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Existence, Uniqueness and Asymptotic Stability of Solutions to Non-Autonomous Semi-Linear Differential Equations with Deviated Arguments

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Abstract: We consider a non-autonomous semi-linear differential equation of parabolic type with a deviated argument in an arbitrary Banach space. Using the Sobolevskiǐ-Tanabe theory of parabolic equations, we prove the existence and uniqueness of a solution. We also discuss the asymptotic stability of a solution. As an application, we give an example to illustrate the main results.

Keywords: *analytic semigroup, parabolic equation, differential equation with a deviated argument, Banach fixed point theorem.*

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1 Introduction

The purpose of this article is to study the following differential equation in a Banach space $(X, \|\cdot\|)$:

$$\left.\begin{aligned} \frac{du}{dt} + A(t)u(t) &= f(t, u(t), u(h(u(t), t))),\ t > 0; \\ u(0) &= u_0,\ u_0 \in X. \end{aligned}\right\} \tag{1}$$

We assume that for each $t \geq 0$, $-A(t)$ generates an analytic semigroup of bounded linear operators on X, $f\colon [0,\infty) \times X \times X \to X$ and $h\colon X \times [0,\infty) \to [0,\infty)$. The nonlinear continuous functions f and h satisfy suitable growth conditions in their arguments stated in Section 2.

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Differential equations with deviated arguments model certain real world systems in the theory of automatic control, the study of problems related with combustion in rocket motion, the theory of self-oscillating systems, problems of long-term planning in economics, biological systems, and many other systems in the areas of science and technology [3]. Recently, many authors have studied the existence, uniqueness and continuous dependence of a solution of the differential equation of the type (1) (see e.g. Gal [6, 7]; Grimm [8]; Jankowski [12]; Oberg [16]). More details of differential equation with deviated arguments can be found in Bahuguna and Muslim [1], Dubey [2], El'sgol'ts and Norkin [3], Gal [6,7], Grimm [8], Jankowski [12], Kwaspisz [14] and Pandey et. al [17,18].

Oberg [16] has studied the following problem in $\mathbb{R}^n$:

$$\left.\begin{aligned} \frac{du(t)}{dt} &= f(t, u(t), u(h(t, u(t)))),\ t > 0; \\ u(0) &= u_0, u_0 \in \mathbb{R}^n, \end{aligned}\right\} \tag{2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. The existence theorem for a solution to Problem (2) has been obtained by the Banach fixed point theorem, when f and h are continuous and uniformly locally Lipschitz on all of their variables.

The following problem with a deviated argument in a Banach space $(X, \|\cdot\|)$ has been studied by Gal [6],

$$\left.\begin{aligned} \frac{du}{dt} - Au(t) &= f(t, u(t), u(h(u(t), t))), \quad t > 0; \\ u(0) &= u_0,\ u_0 \in X, \end{aligned}\right\} \tag{3}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. The existence and uniqueness of a solution of (3) has been established under the following conditions on the functions f and h:

(a) $f : [0, \infty) \times X_\alpha \times X_{\alpha-1} \to X$ satisfies

$$\|f(t, x, x') - f(s, y, y')\| \le L_f(|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x' - y'\|_{\alpha-1})$$

for all $x, y \in X_\alpha,\ x', y' \in X_{\alpha-1},\ s, t \in [0, \infty)$, for some constants $L_f > 0$ and $\theta_1 \in (0, 1]$.

(b) $h : X_\alpha \times [0, \infty) \to [0, \infty)$ satisfies

$$|h(x, t) - h(y, s)| \le L_h(\|x - y\|_\alpha + |t - s|^{\theta_2})$$

for all $x, y \in X_\alpha,\ s, t \in [0, \infty)$, for some constants $L_h > 0$ and $\theta_2 \in (0, 1]$.

For $0 < \alpha \le 1$, $\|x\|_\alpha = \|(-A)^\alpha x\|$, denotes the norm on X_α, the domain of $(-A)^\alpha$.

The main objective is to establish the existence, uniqueness and asymptotic stability of a solution to Problem (1) generalizing some results of Gal [6]. In addition, we establish a stability theorem.

The article is organized as follows. We provide preliminaries, assumptions and lemmas needed for proving the main results in Section 2. We prove the local and global existence, and stability of a solution in Section 3. An example is considered to illustrate the application of the main results.

2 Preliminaries and Assumptions

In this section, we give basic assumptions, preliminaries and lemmas necessary to prove the main results. The material presented here can be found in more details by Friedman [4], Henry [9], Krien [13], Ladas and Lakshmikantham [15], Sobolevskiǐ [19] and Tanabe [20].

Let $(X, \|\cdot\|)$ be a complex Banach space. Let $T \in [0, \infty)$ and $\{A(t) : 0 \le t \le T\}$ be a family of closed linear operators on the Banach space X. We will use the following assumptions [4].

(A1) The domain $D(A)$ of $A(t)$ is dense in X and independent of t.

(A2) For each $t \in [0, T]$, the resolvent $R(\lambda; A(t))$ exists for all $\mathrm{Re}\lambda \le 0$ and there is a constant $C > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \le \frac{C}{|\lambda| + 1}, \ \mathrm{Re}\lambda \le 0, \ t \in [0, T].$$

(A3) For each fixed $s \in [0, T]$, there are constants $C > 0$ and $\rho \in (0, 1]$, such that

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \le C|t - \tau|^{\rho}$$

for any $t, \tau \in [0, T]$. Here C and ρ are independent of t, τ and s.

The assumption **(A2)** implies that for each $s \in [0, T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \ge 0\}$ in $B(X)$, where $B(X)$ denotes the Banach algebra of all bounded linear operators on X. Then there exist positive constants C and d such that

$$\|e^{-tA(s)}\| \le Ce^{-dt}, \quad t \ge 0; \tag{4}$$

$$\|A(s)e^{-tA(s)}\| \le \frac{Ce^{-dt}}{t}, \quad t > 0, \tag{5}$$

for all $s \in [0, T]$ [4].

The assumptions **(A1)**, **(A2)** and **(A3)** imply the existence of a unique fundamental solution $\{U(t, s) : 0 \le s \le t \le T\}$ to the homogeneous Cauchy problem that possesses the following properties [4]:

(i) $U(t, s) \in B(X)$ and $U(t, s)$ is strongly continuous in t, s for all $0 \le s \le t \le T$.

(ii) $U(t, s)x \in D(A)$ for each $x \in X$, for all $0 \le s \le t \le T$.

(iii) $U(t, r)U(r, s) = U(t, s)$ for all $0 \le s \le r \le t \le T$.

(iv) the derivative $\partial U(t, s)/\partial t$ exists in the strong operator topology and belongs to $B(X)$ for all $0 \le s < t \le T$, and strongly continuous in t, where $s < t \le T$.

(v) $\dfrac{\partial U(t, s)}{\partial t} + A(t)U(t, s) = 0$ and $U(s, s) = I$ for all $0 \le s < t \le T$.

For $\alpha > 0$, we define negative fractional powers $A(t)^{-\alpha}$ [4][cf. inequality 4] by

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\tau A(t)} \tau^{\alpha - 1} d\tau.$$

Then $A(t)^{-\alpha}$ is bijective and bounded linear operator on X. We define the positive fractional powers of $A(t)$ by $A(t)^{\alpha} \equiv [A(t)^{-\alpha}]^{-1}$. Then $A(t)^{\alpha}$ is a closed linear operator with the domain $D(A(t)^{\alpha})$ dense in X and $D(A(t)^{\alpha}) \subset D(A(t)^{\beta})$ if $\alpha > \beta$. For $0 < \alpha \leq 1$, let $X_{\alpha} = D(A(0)^{\alpha})$ and equip this space with the graph norm

$$\|x\|_{\alpha} = \|A(0)^{\alpha}x\|.$$

Then X_{α} is a Banach space endowed with the norm $\|\cdot\|_{\alpha}$. If $0 < \alpha \leq 1$, the embedding $X_1 \hookrightarrow X_{\alpha} \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, define $X_{-\alpha} = (X_{\alpha})^*$, the dual space of X_{α}, and endow with the natural norm

$$\|x\|_{-\alpha} = \|A(0)^{-\alpha}x\|.$$

Also the assumption **(A3)** implies that there exists a constant $C > 0$ such that

$$\|A(t)A(s)^{-1}\| \leq C$$

for all $0 \leq s, t \leq T$. Hence, for each t, the functional $y \to \|A(t)y\|$ defines an equivalent norm on $D(A) \equiv D(A(0))$ and the mapping $t \to A(t)$ from $[0, T]$ into $\mathcal{L}(X_1, X)$ is uniformly Hölder continuous [10].

Let f and h be two continuous functions. For $0 < \alpha \leq 1$, let W_{α} and $W_{\alpha-1}$ be open sets in X_{α} and $X_{\alpha-1}$, respectively. For each $u' \in W_{\alpha}$ and $u'' \in W_{\alpha-1}$, there are balls such that $B_{\alpha}(u', r') \subset W_{\alpha}$ and $B_{\alpha-1}(u'', r'') \subset W_{\alpha-1}$, for some positive numbers r' and r''. We will use the following assumptions:

(A4) (a) There exist constants $L_f \equiv L_f(t, u', u'', r', r'') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0, T] \times W_{\alpha} \times W_{\alpha-1} \to X$ satisfies the following condition

$$\|f(t, x, x') - f(s, y, y')\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_{\alpha} + \|x' - y'\|_{\alpha-1}) \tag{6}$$

for all $x, y \in B_{\alpha}$, $x', y' \in B_{\alpha-1}$ and for all $s, t \in [0, T]$.

(b) There exist constants $L_h \equiv L_h(t, u', r') > 0$ and $0 < \theta_2 \leq 1$ such that $h(\cdot, 0) = 0$, $h : W_{\alpha} \times [0, T] \to [0, T]$ satisfies the following condition

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_{\alpha} + |t - s|^{\theta_2}) \tag{7}$$

for all $x, y \in B_{\alpha}$ and for all $s, t \in [0, T]$.

For $t_0 \geq 0$ and $0 < \beta \leq 1$, let $C^{\beta}([t_0, T]; X)$ denote the space uniformly Hölder continuous on $[t_0, T]$ with exponent β. Then $C^{\beta}([t_0, T]; X)$ is a Banach space endowed with the norm

$$\|h\|_{C^{\beta}([t_0,T];X)} = \sup_{t_0 \leq t \leq T} \|h(t)\| + \sup_{t,s \in [t_0,T], t \neq s} \frac{\|h(t) - h(s)\|}{|t - s|^{\beta}}.$$

Now we consider the following inhomogeneous Cauchy problem

$$\frac{du}{dt} + A(t)u = f(t), \quad u(t_0) = u_0. \tag{8}$$

Theorem 2.1 *[4, Theorem II. 3.1] Suppose that the assumptions **(A1)**–**(A3)** hold. If $f \in C^{\beta}([t_0, T]; X)$, then the unique solution of (8) is given by*

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^{t} U(t, s)f(s)ds, \quad t_0 \leq t \leq T.$$

Indeed, $u : [t_0, T] \to X$ is strongly continuously differentiable on $(t_0, T]$.

The following lemmas will be used in the subsequent sections.

Lemma 2.1 *[5, Lemma 1.1] For $h \in C^\beta([t_0,T];X)$, we define $Q : C^\beta([t_0,T];X) \to C([t_0,T];X_1)$ by*

$$Qh(t) = \int_{t_0}^{t} U(t,s)h(s)ds, \ t_0 \le t \le T.$$

Then Q is a bounded mapping and $\|Qh\|_{C([t_0,T];X_1)} \le C\|h\|_{C^\beta([t_0,T];X)}$ for some $C > 0$.

We have the following corollary from Lemma 2.1.

Corollary 2.1 *For $y \in X_1$, we define*

$$H(y;h) = U(t,0)y + \int_0^t U(t,s)h(s)ds, \ 0 \le t \le T.$$

Then H is a bounded linear mapping from $X_1 \times C^\beta([t_0,T];X)$ into $C([t_0,T];X_1)$.

Lemma 2.2 *[10, Lemma 2] Let $0 < \alpha \le 1$ and $f \in C([t_0,T];X_\alpha)$. We define*

$$v(t) = \int_{t_0}^{t} U(t,s)f(s)ds, \ t_0 \le t \le T.$$

Then $v \in C([t_0,T];X_1) \cap C^1((t_0,T];X)$ and $v'(t) + A(t)v(t) = f(t), \ t_0 < t \le T$.

3 Main Results

In this section, we establish the main results. Let $I = [0,\delta]$ for some positive number δ to be specified later. Let $\mathcal{C}_\alpha$, $0 \le \alpha \le 1$ denote the space of all X_α-valued continuous functions on I, endowed with the sup-norm, $\sup_{t\in I} \|\psi(t)\|_\alpha$, $\psi \in C(I;X_\alpha)$. Let

$$Y_\alpha = \mathcal{C}_{L_\alpha}(I;X_{\alpha-1}) = \{\psi \in \mathcal{C}_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \le L_\alpha|t-s|, \text{ for all } t,s \in I\},$$

where L_α is a positive constant to be specified later. It is clear that Y_α is a Banach space under the sup-norm of $\mathcal{C}_\alpha$.

Definition 3.1 A continuous function $u : I \to X$ said to be a solution of Problem (1) if the following are satisfied:

(i) $u(\cdot) \in \mathcal{C}_{L_\alpha}(I;X_{\alpha-1}) \cap C^1((0,\delta);X) \cap C(I;X)$;

(ii) $u(t) \in W_\alpha$, for all $t \in (0,\delta)$;

(iii) $\dfrac{du}{dt} + A(t)u(t) = f(t,u(t),u(h(u(t),t)))$ for all $t \in (0,\delta)$;

(iv) $u(0) = u_0$.

For $0 < \alpha < \beta \le 1$, let $u_0 \in X_\alpha$. Let $r > 0$ be chosen small enough such that the assumption **(A4)** holds for the closed balls $B_\alpha \equiv B_\alpha(u_0,r)$ and $B_{\alpha-1} \equiv B_{\alpha-1}(u_0,r)$. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\mathcal{S}_\alpha = \{y \in \mathcal{C}_\alpha \cap Y_\alpha : y(0) = u_0, \sup_{t\in I}\|y(t) - u_0\|_\alpha \le r, \|y(t) - y(s)\|_\alpha \le K|t-s|^\eta \ \forall\, t,s \in I\}.$$

Then $\mathcal{S}_\alpha$ is a non-empty closed and bounded subset of $\mathcal{C}_\alpha$.

3.1 Local existence of solution

Now we prove the following theorem of the local existence of a solution to Problem (1). The proof is based on the ideas of Friedman [4] and Gal [6].

Theorem 3.1 *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \leq 1$. If the assumptions **(A1)-(A4)** hold, then there exist a positive number $\delta \equiv \delta(\alpha, u_0)$ and a unique solution $u(t)$ to Problem (1) on the interval $[0, \delta]$ such that $u \in \mathcal{S}_\alpha \cap C^1((0, \delta); X)$.*

Proof. Let $v \in \mathcal{S}_\alpha$. We define $f_v(t) = f(t, v(t), v(h(v(t), t)))$. Then the assumption **(A4)** implies that f_v is Hölder continuous on I of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. We consider the following problem:

$$\left.\begin{aligned} \frac{du}{dt} + A(t)u(t) &= f_v(t), \quad t \in I; \\ u(0) &= u_0. \end{aligned}\right\} \tag{9}$$

Then by Theorem 2.1, there exists a unique solution u_v of (9) which is given by

$$u_v(t) = U(t, 0)u_0 + \int_0^t U(t, s) f_v(s) ds, \quad t \in I.$$

We define a map F by

$$Fv(t) = U(t, 0)u_0 + \int_0^t U(t, s) f_v(s) ds, \quad \text{for each } t \in I.$$

We will claim that F maps from $\mathcal{S}_\alpha$ into itself, for sufficiently small $\delta > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned} \|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \leq{}& \|[U(t_2, 0) - U(t_1, 0)]u_0\|_{\alpha-1} \\ &+ \left\| \int_0^{t_2} U(t_2, s) f_v(s) ds - \int_0^{t_1} U(t_1, s) f_v(s) ds \right\|_{\alpha-1}. \end{aligned} \tag{10}$$

We will use the bounded inclusion $X \subset X_{\alpha-1}$ to estimate each of the terms on the right hand side of (10). The first term on the right hand side of (10) is estimated as follows [4, see Lemma II. 14.1],

$$\|(U(t_2, 0) - U(t_1, 0))u_0\|_{\alpha-1} \leq C_1 \|u_0\|_\alpha (t_2 - t_1), \tag{11}$$

where C_1 is some positive constant. We have the following estimate for the second term on the right hand side of (10) [4, Lemma II. 14.4],

$$\begin{aligned} &\left\| \int_0^{t_2} U(t_2, s) f_v(s) ds - \int_0^{t_1} U(t_1, s) f_v(s) ds \right\|_{\alpha-1} \\ &\quad \leq C_2 N_1 (t_2 - t_1)(|\log(t_2 - t_1)| + 1), \end{aligned} \tag{12}$$

where $N_1 = \sup_{s \in [0,T]} \|f_v(s)\|$ and C_2 is some positive constant.

Using the estimates (11) and (12), we get from the inequality (10),

$$\|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \leq L_\alpha |t_2 - t_1|,$$

where $L_\alpha = \max\{C_1\|u_0\|_\alpha, C_2 N_1(|\log(t_2-t_1)|+1)\}$ that depends on C_1, C_2, N_1, δ.

Next our aim is to show that $\sup_{t\in I}\|F(y)(t)-u_0\|_\alpha \le r$, for sufficiently small $\delta>0$. Since $u_0 \in X_\alpha$, we can choose sufficiently small $\delta_1>0$ such that [4, Lemma II.14.1],

$$\|U(t,0)u_0 - u_0\|_\alpha \le \frac{r}{3}, \quad \text{for all } t\in[0,\delta_1]. \tag{13}$$

We choose $\delta_2>0$ such that

$$\left(\frac{C(\alpha)}{1-\alpha}L_f[(1+L_\alpha L_h)r+\delta_2^{\theta_2}]+\frac{C(\alpha)K_1}{1-\alpha}\right)\delta_2^{1-\alpha} \le \frac{2r}{3}.$$

Let $K_1 := \sup_{0\le t\le T}\|f(t,u_0,u_0)\|$. For $v\in\mathcal{S}_\alpha$ and $t\in[0,\delta_2]$, it follows from the assumption **(A4)** [19, cf. inequality (1.65), p. 23], (6), (7) and $h(u_0,0)=0$ that

$$\begin{aligned}
&\left\|\int_0^t U(t,s)f_v(s)ds\right\|_\alpha \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + \|v([h(v(s),s)])-u_0\|_{\alpha-1}]ds \\
&\quad + C(\alpha)K_1\int_0^t (t-s)^{-\alpha}ds \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + L_\alpha|h((v(s),s))-h(u(0),0)|]ds \\
&\quad + C(\alpha)K_1\int_0^t (t-s)^{-\alpha}ds \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + L_\alpha|h((v(s),s))-h(u(0),0)|]ds \\
&\quad + \frac{C(\alpha)K_1\delta^{1-\alpha}}{1-\alpha} \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[r+L_\alpha L_h(\|v(s)-u_0\|_\alpha + s^{\theta_2})]ds + \frac{C(\alpha)K_1\delta_2^{1-\alpha}}{1-\alpha} \\
&\le C(\alpha)L_f[(1+L_\alpha L_h)r+\delta_2^{\theta_2}]\int_0^t (t-s)^{-\alpha}ds + \frac{C(\alpha)K_1\delta_2^{1-\alpha}}{1-\alpha} \\
&\le \left(\frac{C(\alpha)}{1-\alpha}L_f[(1+L_\alpha L_h)r+\delta_2^{\theta_2}]+\frac{C(\alpha)K_1}{1-\alpha}\right)\delta_2^{1-\alpha}.
\end{aligned} \tag{14}$$

Combining (13) and (14), we obtain $\sup_{t\in I}\|Fv(t)-u_0\|_\alpha \le r$, where $\delta_3=\min\{\delta_1,\delta_2\}$ [6, cf. p. 977].

Next we show that $\|Fv(t+h)-Fv(t)\|_\alpha \le Kh^\eta$ for some constant $K>0$ and $0<\eta<1$. If $0\le\alpha<\beta\le1$ and $0\le t\le t+h\le\delta$, then we have

$$\begin{aligned}
\|Fv(t+h)-Fv(t)\|_\alpha \le & \|[U(t+h,0)-U(t,0)]u_0\|_\alpha \\
&+\left\|\int_0^{t+h}U(t+h,s)f_v(s)ds-\int_0^t U(t,s)f_v(s)ds\right\|_\alpha.
\end{aligned}$$

Using [4, Lemma II.14.1 and Lemma II.14.4], we get the following estimates

$$\|[U(t+h,0)-U(t,0)]u_0\|_\alpha \leq C(\alpha,u_0)h^{\beta-\alpha}; \tag{15}$$

$$\left\|\int_0^{t+h} U(t+h,s)f_v(s)ds - \int_0^t U(t,s)f_v(s)ds\right\|_\alpha \leq C(\alpha)N_1 h^{1-\alpha}(1+|\log h|). \tag{16}$$

From (15) and (16), it is clear that

$$\|Fv(t+h)-Fv(t)\|_\alpha \leq h^\eta[C(\alpha,u_0)\delta^{\beta-\alpha-\eta} + C(\alpha)N_1\delta^\nu h^{1-\alpha-\eta-\nu}(|\log h|+1)]$$

for any $\nu > 0$ and $\nu < 1-\alpha-\eta$. Hence, for sufficiently small $\delta > 0$, we have

$$\|Fv(t+h)-Fv(t)\|_\alpha \leq Kh^\eta$$

for some $K > 0$. Thus F maps $\mathcal{S}_\alpha$ into itself.

Finally, we show that F is a contraction map. We choose $\delta_4 > 0$ such that

$$\frac{C(\alpha)}{1-\alpha}L_f(2+L_\alpha L_h)\delta_4^{1-\alpha} < \frac{1}{2}.$$

Let $v_1, v_2 \in S_\alpha$ and $t \in [0,\delta_4]$. Then we have [19, cf. inequality (1.65), page 23],

$$\begin{aligned}\|Fv_1(t)-Fv_2(t)\|_\alpha &\leq C(\alpha)L_f\int_0^t (t-s)^{-\alpha}(\|v_1(s)-v_2(s)\|_\alpha \\ &\quad + \|v_1([h(v_1(s),s)]) - v_2([h(v_2(s),s)])\|_{\alpha-1})ds \\ &\leq C(\alpha)L_f(2+L_\alpha L_h)\int_0^t (t-s)^{-\alpha}\|v_1(s)-v_2(s)\|_\alpha ds \\ &\leq \frac{C(\alpha)}{1-\alpha}L_f(2+L_\alpha L_h)\delta_4^{1-\alpha}\sup_{t\in I}\|v_1(t)-v_2(t)\|_\alpha. \end{aligned} \tag{17}$$

Then, from (17), it is clear that F is a contraction map. Since $\mathcal{S}_\alpha$ is a complete metric space, by the Banach fixed-point theorem, there exists $u \in \mathcal{S}_\alpha$ such that $Fu = u$. From Lemma 2.1 and Theorem 2.1, it follows that $u \in C^1((0,\delta);X)$. Thus u is a solution to Problem (1) on $[0,\delta]$, where $\delta = \min\{\delta_3,\delta_4\}$.

3.2 Global existence of solution

In this section, we prove the global existence of a solution to Problem (1).

Theorem 3.2 *Assume that **(A1)–(A4)** hold. Suppose that there are positive constants k_1 and k_2 such that*

$$\|f(t,x,y)\| \leq k_1(1+\|x\|_\alpha+\|y\|_{\alpha-1}) \text{ for } 0<\alpha<1, \tag{18}$$

$$|h(z,t)| \leq k_2(1+\|z\|_\alpha) \tag{19}$$

for all t, where $0 \leq t \leq T$, $x, z \in X_\alpha$ and $y \in X_{\alpha-1}$, then the initial value problem (1) has a unique solution that exists for all $t \in [0,T]$, for each $u_0 \in W_\beta$, where $0 < \alpha < \beta \leq 1$.

Proof. Let $\delta > 0$ be sufficiently small such that $u(t)$, $t \in (0, \delta]$, be the local solution of (1) obtained in Theorem 3.1. So for the global existence of a solution to problem (1), it is enough to show that $\|u(t)\|_\alpha$ is bounded as $t \uparrow \delta$ and this bound is independent of t.

Now using (6), (7), (18) and (19), we get, for $u(.) \in X_1$,

$$\begin{aligned} \|u(t)\|_\alpha &\le \|U(t,0)u_0\|_\alpha + \left\| \int_0^t U(t,s) f(s, u(s), u(h(u(s), s))) ds \right\|_\alpha \\ &\le \|A(0)^\alpha A(t)^{-\beta} A(t)^\beta U(t,0) A(0)^{-\beta} A(0)^\beta u_0\| \\ &\quad + k_1 \int_0^t (t-s)^{-\alpha} [(1 + \|u(s)\|_\alpha + L_\alpha |h(u(s), s) - h(u_0, 0)| + \|u_0\|_{\alpha-1}] ds. \end{aligned} \tag{20}$$

Using [4, inequality (II.14.12) and (II.14.14)] in (20), we get

$$\|u(t)\|_\alpha \le (C' + D)\|u_0\|_\alpha + k_1[1 + (1 + L_\alpha k_2)] \int_0^t (t-s)^{-\alpha} (1 + \|u(s)\|_\alpha) ds,$$

where $D = \sup_{t \in [0,T]} K k_1 \int_0^t (t-s)^{-\alpha} ds$, K is the constant in the bounded inclusion $X \subset X_{\alpha-1}$ and C' is some positive constant. Applying the Gronwall lemma, we get that $\|u(t)\|_\alpha$ is bounded as $t \uparrow \delta$.

Remark 3.1 In the case when $A(t)$ is a self adjoint positive definite operator in a Hilbert space X, Theorem 3.1 and Theorem 3.2 can be strengthened. Assumptions **(A1), (A2)** and **(A3)** imply that, for $0 \le \alpha \le 1$ and for all $s, t \in [0, T]$ [13, p. 185],

$$\|A(t)^\alpha A(s)^{-\alpha}\| \le C \|A(t) A(s)^{-1}\|^\alpha \le C', \tag{21}$$

where $C, C' > 0$. Then we can prove Theorem 3.1 and Theorem 3.2 with a less regularity assumption on u_0.

3.3 Existence of solution with regularity

In this section, we give a theorem with more regularity on f and u_0. We denote $D(A(0))$ by X_1. We equipped this space X_1 with the graph norm

$$\|x\|_1 := (\|x\|^2 + \|A(0)x\|^2)^{\frac{1}{2}},$$

that is equivalent to the usual norm $\|A(0)x\|$ for $x \in D(A(0))$.

Let f and h be two continuous functions. Let W_1 and W be open sets in X_1 and X, respectively. For each $u \in W_1$ and $u' \in W$, there are balls such that $B_1(u, r) \subset W_1$ and $B(u', r') \subset W$. We will make use of the following stronger assumptions:

$(\mathbf{A4})'$ (a) There exist constants $L_f \equiv L_f(t, u, u', r, r') > 0$ and $0 < \theta_1 \le 1$, such that the nonlinear map $f : [0, T] \times W_1 \times W \to X_\alpha$ satisfies:

$$\|f(t, x, x') - f(s, y, y')\|_\alpha \le L_f(|t-s|^{\theta_1} + \|x - y\|_1 + \|x' - y'\|) \tag{22}$$

for all $x, y \in B_1$, $x', y' \in B$, for all $s, t \in [0, T]$ and $\alpha \in (0, 1)$.

(b) There exist constants $L_h \equiv L_h(t, u', r') > 0$ and $0 < \theta_2 \le 1$, such that $h(\cdot, 0) = 0$, $h : W_1 \times [0, T] \to [0, T]$ satisfies:

$$|h(x, t) - h(y, s)| \le L_h(\|x - y\|_1 + |t - s|^{\theta_2}) \tag{23}$$

for all $x, y \in B_1$ and for all $s, t \in [0, T]$.

Then we have the following theorem.

Theorem 3.3 *Let* $u_0 \in W_1$*. Suppose that the assumptions* ***(A1)-(A3)*** *and* $(\mathbf{A4})'$ *hold. Then there exist a positive number* $\delta \equiv \delta(u_0)$ *and a unique solution* $u(t)$ *of Problem (1) on the interval* $[0,\delta]$ *such that* $\in C_L(I;X) \cap C^1((0,\delta);X) \cap C(I;X)$*, where*

$$C_L(I;X) = \{\psi \in C(I;X_1) : \|\psi(t) - \psi(s)\| \leq L|t-s|, \text{ for all } t, s \in I\},$$

for some $L > 0$*. Further, we assume that there are positive constants* k_1 *and* k_2 *such that*

$$\|f(t,x,y)\|_\alpha \leq k_1(1 + \|x\|_1 + \|y\|) \text{ for } 0 < \alpha < 1, \tag{24}$$
$$|h(z,t)| \leq k_2(1 + \|z\|_1), \tag{25}$$

for all t, $x, z \in X_1$ *and* $y \in X$*, where* $0 \leq t \leq T$*. Then the unique solution of (1) exists for all* $t \geq 0$*.*

Proof. We denote the interval $[0,\delta]$ by I. For each $v \in C(I, B_1)$, we define a map F by

$$Fv(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s, v(s), v(h(v(s),s)))ds \quad \text{for each } t \in I.$$

By Lemma 2.2, the map F from $C(I, B_1)$ into $C(I;X_1)$ is well defined. The proof of this Theorem can be obtained by the same argument as in the proof of Theorem 3.1 and Theorem 3.2. Thus, we omit the details of the proof.

3.4 Asymptotic stability of solution

In this section, we discuss the asymptotic stability of a solution to Problem (1) in X. The proof is based on the ideas of Friedman [4] and Webb [21].

Theorem 3.4 *Suppose that the assumptions* ***(A1)-(A4)*** *hold,* $u_0 \in X_\beta$*, where* $0 < \alpha < \beta \leq 1$ *and there exists a continuous solution* $u \in X_\alpha$*. Suppose there exist a continuous function* $\epsilon : [0,\infty) \to [0,\infty)$ *and a constant* $k_3 > 0$ *such that*

$$\|f(t, u(t), u(h(u(t),t)))\| \leq k_3(\epsilon(t) + \|u(t)\|_\alpha + \|u(t)\|_{\alpha-1}) \text{ for } 0 < \alpha < 1, t \geq 0. \tag{26}$$

Then

(i) if $\epsilon(t)$ *is bounded on* $[0,\infty)$*, then* $\|u(t)\|_\alpha$ *is bounded on* $[0,\infty)$*;*

(ii) if $\epsilon(t) = \mathrm{O}(e^{\sigma t})$ *for some* $-1 < \sigma < 0$*, then* $\|u(t)\|_\alpha = \mathrm{O}(e^{\sigma t})$*;*

(iii) if $\epsilon(t) = \mathrm{o}(1)$*, then* $\|u(t)\|_\alpha = \mathrm{o}(1)$*.*

Proof. It is known [4, p. 176] that there exists $0 < \theta < d$, such that

$$\|A^\gamma(t)U(t,0)\| \leq \frac{C}{t^\gamma}e^{-\theta t} \text{ if } t > 0, \tag{27}$$

for any $0 \leq \gamma \leq 1$.

Now, for $t > 0$, put $\varphi(t) = e^{\theta t}\|u(t)\|_\alpha$. Using (27) to the solution of Problem (1), we obtain

$$\begin{aligned}\varphi(t) &\le Ct^{-\alpha}\|u_0\| + C\int_0^t e^{\theta s}(t-s)^{-\alpha}k_3[\epsilon(s) + \|u(s)\|_\alpha + \|u(s)\|_{\alpha-1}]ds\\ &\le Ct^{-\alpha}\|u_0\| + Ck_3\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds + Ck_3(1+K)\int_0^t (t-s)^{-\alpha}\varphi(s)ds\\ &\le \left\{C_0t^{-\alpha}\|u_0\| + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds\right\} + C_0\int_0^t (t-s)^{-\alpha}\varphi(s)ds, \qquad (28)\end{aligned}$$

where $C_0 = \max\{C, Ck_3, Ck_3(1+K)\}$. We denote

$$\chi(t) = C_0t^{-\alpha}\|u_0\| + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds.$$

Then it is clear that

$$\chi(t) \le C_0t^{-\alpha}\|u_0\| + \tilde{C}e^{\theta t}\sup_{0\le s<\infty}\epsilon(s), \qquad (29)$$

for some constant $\tilde{C} > 0$. We get from (28) by the method of iteration that [21],

$$\varphi(t) \le \chi(t) + \int_0^t \left[\sum_0^\infty \frac{(t-s)^{j-1-j\alpha}[\Gamma(1-\alpha)]^j}{\Gamma(j-j\alpha)}\right]\chi(s)ds.$$

We note that the series in the bracket is bounded by $B_1(t-s)^{-\alpha}\exp[B_2(t-s)^{1-\alpha}]$ for some constants $B_1, B_2 > 0$. Thus it follows that, for $t \ge 1$ and for any $\lambda > 0$,

$$\varphi(t) \le B_3e^{\lambda t}\|u_0\| + B_4e^{\theta t}\sup_{0\le s<\infty}\epsilon(s), \qquad (30)$$

where B_3 and B_4 are some positive constants. Thus, for any $0 < \theta_0 < \theta$, we get

$$\|u(t)\|_\alpha \le B_3e^{-\theta_0 t}\|u_0\| + B_4\sup_{0\le s<\infty}\epsilon(s). \qquad (31)$$

The proof follows from the inequality (31).

4 Example

Consider the following differential equation with deviated argument [6, 10]:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(k(t,x)\frac{\partial}{\partial x}u(x)) &= \widetilde{H}(x, u(t,x)) + \widetilde{G}(t, x, u(t,x));\\ u(t,0) &= u(t,1), \quad t > 0;\\ u(0,x) &= u_0(x), \quad x \in (0,1).\end{aligned}\right\} \qquad (32)$$

Here, $\widetilde{H}(x, u(t,x)) = \int_0^x K(x,y)u(\widetilde{g}(t)|u(t,y)|, y)dy$ for all $(t,x) \in (0,\infty)\times(0,1)$. Assume that $\widetilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $\widetilde{g}(0) = 0$ and $K \in C^1([0,1]\times[0,1];\mathbb{R})$. The function $\widetilde{G} : \mathbb{R}_+\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x [6].

We assume that k is positive function with continuous partial derivative k_x such that, for all $0 \le t < \infty$ and $x \in (0,1)$,

(i) $0 < k_0 \leq k(t,x) < k_0^{'}$,

(ii) $|k_x(t,x)| \leq k_1$,

(iii) $|k(t,x) - k(s,x)| \leq C|t-s|^{\epsilon}$,

(iv) $|k_x(t,x) - k_x(s,x)| \leq C|t-s|^{\epsilon}$,

for some ϵ with $0 < \epsilon \leq 1$, some constants k_0, $k_0^{'}$, and $C > 0$.

Let $X = L^2((0,1);\mathbb{R})$. We define $X_1 = D(A(0)) = H^2(0,1) \cap H_0^1(0,1)$ and $A(t)u(t) = -\frac{\partial}{\partial x}(k(t,x)\frac{\partial}{\partial x}u(x))$. Then $X_{1/2} = D((A(0))^{1/2}) = H_0^1(0,1)$. Then the family $\{A(t) : t > 0\}$ satisfies the assumptions **(A1)**, **(A2)** and **(A3)** on each bounded interval $[0,T]$ [10].

For $x \in (0,1)$, we define $f : \mathbb{R}_+ \times H^2(0,1) \times L^2(0,1) \to H_0^1(0,1)$ by

$$f(t,\phi,\psi) = \widetilde{H}(x,\psi) + \widetilde{G}(t,x,\phi),$$

where $\widetilde{H}(x,\psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy$ and $\widetilde{G} : \mathbb{R}_+ \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$ satisfies $\|\widetilde{G}(t,x,u)\|_{H_0^1(0,1)} \leq C(1 + \|u\|_{H^2(0,1)})$, for some $C > 0$. Then it can be shown that f satisfies the condition (22)(see Gal [6]) and $h : H^2(0,1) \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(\phi(x,t),t) = \widetilde{g}(t)|\phi(x,t)|$ satisfies (23) (see Gal [6]). Thus, we can apply the results of previous sections to study the existence, uniqueness and asymptotic stability of solution of (32).

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