu(\xi_1) \\
&= B(\sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^s p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1} (\int_r^T \Phi_q(\int_{\rho(0)}^s p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{c}{E}\{B_1 \sum_1^{m-2} a_i \int_{\xi_i}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1} (\int_{\rho(0)}^T \Phi_q(\int_{\rho(0)}^T p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= c.
\end{aligned}$$

Hence condition (i) of Theorem 3.5 is satisfied.

Secondly we show that (ii) of Theorem 3.5 is fulfilled. For this, we select $u \in \partial P(\beta, b)$. Then $\beta(u) = \min_{t\in[\xi_1,\xi_{m-2}]} u(t) = u(\xi_1) = b$. This means $u(t) > b \;\; t \in [\xi_1, T]$ and since $u \in P$, we have $b \le u(t) \le \|u\| \le \frac{T-\rho(0)}{\xi_1-\rho(0)} b$ for all $u \in P$. So we have

$$b \le u(t) \le \frac{T-\rho(0)}{\xi_1-\rho(0)} b,$$

for all $t \in [\xi_1, T]$. Then assumption (ii) of Theorem 3.6 implies $f(s, u) > \Phi_p(\frac{b}{D})$ for all $s \in [\xi_1, T]$,

$$\begin{aligned}
\beta(Lu) &= Lu(\xi_1) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau, u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)f(\tau, u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_1}(\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)f(\tau, u(\tau))\nabla\tau)\nabla s)\triangle r \\
&> \frac{b}{D}[B_0 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s + \int_{\rho(0)}^{\xi_1}(\int_{\xi_1}^{T} \Phi_q(\int_{\rho(0)}^{\xi_1} p(\tau)\nabla\tau)\nabla s)\triangle r] \\
&= b.
\end{aligned}$$

Then condition (ii) of Theorem 3.5 holds.

Finally we verify that (iii) of Theorem 3.5 is also satisfied. We note that $u(t) \equiv \frac{a}{2}$ is a member of $P(\alpha, a)$ and $\alpha(u) = \frac{a}{2} < a$ for $t \in [\rho(0), T]$. So $P(\alpha, a) \neq \emptyset$. Now let $u \in \partial P(\alpha, a)$, then $\alpha(u) = a$. This implies that $0 \le u(t) \le a$ for $t \in [\rho(0), \xi_{m-2}]$. Note that $\|u\| \le \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)}\alpha(u) = \frac{T-\rho(0)}{\xi_{m-2}-\rho(0)} a$ for all $t \in [\rho(0), \xi_{m-2}]$. So

$$0 \le u(t) \le \frac{T-\rho(0)}{\xi_1-\rho(0)} a,$$

for all $s \in [\rho(0), \xi_{m-2}]$. As before, we get

$$\begin{aligned}
\alpha(Lu) &= Lu(\xi_{m-2}) \\
&= B(\sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s) \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{r}^{T} \Phi_q(\int_{\rho(0)}^{s} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)f(\tau,u(\tau))\nabla\tau)\nabla s)\triangle r \\
&< \frac{a}{G}\{B_1 \sum_{1}^{m-2} a_i \int_{\xi_i}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s \\
&+ \int_{\rho(0)}^{\xi_{m-2}} (\int_{\rho(0)}^{T} \Phi_q(\int_{\rho(0)}^{T} p(\tau)\nabla\tau)\nabla s)\triangle r\} \\
&= a.
\end{aligned}$$

The condition (iii) of Theorem 3.5 is satisfied. Therefore Theorem 3.5 implies that L has at least three fixed points which are positive solutions $u_1, u_2, u_3 \in P(\overline{\gamma, c})$ such that

$$0 \leq \alpha(u_1) < a < \alpha(u_2),\ \beta(u_2) < b < \beta(u_3),\ \gamma(u_3) < c.$$

The proof of Theorem 3.6 is complete. □

We can illustrate our result which is given in Theorem 3.4 in the following example.

Example 3.1 Let **T**$= [0,1] \cup [2,3]$. We consider the following p-Laplacian dynamic equation:

$$(\Phi_p(u^{\triangle\nabla}))^{\nabla}(t) + p(t)f(t,u(t)) = 0,\ t \in [0,3]_{\mathbf{T_k}\cap\mathbf{T^{k^2}}} \tag{19}$$

satisfying the boundary conditions

$$u^{\triangle\nabla}(0) = 0, u^{\triangle}(3) = 0, u(0) = \sum_{1}^{2} \alpha_i u^{\triangle}(\xi_i), \tag{20}$$

where $p = q = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $m = 4$, $p(t) \equiv 1$, $B_0 = B_1 = 1$ and

$$f(t,u) = f(u) = \begin{cases} \frac{u^2}{10^4} + \frac{6}{10}, & 0 \leq u \leq 10^3, \\ 100.6 + 2(u - 10^3), & u > 10^3. \end{cases}$$

Taking $x = 1$, $y = 10$, $z = 10^4$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{5}{2}$; it is easy to see that $D = \frac{15}{8}$, $E = 12$, $F = 10$, $x < \frac{F}{E}y < \frac{F}{6E}z$ and then f(u) satisfies

$$\begin{aligned}
f(u) &> \Phi_2(\frac{z}{D}) = 5334 \quad u \in [10^4, 6 \times 10^4], \\
f(u) &< \Phi_2(\frac{y}{E}) = 0.84 \quad u \in [0, 60], \\
f(u) &> \Phi_2(\frac{x}{F}) = 0.1 \quad u \in [0, \frac{6}{5}].
\end{aligned}$$

The use of Theorem 3.4 implies four point BVP (19) − (20) has at least two positive solutions u_1, u_2 satisfying

$$u_1(\tfrac{1}{2}) < 10 \text{ and } u_1(\tfrac{5}{2}) > 1,\ u_2(\tfrac{1}{2}) > 10 \text{ and } u_2(\tfrac{5}{2}) < 10^4.\ \square$$

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