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Euler Solutions for Integro Differential Equations with Retardation and Anticipation

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Abstract: In this paper, we obtain results for Euler solution for integro differential equation with retardation and anticipation.

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1 Introduction

Integro differential equations arise quite frequently as mathematical models in diverse disciplines. The study of integro differential equations has been attracting the attention of many scientific researchers due to its potential as a better model to represent physical phenomena in various disciplines. Much work has been done in the existence and uniqueness of solutions for integro differential equations see [2, 3, 6, 7, 8, 12]. All these results are abstract in the sense that there is no specific procedure to obtain a solution of the considered equations, so the Euler solutions for integro differential equations are studied [4].

In many physical phenomena the both past history and future play an important role along with the present state and hence an appropriate model of the phenomena will be one that involves past history and future expectation also. This led to the study of systems involving both retardation and anticipation, for example, see [1]. The existence of Euler solutions have been studied for set differential equations [11], for causal differential equations [10], for delay differential equations [5], due to the inherited simplicity in its idea which paves a path for obtaining a solution of the given system. In this paper,

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we give an approach to obtaining the solution of the integro differential equation with retardation and anticipation under continuity conditions.

In this paper we consider the integro differential equations with retardation and anticipation of the type

$$x' = f(t, x, Sx, x_t, x^t), \quad t \in I = [t_0, T], \tag{1}$$

$$x_{t_0}(0) = \phi_0(0), \quad x^T(0) = \psi_0(0), \tag{2}$$

where the retardation function x_t is defined as $x_t \in C_0 = C[[-h_1, 0], \mathbb{R}]$ such that $x_t(s) = x(t+s)$, $s \in [-h_1, 0]$ and the anticipation is defined as $x^t \in C_1 = C[[0, h_2], \mathbb{R}]$ such that $x^t(\sigma) = x(t+\sigma)$ where $\sigma \in [0, h_2]$ and construct Euler solution for the fore mentioned integro differential equation with retardation and anticipation.

2 Preliminaries

In this section we begin with the integro differential equation given by

$$x' = f(t, x) + \int_{t_0}^{t} K(t, s, x(s))ds, \tag{3}$$

$$x(t_0) = x_0. \tag{4}$$

We begin with the following known results corresponding to integro differential equations which are prerequisite to obtain the Euler solutions for integro differential equations with retardation and anticipation. These results are from [9].

Theorem 2.1 *Assume that*

A(1) $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}]$, $H \in C[\mathbb{R}_+^2 \times \mathbb{R}, \mathbb{R}]$ *and* $H(t, s, u)$ *is monotone non decreasing in* u *for each* $(t, s) \in \mathbb{R}_+^2$;

A(2) $v' \leq g(t, v) + \int_{t_0}^{t} H(t, s, v(s))ds$ *and* $w' \geq g(t, w) + \int_{t_0}^{t} H(t, s, w(s))ds$;

A(3) for $(t, s) \in \mathbb{R}_+^2$, $x \geq y$ *and* $L \geq 0$,

$$g(t, x) - g(t, y) \leq L(x - y), \quad H(t, s, x) - H(t, s, y) \leq L^2(x - y).$$

Then we have $v(t) \leq w(t)$, *for* $t \geq t_0$, *provided* $v(t_0) \leq w(t_0)$.

Next we state the following result which gives existence of extremal solutions.

Theorem 2.2 *Assume that* $g \in C[[t_0, t_0 + a] \times \mathbb{R}, \mathbb{R}]$, $H \in C[[t_0, t_0 + a] \times [t_0, t_0 + a] \times \mathbb{R}, \mathbb{R}]$, $H(t, s, u)$ *is non decreasing in* u *for each* (t, s) *and* $\int_t^s |H(\sigma, s, u(s))| \, d\sigma \leq N$ *for* $t_0 \leq s \leq t \leq t_0 + a$, $u \in \Omega^0 = \{u \in C[[t_0, t_0 + a], \mathbb{R}] : |u(t) - u_0| \leq b\}$. *Then there exists a maximal and minimal solutions for the scalar IVP*

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s)) \, ds, \tag{5}$$

$$u(t_0) = u_0. \tag{6}$$

on $[t_0, t_0 + \alpha]$, *for some* $0 < \alpha < a$.

We now give the comparison theorem, which is used in the proof of our main result.

Theorem 2.3 *Assume that* $g \in C[\mathbb{R}_+^2,\ \mathbb{R}]$, $H \in C[\mathbb{R}_+^3,\ \mathbb{R}]$, $H(t,s,u)$ *is non decreasing in* u *for each* (t,s) *and for* $t \geq t_0$, $D_-m(t) \leq g(t, m(t)) + \int_{t_0}^t H(t,s,m(s))ds$, *where* $m \in C[\mathbb{R}_+,\ \mathbb{R}]$ *and* $D_-m(t) = lim_{h\to 0^-} inf[\ \frac{m(t+h)-m(t)}{h}]$. *Suppose that* $\gamma(t)$ *is the maximal solution of* $u' = g(t, u(t)) + \int_{t_0}^t H(t,s,u(s))ds$, $u(t_0) = u_0 \geq 0$, *existing on* $[t_0, \infty)$. *Then* $m(t) \leq \gamma(t)$, *for* $t \geq t_0$, *provided* $m(t_0) \leq u_0$.

Before we proceed further, we state the following known result relating to integro differential equations, which is indirectly used in our work.

Theorem 2.4 *Let* E_1 *be an open* (t,u)*-set in* $\mathbb{R}^{n+1}$ *and let* $f \in C[E_1, \mathbb{R}^n]$, $K \in C[E_1 \times \mathbb{R}_+^n, \mathbb{R}_+^n]$ *and* $x(t)$ *be a solution of* (3) *and* (4)*on some interval* $t_0 \leq t \leq a_0$. *Then* $x(t)$ *can be extended as a solution to the boundary of* E_1.

We now present a theorem relating to the largest interval of existence of maximal solutions in a particular setup.

Theorem 2.5 *Let the hypothesis of Theorem* 2.2 *hold. Suppose that the largest interval of existence of the maximal solution* $r(t)$ *of* (5) *and* (6) *is* $[t_0,\ t_0 + a)$. *Then there is an* $\epsilon_0 > 0$ *such that* $0 < \epsilon < \epsilon_0$, *the maximal solution* $r(t,\epsilon)$ *of*

$$u' = g(t,u) + \int_{t_0}^t H(t,s,u(s))\ ds + \epsilon, \tag{7}$$

$$u(t_0) = u_0 + \epsilon \geq 0, \tag{8}$$

exists over $[t_0, t_1] \subset [t_0, t_0 + a)$ *and* $\lim_{\epsilon\to 0} r(t,\epsilon) = r(t)$ *uniformly on* $[t_0, t_1]$.

3 Comparison Theorems

In order to construct the Euler solutions for the integro differential equation with retardation and anticipation. We need the following comparison theorems. We begin with the following result which deals with the existence of maximal solution in our setup, which is required for our main result.

Theorem 3.1 *Let* E *be the product space* $[t_0, t_0 + a) \times \mathbb{R}^2$ *and* $g \in C[E,\ \mathbb{R}]$, $H \in C[[t_0, t_0 + a) \times [t_0, t_0 + a) \times \mathbb{R},\ \mathbb{R}]$. *Assume that* $g(t,u,v)$ *is non decreasing in* v *for each* (t,u), *and* $H(t,s,u)$ *is non decreasing in* u *for each* (t,s). *Suppose that* $r(t)$ *is the maximal solution of the integro differential equation*

$$u' = g(t,u,u) + \int_{t_0}^t H(t,s,u(s))ds, \tag{9}$$

$$u(t_0) = u_0 \geq 0, \tag{10}$$

existing on $[t_0, t_0 + a)$ *and*

$$r(t) \geq 0, \tag{11}$$

on $[t_0, t_0 + a)$. *Then the maximal solution* $r_1(t)$ *of*

$$u' = g_1(t,u) + \int_{t_0}^t H(t,s,u(s))ds, \tag{12}$$

$$u(t_0) = u_0 \geq 0, \tag{13}$$

where $g_1(t,u) = g(t,u,r(t))$ *exists on* $[t_0, t_0 + a)$ *and* $r_1(t) = r(t)$ *for* $t \in [t_0, t_0 + a)$, $\int_s^t |H(\sigma, s, u(s))| d\sigma \leq N$ *for* $t_0 \leq s \leq t \leq t_0 + a$.

Proof. Consider the scalar integro differential equation (12) and (13). By Theorem 2.2 there exists a maximal solution $r_1(t)$ of (12) and (13) in the interval $[t_0, t_0+\alpha)$, where $0 < \alpha < a$ and by Theorem 2.4 this maximal solution can be extended from $[t_0, t_0+\alpha)$ to $[t_0, t_0+a)$. This implies that either $r_1(t)$ is defined over $[t_0, t_0+a)$ or there exists a $t_1 < t_0 + a$ such that

$$|r_1(t_k)| \to \infty, \tag{14}$$

for a certain sequence $\{t_k\}$, such that $t_k \to t_1^-$ as $k \to \infty$. Observe that

$$r'(t) = g(t, r(t), r(t)) + \int_{t_0}^{t} H(t, s, r(s))ds = g_1(t, r(t)) + \int_{t_0}^{t} H(t, s, r(s))ds,$$

and Theorem 2.3 yields that

$$r(t) \leq r_1(t), \tag{15}$$

as far as $r_1(t)$ exists. Now using the relations (11), (14) and (15), we have

$$|r_1(t_k)| \to +\infty \tag{16}$$

for some sequence $\{t_k\}$, such that $t_k \to t_1^-$ as $k \to \infty$. We shall prove that (16) does not hold. Since the largest interval of existence of maximal solution $r(t)$ of the scalar integro differential equaiton (9) and (10) is $[t_0, t_0+a)$, so by Theorem 2.5 there is an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ and the maximal solution $r(t, \epsilon)$ of

$$u' = g(t, u, u) + \int_{t_0}^{t} H(t, s, u(s))\ ds + \epsilon, \tag{17}$$

$$u(t_0) = u_0 + \epsilon \geq 0, \tag{18}$$

exists over $[t_0,\ t_1+\nu] \subset [t_0,\ t_0+a),\ \ \nu > 0,\ \ t_1+\nu < t_0+a$. From the relations (17), (18) we get

$$r'(t, \epsilon) > g(t, r(t, \epsilon), r(t, \epsilon)) + \int_{t_0}^{t} H(t, s, r(s, \epsilon))ds$$

and $r(t_0) = u_0 < u_0 + \epsilon = r(t_0, \epsilon)$. So

$$r(t_0) < r(t_0, \epsilon).$$

Now applying Theorem 2.1 we conclude that

$$r(t) < r(t, \epsilon), \tag{19}$$

for $t \in [t_0, t_1 + \nu]$. Since g is non decreasing in v, we arrive at $r'(t, \epsilon) > g_1(t, r(t, \epsilon)) + \int_{t_0}^{t} H(t, s, r(s, \epsilon))ds$, for $t \in [t_0, t_1 + \nu]$. But

$$r_1'(t) = g_1(t, r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds,$$

for $t \in [t_0, t_1]$ and $r_1(t_0) = u_0 < u_0 + \epsilon = r(t_0, \epsilon)$, so

$$r_1(t) < r(t, \epsilon),$$

for $t \in [t_0, t_1]$. Since $r(t, \epsilon)$ exists on $[t_0, t_1 + \nu]$, $\nu > 0$. This leads to a contradiction to (16). Hence $r_1(t)$ exists on $[t_0, t_0 + a)$. Thus $r(t) \le r_1(t)$ for $t \in [t_0, t_0 + a)$. Furthermore,

$$\begin{aligned} r_1'(t) &= g_1(t, r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds \\ &= g(t, r_1(t), r(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds. \end{aligned}$$

From the monotonic character of g in v, and from the relation (15), we get

$$\begin{aligned} r_1'(t) &= g(t, r_1(t), r(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds \\ &\le g(t, r_1(t), r_1(t)) + \int_{t_0}^{t} H(t, s, r_1(s))ds. \end{aligned}$$

Now using Theorem 2.3, we find that

$$r_1(t) \le r(t) \tag{20}$$

on $t \in [t_0, t_0 + a)$, which implies along with the relation (15) that $r_1(t) = r(t)$ for $t \in [t_0, t_0 + a)$.

We need the following known result in suitable form.

Theorem 3.2 *Let the hypothesis of Theorem* 3.1 *hold and* $m \in C[[t_0,\ t_0 + a),\ \mathbb{R}]$ *such that* $(t, m(t), \nu) \in E$, $t \in [t_0,\ t_0 + a)$ *and* $m(t_0) \le u_0$. *Assume that for a fixed Dini Derivative the inequality* $Dm(t) \le g(t, m(t), \nu) + \int_{t_0}^{t} H(t, s, m(s))ds$, *is satisfied for* $t \in [t_0,\ t_0 + a) - S$, *where* S *denotes an at most countable subset of* $[t_0, t_0 + a)$. *Then for all* $\nu \le r(t)$, $t \in [t_0, t_0 + a)$, *we have* $m(t) \le r(t)$, *for* $t \in [t_0, t_0 + a)$.

Proof. Since the hypothesis of Theorem 3.1 holds, so there exists a maximal solution $r_1(t)$ of the scalar integro differential equation (12) and (13) with $g_1(t, u) = g(t, u, r(t))$ exists on $[t_0, t_0 + a)$ and $r(t) = r_1(t)$ for $t \in [t_0, t_0 + a)$. Let $\nu \le r(t)$, $t \in [t_0, t_0 + a)$. Then using the monotonicity of g in ν we get

$$\begin{aligned} Dm(t) &\le g(t, m(t), \nu) + \int_{t_0}^{t} H(t, s, m(s))ds \\ &\le g(t, m(t), r(t)) + \int_{t_0}^{t} H(t, s, m(s))ds \\ Dm(t) &\le g_1(t, m(t)) + \int_{t_0}^{t} H(t, s, m(s))ds, \end{aligned}$$

for $t \in [t_0, t_0 + a) - S$, which on using Theorem 2.3 gives $m(t) \le r(t)$, for $t \in [t_0, t_0 + a)$.

The following theorem is needed before we proceed further.

Theorem 3.3 *Assume that* $m \in C[I,\ \mathbb{R}_+]$, $g \in C[I \times \mathbb{R}_+,\ \mathbb{R}_+]$, $H \in C[I \times I \times \mathbb{R}_+,\ \mathbb{R}_+]$, H *is non decreasing in* u *for each* (t, s) *and for* $t \in I = [t_0, T]$,

$$D_{-}m(t) \le g(t, |m|_0(t)) + \int_{t_0}^{t} H(t, s, |m|(s))ds, \tag{21}$$

where $|m|_0(t) = sup_{t_0 \le s \le t}|m(s)|$. *Suppose that* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s))ds, \tag{22}$$

$$u(t_0) = u_0 \ge 0, \tag{23}$$

existing on $[t_0, T)$. *Then* $m(t) \le r(t), \quad t \ge t_0$, *provided* $\mid m(t_0) \mid_0 \le u_0$.

Proof. Since the largest interval of existence of maximal solution is $[t_0, T)$ for the integro differential equation (22) so there exists an $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$, the maximal solution $r(t, t_0, u_0, \epsilon)$ of

$$u' = g(t, u) + \int_{t_0}^{t} H(t, s, u(s))ds + \epsilon, \tag{24}$$

$$u(t_0) = u_0 + \epsilon \ge 0, \tag{25}$$

existing on $[t_0,\ t_1] \subset [t_0,\ T)$, for $t_1 < T$ and $lim_{\epsilon \to 0} r(t, t_0, u_0, \epsilon) = r(t, t_0, u_0)$ uniformly on $[t_0, t_1]$. To prove the conclusion of the theorem, it is sufficient to show that

$$m(t) < r(t, t_0, u_0, \epsilon), \tag{26}$$

for $t_0 \le t \in I$. Suppose that the relation (26) does not hold then there exists $t_\alpha > t_0$ such that $m(t_\alpha) = r(t_\alpha, t_0, u_0, \epsilon)$ *and* $m(t) < r(t, t_0, u_0, \epsilon)$ for $t_0 \le t < t_\alpha$. this yields on computation,

$$D_{-}m(t_\alpha) > g(t_\alpha, r(t_\alpha, t_0, u_0, \epsilon)) + \int_{t_0}^{t} H(t_\alpha, s, r(t_\alpha, t_0, u_0, \epsilon))ds \tag{27}$$

which is contradiction. Observe that we have used the fact that $g(t, u) \ge 0,\ \ H(t, s, u) \ge 0$ implies that $r(t_\alpha, t_0, u_0, \epsilon)$ is non decreasing in t and

$$|m|_0(t_\alpha) = sup_{t_0 \le s \le t_\alpha}|m(s)| = r(t_\alpha, t_0, u_0, \epsilon) = m(t_\alpha),$$

which yields

$$\begin{aligned} D_{-}m(t_\alpha) &\le g(t_\alpha, |m|_0(t_\alpha)) + \int_{t_0}^{t_\alpha} H(t_\alpha, s, |m|_0(s))ds, \\ &= g(t_\alpha, r(t_\alpha, t_0, u_0, \epsilon)) + \int_{t_0}^{t} H(t_\alpha, s, r(t_\alpha, t_0, u_0, \epsilon))ds \end{aligned}$$

which is contradiction to (27), and the proof is complete.

4 Euler Solutions

In this section we define an Euler solution and prove a result for its existence of integro differential equation with retardation and anticipation. Further we give a result which gives conditions under which the Euler solution becomes a solution of the IVP of the integro differential equation with retardation and anticipation.

Consider the integro differential equation with retardation and anticipation:

$$x' = f(t, x, Sx, x_t, x^t), \tag{28}$$

$$x_{t_0}(0) = \phi_0(0), \quad x^T(0) = \psi_0(0), \tag{29}$$

where $t \in I = [t_0,\ T]$, $\phi_0 \in C_0$, $\psi_0 \in C_1$, $f \in C[I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1,\ \mathbb{R}]$, $Sx(t) = \int\limits_{t_0}^{t} K(t, s, x)ds$, $K(t, s, x) \in C[I^2 \times \mathbb{R},\ \mathbb{R}_+]$ and $C_0 = C[[-h_1, 0],\ \mathbb{R}]$, $C_1 = C[[0, h_2],\ \mathbb{R}]$.

In order to construct the Euler Solution we consider a partition π of the interval I and on each subinterval of the partition, we obtain a differential equation where the right hand side is a constant. This will help us to define Euler solution as a limit of a sequence of polygonal arcs.

In order to do so we have to find a reasonable estimate of x^t in the right hand side of the differential equation (28). For this we take the anticipation as

$$z(t) = \begin{cases} x^t(0), \ \textit{wherever} \quad | \xi^t(0) - \phi_0(0) | < M, \\ x^t(0) + \dfrac{\xi(t)}{j}, \end{cases} \tag{30}$$

where j is the number of points in the partition π and

$$\xi(t) = \begin{cases} \phi_0(0), \quad t \in [t_0 - h_1,\ t_0], \\ \phi_0(0) + \dfrac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t - t_0), \quad t \in [t_0,\ T], \\ \psi_0(0), \quad t \in [T,\ T + h_2]. \end{cases} \tag{31}$$

With this approximation the integro differential equation with retardation and anticipation reduces to the integro differential equation with retardation only, i.e.,

$$x' = f(t, x, Sx, x_t, z(t)), \tag{32}$$

$$x_{t_0}(0) = \phi_0(0), \quad z(T) = \psi_0(0), \tag{33}$$

for $t \in I = [t_0, T]$. Let partition of the interval $[t_0, T]$ be given by

$$\pi = \{t_0, t_1, t_2, ..., t_N = T\}. \tag{34}$$

Consider the sub interval $[t_0, t_1]$ and the differential equation (32), in that subinterval. In the right hand side of (32) replace t by t_0, x by x_0, x_t by $\phi_0(0)$, $z(t)$ by $z(t_0)$ and Sx by $(Sx(t_0),\ t_0)$ ie., in the integral replace t with t_0, s with t_0, x with x_0, so (32) reduces to

$$x' = f(t_0, x_0, (Sx(t_0), t_0), \phi_0(0), z(t_0)). \tag{35}$$

Then the right hand side of the differential equation (35) is a constant and hence (35) posses a unique solution $x(t) = x(t, t_0, \phi_0(0))$ on $[t_0, t_1]$.

Set $x_1 = x(t_1) = x(t_1, t_0, \phi_0(0))$. We now choose the next subinterval $[t_1, t_2]$ and consider the differential equation (32) by setting $t = t_1$, $x = x_1$, $x_t = \phi_1(t_1)$, $z(t) = z(t_1)$ and $Sx = (Sx(t_1),\ t_1)$, i.e., in the integral replace t with t_1, s with t_1, x with x_1. Then the system (32) reduces to

$$x' = f(t_1, x_1, (Sx(t_1), t_1), \phi_1(t_1), z(t_1)), \tag{36}$$

where

$$\phi_1(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ x(t, t_0, \phi_0(0)), & t \in [t_0, t_1], \end{cases} \tag{37}$$

$$z(t) = \begin{cases} x^{t_1}(0), & \mid \xi^{t_1}(0) - \phi_0(0) \mid < M, \\ x^{t_1}(0) + \dfrac{\xi(t_1)}{N+1}, \end{cases}$$

$$\xi(t_1) = \phi_0(0) + \frac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t_1 - t_0). \tag{38}$$

Clearly the right hand side of (36) is a constant hence there exists a unique solution $x(t) = x(t, t_1, \phi_1(t_1))$ on $[t_1, t_2]$.

Set $x_2 = x(t_2) = x(t_2, t_1, \phi_1(t_1))$. Again consider the integro differential equation with retardation (32) on $[t_2, t_3]$ and as earlier replacing t by t_2, x by x_2, x_t by $\phi_2(t_2)$, $z(t) = z(t_2)$ and Sx by $(Sx(t_2),\ t_2)$, i.e., in the integral replace t by t_2, s by t_2, x by x_2. Then the system (32) reduces to

$$x' \ = \ f(t_2, x_2, (Sx(t_2), t_2), \phi_2(t_2), z(t_2)), \tag{39}$$

where

$$\phi_2(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ \phi_1(t), & t \in [t_0, t_1], \\ x(t, t_1, \phi_1(t_1)), & t \in [t_1, t_2], \end{cases} \tag{40}$$

$$z(t) = \begin{cases} x^{t_2}(0), & \mid \xi^{t_2}(0) - \phi_0(0) \mid < M, \\ x^{t_2}(0) + \dfrac{\xi(t_2)}{N+1}, \end{cases}$$

$$\xi(t_2) = \phi_0(0) + \frac{(\psi_0(0) - \phi_0(0))}{(T - t_0)}(t_2 - t_0). \tag{41}$$

We observe that the right hand side of (39) is a constant and proceeding as earlier we get a solution $x(t, t_2, \phi_2(t_2))$ in the interval $[t_2, t_3]$. Set $x_3 = x(t_3) = x(t_3, t_2, \phi_2(t_2))$.

Now proceeding in this fashion, we construct a sequence of arcs $x(t, t_0, \phi_0(0))$, $x(t, t_1, \phi_1(t_1)), ..., x(t, t_{N-1}, \phi_{N-1}(t_{N-1}))$ on the sub intervals $[t_0, t_1]$, $[t_1, t_2]$, $..., [t_{N-1}, t_N]$ respectively, which is the Euler polygonal arcs defined on the partition $\pi = \{t_0, t_1, t_2, ..., t_N = T\}$. Thus the entire arc on I is defined by

$$x_\pi = x_\pi(t) = \{x(t, t_i, \phi_i(t_i)) : t_i \le t \le t_{i+1}, \ \ i = 0, 1, 2, ..., N-1\}, \tag{42}$$

where

$$\phi_i(t) = \begin{cases} \phi_0(t), & t \in [t_0 - h_1, t_0], \\ \phi_1(t), & t \in [t_0, t_1], \\ \vdots \\ x(t, t_{i-1}, \phi_{i-1}(t_{i-1})), & t \in [t_{i-1}, t_i]. \end{cases} \tag{43}$$

In (42) the notation emphasizes the fact that the arc corresponds to the partition π. The diameter μ_π of the partition π is given by

$$\mu_\pi = \max\{t_i - t_{i-1} : 1 \le i \le N\}. \tag{44}$$

Definition 4.1 An Euler solution for the integro differential equation with retardation and anticipation (28), (29) is any arc $x = x(t)$ which is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding to some sequence π_j such that $\pi_j \to 0$, as the diameter $\mu_{\pi_j} \to 0$, as $j \to \infty$.

Remark 4.1 Observe that the number of points N_j of the partition π_j must tend to ∞ as $\pi_j \to 0$ and also that the Euler arc satisfies the conditions $x_{t_0}(0) = \phi_0(0)$, $x^T(0) = \psi_0(0)$.

We now state a result which guarantees the existence of an Euler solution.

Theorem 4.1 *Assume that*

$$| f(t, x, Sx, x_t, z^t) | \leq g(t, | x |_0 (t), | z(t) |) + \int_{t_0}^{t} H(t, s, | x(s) |) ds, \tag{45}$$

where $f : I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1 \to \mathbb{R},\ K : I^2 \times \mathbb{R} \to \mathbb{R}_+,\ g \in C[I \times \mathbb{R}_+ \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (u, v), *is non decreasing in* u *for each* (t, v), *is non decreasing in* v *for each* (t, u) ,$H \in C[I^2 \times \mathbb{R}_+,\ \mathbb{R}_+]$ *is non decreasing in* t *for each* (s, u), *is non decreasing in* s *for each* (t, u), *is non decreasing in* u *for each* (t, s),
$| x |_0 (t) = max_{t-h_1 \leq t+s \leq t} | x(t + s) |$ *and* $r(t, t_0, u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t, u, u) + \int_{t_0}^{t} H(t, s, u) ds, \tag{46}$$

$$u(t_0) = u_0, \quad u(T) = \psi_0(0), \tag{47}$$

existing on $[t_0, T]$ *and* $|z(t)| \leq r(t)$, *and* z^t *is the reasonable estimate of* x^t. *Then,*
(a) there exists at least one Euler solution $x(t) = x(t, t_0, \phi_0(0))$ *of the IVP* (28), (29) *which satisfies the Lipschitz condition;*
(b) any Euler solution $x(t)$ *of* (28), (29) *satisfies the relation*

$$| x(t) - \phi_0(0) | \leq r(t, t_0, u_0) - u_0, \quad t \in [t_0, T], \tag{48}$$

where $u_0 =| \phi_0 |$.

Proof. Let π be the partition of $[t_0, T]$ defined by (34) and let $x_\pi = x_\pi(t)$ denote the corresponding arc with nodes of x_π represented by $x_1, x_2, x_3, ..., x_N$. Writing $x_\pi(t) = x_i(t) = x(t, t_i, \phi_i(t_i)), \quad t_i \leq t \leq t_{i+1},\ i = 0, 1, 2, ..., N - 1$, where $\phi_i(t_i)$ is given by (43) and observe that $x_i(t_i) = x_i, \quad i = 0, 1, 2, ..., N - 1$. Further for any $t \in [t_i, t_{i+1}]$, we have from the definition of Euler solution

$$\begin{aligned} | x'_\pi(t) | &=| f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) | \\ &\leq g(t_i, | x_{t_i}(0) |, | z(t_i) |) + \int_{t_0}^{t_i} H(t_i, t_i, | x(t_i) |) ds \end{aligned}$$

thus

$$| x'_\pi(t) | \leq g(t_i, | x_{t_i}(0) |, | z(t_i) |) + \int_{t_0}^{t_i} H(t_i, t_i, | x(t_i) |) ds, \quad i = 0, 1, 2, ..., N - 1. \tag{49}$$

Consider the interval $[t_0, t_1]$ and applying the properties of norm, integral and the non decreasing nature of g and H, along with the fact that both g and H are non-negative, we get

$$\begin{aligned}
\mid x_1(t) - \phi_0(0) \mid = &\ \mid \phi_0(0) + \int_{t_0}^{t} f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0))ds - \phi_0(0) \mid \\
&\leq \int_{t_0}^{t} \mid f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0)) \mid ds \\
&\leq \int_{t_0}^{t} [g(s, r(s), r(s)) + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
&\leq r(T, t_0, \mid \phi_0 \mid) - \mid \phi_0 \mid = \psi_0(0) - \phi_0(0) = M\ (say).
\end{aligned}$$

Next consider the interval $[t_1, t_2]$ again as before, using the properties of norm and integral, the monotone character of g and H and the fact that both g and H are non negative, we obtain,

$$\begin{aligned}
\mid x_2(t) - \phi_0(0) \mid = &\mid x_1(t_1) + \int_{t_1}^{t} f(t_1, x_1, Sx_1, x_{t_1}(0), z(t_1))ds - \phi_0(0) \mid \\
&\leq \int_{t_0}^{t_1} \mid f(t_0, x_0, Sx_0, x_{t_0}(0), z(t_0)) \mid ds \\
&+ \int_{t_1}^{t} \mid f(t_1, x_1, Sx_1, x_{t_1}(0), z(t_1)) \mid ds \\
&= \int_{t_0}^{t} [g(s, r(s), r(s))ds + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
&\leq r(T, t_0, \mid \phi_0 \mid) - \mid \phi_0 \mid = \psi_0(0) - \phi_0(0) = M\ (say).
\end{aligned}$$

Proceeding in this manner, on each subinterval $[t_i, t_{i+1}]$, we arrive at

$$\mid x_i(t) - \phi_0(0) \mid \leq r(T, t_0, \mid \phi_0 \mid) - \mid \phi_0 \mid = M.$$

Thus combining the relations of all polygonal arcs over the partition π, we deduce that

$$\mid x_\pi(t) - \phi_0(0) \mid \leq r(T, t_0, \mid \phi_0 \mid) - \mid \phi_0 \mid = M, \tag{50}$$

on $[t_0, T]$. Now from the relation (49), we have

$$\begin{aligned}
\mid x'_\pi(t) \mid &\leq g(t_i, \mid x_{t_i}(0) \mid, \mid z(t_i) \mid) + \int_{t_0}^{t_i} H(t_i, t_i, \mid x(t_i) \mid)ds \\
&\leq g(T, r(T), r(T)) + \int_{t_0}^{t} H(t, s, r(s))ds \\
&= r'(T, t_0, \mid \phi_0 \mid) = L\ (say).
\end{aligned}$$

We next show that x_π is Lipschitz. For this consider $t_0 \leq l \leq t \leq T$, where $l \in [t_i, t_{i+1}]$

and $t \in [t_k, t_{k+1}], \quad i < k$. Then

$$
\begin{aligned}
| x_\pi(t) - x_\pi(l) | =& | x_k + \int_{t_k}^{t} f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k))ds \\
& - \{x_i + \int_{t_i}^{l} f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i))ds\} | \\
& + ... + \int_{t_{k-1}}^{t_k} f(t_{k-1}, x_{k-1}, Sx_{k-1}, x_{t_{k-1}}(0), z(t_{k-1}))ds \\
& + \int_{t_k}^{t} f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k))ds \\
& - \{x_i + \int_{t_i}^{l} f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i))ds\} | \\
& \leq \int_{t_i}^{t_{i+1}} | f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) | ds \\
& + ... + \int_{t_{k-1}}^{t_k} | f(t_{k-1}, x_{k-1}, Sx_{k-1}, x_{t_{k-1}}(0), z(t_{k-1})) | ds \\
& + \int_{t_k}^{t} | f(t_k, x_k, Sx_k, x_{t_k}(0), z(t_k)) | ds \\
& - \int_{t_i}^{l} | f(t_i, x_i, Sx_i, x_{t_i}(0), z(t_i)) | ds \\
& = \int_{l}^{t} [g(s, r(s), r(s)) + \int_{s}^{t} H(\sigma, s, r(s))d\sigma]ds \\
& = \int_{l}^{t} r'(s, t_0, u_0)ds \leq L(t - l),
\end{aligned}
$$

for some $\xi \in (l, t)$. This follows using the relations (45), (46), (47) along with the fact that $g(t, u, v), \ H(t, s, u), \ r(t)$ are positive and non decreasing. Thus x_π satisfies the Lipschitz condition with some constant L on $[t_0, T]$. Now let π_j be a sequence of partitions of $[t_0, T]$ such that $\pi_j \to 0$ as $j \to \infty$. Thus from the earlier construction, we get a sequence of polygonal arcs x_{π_j} on $[t_0, T]$ corresponding to each partition π_j satisfying

$$x_{\pi_j}(t_0) = \phi_0(0), \quad | x_{\pi_j}(t) - \phi_0(0) | \leq M, \quad | x'_{\pi_j}(t) | \leq L.$$

Hence the family $\{x_{\pi_j}\}$ is equicontinuous and uniformly bounded. Then the family $\{x_{\pi_j}\}$ satisfies the hypothesis of the Ascoli–Arzela Theorem and hence we obtain a subsequence which converges uniformly to a continuous function $x(t)$ on $[t_0, T]$ which is absolutely continuous on $[t_0, T]$. Now using the definition of the Euler solution, we conclude that $x(t)$ is an Euler solution for (28), (29) on $[t_0, T]$. To prove the relation in (b), it suffices to observe that $x(t)$ is the uniform limit of the polygonal arcs that satisfy the relation (48) and thus inherits the property. Thus the proof is complete.

Remark 4.2 If f and K are continuous and $K(t, s, x)$ is non decreasing in t for each (s, x), we can show that the Euler solution is a solution. This is the essence of the next result.

Theorem 4.2 *Assume that*

$$| f(t,x,Sx,x_t,z^t) | \leq g(t, | x |_0 (t), | z(t) |) + \int_{t_0}^{t} H(t,s,| x(s) |)ds, \tag{51}$$

where $g \in C[I \times \mathbb{R}_+ \times \mathbb{R}_+, \ \mathbb{R}_+]$ *is non decreasing in* t *for each* (u,v), *is non decreasing in* u *for each* (t,v), *is non decreasing in* v *for each* (t,u) *,*$H \in C[I^2 \times \mathbb{R}_+, \ \mathbb{R}_+]$ *is non decreasing in* t *for each* (s,u), *is non decreasing in* s *for each* (t,u), *is non decreasing in* u *for each* (t,s), $| x |_0 (t) = max_{t-h_1 \leq t+s \leq t} | x(t+s) |$ *and* $r(t,t_0,u_0)$ *is the maximal solution of the scalar integro differential equation*

$$u' = g(t,u,u) + \int_{t_0}^{t} H(t,s,u)ds, \tag{52}$$

$$u(t_0) = u_0, \quad u(T) = \psi_0(0), \tag{53}$$

existing on $[t_0,T]$, $|z(t)| \leq r(t)$, *and* $z(t)$ *is the reasonable estimate of* x^t. *Further suppose that* $f \in C[I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1, \ \mathbb{R}]$, $K \in C[I^2 \times \mathbb{R}, \mathbb{R}_+]$ *is non decreasing in* t *for each* (s,x), $max_{t,s \in [t_0,T]} K(t,s,x) = k_1 \leq \frac{M+\phi_0(0)}{T-t_0}$. *Then the Euler solution* $x(t)$ *is a solution of* (28)*,* (29)*.*

Proof. Since the hypothesis of Theorem 4.1 is satisfied so we obtain a sequence $\{x_{\pi_j}\}$ of polygonal arcs for the integro differential equation with retardation and anticipation (28), (29) that converge uniformly to an Euler solution $x(t)$ on $[t_0,T]$.

Let $\widehat{B}(\phi_0(0),M) = \{(x,Sx,x_t,x^t) : \ x \in C[I,\mathbb{R}], \ \ | x(t) - \phi_0(0) | \leq M,$ $| Sx(t) - \phi_0(0) | \leq k_1(T-t_0) - | \phi_0(0) | \leq M, \ \ \sup_{-h_1 \leq s \leq 0} | x(t+s) - \phi_0(0) | \leq M,$ $\sup_{\sigma \in [0,h_2]} | x(t+\sigma) - \phi_0(0) | \leq M, \ \ t \in [t_0,T]\}$. Then, we observe that all the Euler polygonal arcs belongs to the ball $\widehat{B}(\phi_0(0),M)$, from the proof of Theorem 4.1, also we conclude that all these Euler arcs satisfy Lipschitz condition with some constant L. Now since f is continuous implies that it is uniformly continuous on compact sets $I \times \widehat{B}$. Hence for any given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$| t - t^* | < \delta, \quad | x(t) - x(t^*) | < \delta, \quad | Sx(t) - Sx(t^*) | < \delta, \quad | x_t - x_{t^*} | < \delta, \quad | x^t - x^{t^*} | < \delta,$$

implies

$$| f(t,x,Sx,x_t,x^t) - f(t^*,x^*,Sx^*,x_{t^*},x^{t^*}) | < \epsilon,$$

for any $t, \ t^* \in [t_0,T]$ and $x, x^* \in C[[t_0,T],\mathbb{R}]$ such that $(x,Sx,x_t,x^t) \in \widehat{B}(\phi_0(0),M)$. Let j be sufficiently large so that the diameter of μ_{π_j} corresponding to that j which satisfies $\mu_{\pi_j} < \delta$ and $L\mu_{\pi_j} < \delta, \ \ k_1\mu_{\pi_j} < \delta, \ \ (L + \frac{M}{j(T-t_0)})\mu_{\pi_j} < \delta$. Let $\pi_j = \{t_0,t_1,t_2,...,T\}$. Now for any t, which is not one of the infinitely many points at which $x_{\pi_j}(t)$ is a node, then we have $x'_{\pi_j}(t) = f(\widehat{t}, x_{\pi_j}(\widehat{t}), Sx_{\pi_j}(\widehat{t}), x_{\pi_{j\widehat{t}}}, z(\widehat{t}))$ for some $\widehat{t}$ with in $\mu_{\pi_j} < \delta$ of t. We have $| t - \widehat{t} | < \delta$, using the fact that x_{π_j} is Lipschitz, we get $| x_{\pi_j}(t) - x_{\pi_j}(\widehat{t}) | \leq L(t - \widehat{t}) \leq L\mu_{\pi_j} < \delta$,

$$\begin{aligned} | Sx_{\pi_j}(t) - Sx_{\pi_j}(\widehat{t}) | = & \ | \int_{t_0}^{t} K(t,s,x_{\pi_j}(s)ds - \int_{t_0}^{\widehat{t}} K(\widehat{t},s,x_{\pi_j}(s)ds | \\ \leq & \int_{t_0}^{t} | K(t,s,x_{\pi_j}(s) | ds < \delta. \end{aligned}$$

Now consider $\mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid$ for $t - h_1 \leq t+s \leq t$. Then

$$\mid x_{\pi_{jt}}(s) - x_{\pi_{j\widehat{t}}}(s) \mid = \mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid < \delta,$$
$$\mid x_{\pi_{jt}} - x_{\pi_{j\widehat{t}}} \mid = sup_{t_0+h_1 \leq t+s \leq t} \mid x_{\pi_j}(t+s) - x_{\pi_j}(\widehat{t}+s) \mid \leq L\mu_{\pi_j} < \delta.$$

Also if $\mid \xi^t(0) - \phi_0(0) \mid < M$ then $\mid x^t_{\pi_j}(0) - x^{\widehat{t}}_{\pi_j}(0) \mid = \mid x_{\pi_j}(t) - x_{\pi_j}(\widehat{t}) \mid < \delta$ otherwise

$$\mid z(t) - z(t_1) \mid = \mid x^t_{\pi_j}(0) + \frac{z(t)}{j} - \frac{z(\widehat{t})}{j} - x^{\widehat{t}}_{\pi_j}(0) \mid \leq [L + \frac{M}{j(T-t_0)}]\mu_{\pi_j} < \delta.$$

Hence we have $\mid z(t) - z(\widehat{t}) \mid < \delta$. Thus by uniform continuity of f on compact sets $\mid x'_{\pi_j}(t) - f(t, x_{\pi_j}(t), Sx_{\pi_j}(t), x_{\pi_{jt}}, z(t)) \mid$

$$= \mid f(\widehat{t}, x_{\pi_j}(\widehat{t}), Sx_{\pi_j}(\widehat{t}), x_{\pi_{j\widehat{t}}}, z(\widehat{t})) - f(t, x_{\pi_j}(t), Sx_{\pi_j}(t), x_{\pi_{jt}}, z(t)) \mid < \epsilon.$$

Now for any $t \in [t_0, T]$, consider

$$\mid x_{\pi_j}(t) - \phi_0(0) - \int_{t_0}^{t} f(s, x_{\pi_j}(s), Sx_{\pi_j}(s), x_{\pi_{js}}, z(s))ds \mid$$
$$\leq \int_{t_0}^{t} \mid x'_{\pi_j}(s) - f(s, x_{\pi_j}(s), Sx_{\pi_j}(s), x_{\pi_{js}}, z(s)) \mid ds \leq \epsilon(T - t_0).$$

Letting $j \to \infty$ in the above inequality, we get

$$\mid x(t) - \phi_0(0) - \int_{t_0}^{t} f(s, x(s), Sx(s), x_s, x^s)ds \mid \quad < \epsilon(T - t_0).$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$x(t) = \phi_0(0) + \int_{t_0}^{t} f(s, x(s), Sx(s), x_s, x^s)ds$$

which implies that $x(t)$ is continuously differentiable and hence

$$x'(t) = f(t, x, Sx, x_t, x^t)$$

and $x_{t_0}(0) = \phi_0(0),\ x^T(0) = z(T) = \psi_0(0), \quad t_0 \in [t_0, T]$. Thus the proof is complete.

5 Conclusion

The concepts of anticipation and retardation arise naturally when modeling any goal oriented physical phenomena. Recently, integro differential equations including these concepts, have been studied in [6, 12]. In this paper we provided an existence result, using the concept of Euler solutions and gave criteria under which this Euler solution becomes a solution. In future, we propose to develop the necessary tools to obtain numerical solutions of the considered problem.

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