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Positive Solutions for a Fourth Order Three Point Focal Boundary Value Problem

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Abstract: The authors consider a fourth order three point boundary value problem. Some a priori estimates to positive solutions for the boundary value problem are obtained. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are established.

Keywords: *fixed point theorem; cone; nonlinear boundary value problem; positive solution.*

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1 Introduction

In this paper, we consider the fourth order differential equation

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1}$$

together with the boundary conditions

$$u(0) = u'(p) = u''(1) = u'''(1) = 0. \tag{2}$$

Throughout this paper, we assume that

(H1) p is a real constant such that $1 - \sqrt{3}/3 \le p \le 1$, $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0, 1]$.

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In this paper, we will study positive solutions of the problem (1)–(2). By a *positive solution*, we mean a solution $u(t)$ to the problem (1)–(2) such that $u(t) > 0$ for $t \in (0, 1)$.

The fourth order equation (1), known as the beam equation, has been studied by many authors under various boundary conditions and by different approaches. For example, in 2006, Anderson and Avery [2] considered the fourth order four-point right focal boundary value problem

$$u''''(t) + f(u(t)) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(q) = u''(r) = u'''(1) = 0, \tag{4}$$

under the assumption that $1/2 < q < (1 + q)/2 < r < 1$. We note that if we allow $r = 1$, then (4) reduces to (2). In 2005, Yang [9] considered the boundary value problem

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{5}$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{6}$$

and obtained sufficient conditions for the existence and nonexistence of positive solutions to the problem (5)–(6). We note that if we let $p = 0$, then (2) reduces to (6).

For some other results on boundary value problems for the beam equation, we refer the reader to the papers [1, 3–6, 8].

In this paper, we shall first prove some upper and lower estimates to positive solutions of the problem (1)–(2), and then establish some sufficient conditions for the existence and non-existence of positive solutions.

This paper is organized as follows. In Section 2, we give the Green function for the problem (1)–(2), state the Krasnosel'skii's fixed point theorem, and fix some notations. In Section 3, we present some a priori estimates to positive solutions to the problem. In Section 4, we establish some existence and nonexistence results for positive solutions.

2 Preliminaries

The Green function $G : [0, 1] \times [0, 1] \to [0, \infty)$ for the problem (1)–(2) is

$$\begin{aligned} G(t, s) \quad = \quad & -t[p^2/2 - ps - ((p - s)^2/2)H(p - s)] \\ & -t^2 s/2 + t^3/6 - ((t - s)^3/6)H(t - s). \end{aligned}$$

Here, $H : (-\infty, \infty) \to (-\infty, \infty)$ is the unit step function given by

$$H(t) = \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The problem (1)–(2) is then equivalent to the integral equation

$$u(t) = \int_0^1 G(t, s)g(s)f(u(s))\, ds, \quad 0 \le t \le 1. \tag{7}$$

It is easy to verify that G is a continuous function. Also, we note that if $0 \le s \le p$, then

$$G(p, s) = s^3/6 \ge 0;$$

if $p \le s \le 1$, then

$$G(p, s) = p^2(3s - 2p)/6 \ge 0.$$

In summary, we have $G(p, 0) = 0$ and

$$G(p, s) > 0, \quad 0 < s \le 1.$$

We will need the following simplified version of the Krasnosel'skii fixed point theorem (see [7]) to prove some of our results.

Theorem 2.1 *Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let H_1 and H_2 be distinct positive numbers. If $L : P \to P$ is a completely continuous operator such that*

(K1) If $v \in P$ and $\|v\| = H_1$, then $\|Lv\| \le \|v\|$, and

(K2) If $v \in P$ and $\|v\| = H_2$, then $\|Lv\| \ge \|v\|$.

Then L has a fixed point v in P with $\min\{H_1, H_2\} \le \|v\| \le \max\{H_1, H_2\}$.

For the rest of this paper, we let $X = C[0, 1]$ be equipped with the norm

$$\|v\| = \max_{t \in [0,1]} |v(t)|, \quad \text{for all} \quad v \in X.$$

Clearly, X is a Banach space. We define

$$Y = \{v \in X \mid v(t) \ge 0 \text{ for } 0 \le t \le 1\},$$

and define the operator $T : Y \to X$ by

$$(Tu)(t) = \int_0^1 G(t, s)g(s)f(u(s))\, ds, \quad 0 \le t \le 1. \tag{8}$$

It is easy to see that if (H1) holds, then $T : Y \to Y$ is a completely continuous operator. We also define the constants

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x \to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \frac{f(x)}{x}.$$

These constants, which are associated with the function f, will be used in Sections 4 and 5.

3 Estimates for Positive Solutions

In this section, we derive some upper and lower estimates for positive solutions of the problem (1)–(2).

Lemma 3.1 *If (H1) holds, then $G(t, s) \le G(p, s)$ for $0 \le t, s \le 1$.*

Proof. We take four cases to prove this inequality. If $t \le s \le p$, then

$$G(p, s) - G(t, s) = \frac{(s - t)^3}{6} \ge 0.$$

If $s \le t \le p$ or $s \le p \le t$, then

$$G(p,s) - G(t,s) = 0.$$

If $t \le p \le s$ or $p \le t \le s$, then

$$G(p,s) - G(t,s) = \frac{(t-p)^2}{6}(2s - 2p + s - t) \ge 0.$$

If $p \le s \le t$, then

$$G(p,s) - G(t,s) = \frac{(s-p)^2}{6}(2t - 2p + t - s) \ge 0.$$

The proof is now complete.

We define the function $a : [0,1] \to [0,\infty)$ by

$$a(t) = \frac{3p(2-p)t - 3t^2 + t^3}{p^2(3-2p)}, \quad 0 \le t \le 1.$$

We notice that

$$a(0) = 0, \quad a(1) = \frac{3(\sqrt{3}/3 + 1 - p)(p - (1 - \sqrt{3}/3))}{p^2(3-2p)} \ge 0,$$

and

$$a''(t) = \frac{-6(1-t)}{p^2(3-2p)} \le 0, \quad 0 \le t \le 1.$$

Therefore, $a(t)$ is concave downward on $[0,1]$. Since $a(0) = 0$ and $a(1) \ge 0$, we have

$$a(t) \ge 0, \quad 0 \le t \le 1.$$

It is easy to see that

$$a(t) \ge \min\{t, 1-t\}, \quad 0 \le t \le 1. \tag{9}$$

We leave the verification of (9) to the reader.

Lemma 3.2 *Suppose (H1) holds. Then $G(t,s) \ge a(t)G(p,s)$ for $0 \le t, s \le 1$.*

Proof. We take four cases to prove the lemma.
If $t \le s \le p$, then

$$\begin{aligned} G(t,s) - a(t)G(p,s) \;&=\; \frac{t}{6(3-2p)p^2}[s^2(3-s-2p)(s-p)^2 \\ &\quad + s(s-t)(p-s)(2s - 2s^2 + 2p - 2p^2 + s - sp + p - sp) \\ &\quad + (s-t)^2(2p^2 - 2p^3 + p^2 - s^3)] \\ &\ge\; 0. \end{aligned}$$

If $s \le t \le p$ or $s \le p \le t$, then

$$G(t,s) - a(t)G(p,s) = \frac{s^3(t-p)^2(3-t-2p)}{6(3-2p)p^2} \ge 0.$$

If $t \le p \le s$ or $p \le t \le s$, then

$$G(t,s) - a(t)G(p,s) = \frac{t(t-p)^2(1-s)}{2(3-2p)} \ge 0.$$

If $p \le s \le t$, then

$$\begin{aligned} G(t,s) - a(t)G(p,s) &= \frac{t(p-t)^2(1-s)}{6-4p} + \frac{(s-t)^3}{6} \\ &\ge \frac{1}{6}\left[t(p-t)^2(1-s) + (s-t)^3\right] \\ &\ge \frac{1}{6}\left[t(p-t)^2(1-s) + (p-t)^2(s-t)\right] \\ &= \frac{1}{6}(p-t)^2 s(1-t) \\ &\ge 0. \end{aligned}$$

This completes the proof of the lemma.

Lemma 3.3 *Suppose (H1) holds. Then* $G(t,s) \ge 0$ *for* $0 \le t, s \le 1$.

Proof. The lemma follows easily from Lemma 3.2 and the facts that $a(t) \ge 0$ for $0 \le t \le 1$ and $G(p,s) \ge 0$ for $0 \le s \le 1$.

Lemma 3.4 *Suppose (H1) holds. If* $u \in C^4[0,1]$ *satisfies the boundary conditions* (2)*, and*

$$u''''(t) \le 0 \quad for \quad 0 \le t \le 1, \tag{10}$$

then $\|u\| = u(p)$, $u(t) \ge 0$, *and*

$$a(t)u(p) \le u(t) \le u(p) \quad for \quad 0 \le t \le 1. \tag{11}$$

Proof. Suppose $u \in C^4[0,1]$ satisfies (2) and (10). If $0 \le t \le 1$, then

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \ge 0,$$

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \ge a(t)\int_0^1 G(p,s)(-u''''(s))ds = a(t)u(p),$$

and

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \le \int_0^1 G(p,s)(-u''''(s))ds = u(p),$$

which proves the lemma.

The next theorem follows immediately from Lemma 3.4.

Theorem 3.1 *Suppose (H1) holds. If* $u \in C^4[0,1]$ *is a non-negative solution to the problem* (1)–(2)*, then* $u(t)$ *satisfies* (11).

We now define

$$P = \left\{v \in X : v(p) \ge 0,\ a(t)v(p) \le v(t) \le v(p) \text{ on } [0,1]\right\}.$$

Clearly P is a positive cone in X. It is obvious that if $u \in P$, then $u(p) = \|u\|$. We see from Theorem 3.1 that if $u(t)$ is a nonnegative solution to the problem (1)–(2), then $u \in P$. In a similar fashion to Lemma 3.4, we can show that $T(P) \subset P$. To find a positive solution to the problem (1)–(2), we need only to find a fixed point u of T such that $u \in P$ and $u(p) = \|u\| > 0$.

4 Existence and Nonexistence Results

First, we define some important constants:

$$A = \int_0^1 G(p,s)g(s)a(s)\,ds \quad \text{and} \quad B = \int_0^1 G(p,s)g(s)\,ds.$$

The next two theorems provide sufficient conditions for the existence of at least one positive solution for the problem (1)–(2).

Theorem 4.1 *Suppose that (H1) holds. If $BF_0 < 1 < Af_\infty$, then the problem* (1)–(2) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \le 1$. By the definition of F_0, there exists $H_1 > 0$ such that $f(x) \le (F_0 + \varepsilon)x$ for $0 < x \le H_1$. Now for each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\
&\le \int_0^1 G(p,s)g(s)(F_0+\varepsilon)u(s)\,ds \\
&\le (F_0+\varepsilon)\|u\| \int_0^1 G(p,s)g(s)\,ds \\
&= (F_0+\varepsilon)\|u\|B \le \|u\|.
\end{aligned}$$

Hence, condition (K1) in Theorem 2.1 is satisfied.

Next we choose $\delta > 0$ and $\tau \in (0, 1/4)$ such that

$$\int_\tau^{1-\tau} G(p,s)g(s)a(s)\,ds \cdot (f_\infty - \delta) \ge 1.$$

There exists $H_3 > 2H_1$ such that $f(x) \ge (f_\infty - \delta)x$ for $x \ge H_3$. Let $H_2 = H_3/\tau$. If $u \in P$ and $\|u\| = H_2$, then for each $t \in [\tau, 1-\tau]$, we have

$$u(t) \ge H_2 a(t) \ge H_2 \min\{t, 1-t\} \ge H_2\tau = H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)f(u(s))\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)(f_\infty-\delta)u(s)\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)a(s)\,ds \cdot (f_\infty-\delta)\|u\| \ge \|u\|.
\end{aligned}$$

Thus, condition (K2) of Theorem 2.1 is satisfied. By Theorem 2.1, T has a fixed point u such that $\min\{H_1, H_2\} = H_1 \le \|u\| \le \max\{H_1, H_2\} = H_2$. This completes the proof of the theorem.

The proof of the following companion result is very similar to that of Theorem 4.1 and is therefore omitted.

Theorem 4.2 *Suppose that (H1) holds. If $BF_\infty < 1 < Af_0$, then the problem* (1)–(2) *has at least one positive solution.*

The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (1)–(2).

Theorem 4.3 *Suppose (H1) holds. If $Bf(x) < x$ for all $x > 0$, then the problem* (1)–(2) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1)–(2). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(p,s)g(s)u(s)\,ds \\ &\le B^{-1}u(p)\int_0^1 G(p,s)g(s)\,ds \\ &= B^{-1}u(p)B = u(p), \end{aligned}$$

which is a contradiction.

The proof of our next theorem is similar to the one above.

Theorem 4.4 *Suppose (H1) holds. If $Af(x) > x$ for all $x > 0$, then the problem* (1)–(2) *has no positive solutions.*

We conclude this paper with an example.

Example 4.1 Consider the fourth order boundary value problem

$$u''''(t) = \lambda(1+t)u(t)(1+3u(t))/(1+u(t)), \quad 0 \le t \le 1, \tag{12}$$

$$u(0) = u'(3/4) = u''(1) = u'''(1) = 0. \tag{13}$$

Here $\lambda > 0$ is a parameter. In this example, $p = 3/4$, $g(t) = 1 + t$, and

$$f(u) = \lambda u(1+3u)/(1+u).$$

It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 3\lambda$, and

$$\lambda x < f(x) < 3\lambda x \quad \text{for} \quad x > 0.$$

Calculations indicate that

$$A = 142837/2064384, \quad B = 363/5120.$$

By Theorem 4.1, if

$$4.8176 \approx 1/(3A) < \lambda < 1/B \approx 14.1046,$$

then the problem (12)–(13) has at least one positive solution. From Theorems 4.3 and 4.4 we see that if

$$\lambda \le 1/(3B) \approx 4.7015 \quad \text{or} \quad \lambda \ge 1/A \approx 14.4528,$$

then the problem (12)–(13) has no positive solutions.

This example shows that our existence and nonexistence results can be quite sharp.

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References

[1] Agarwal, R. P. On a fourth-order boundary value problems arising in beam analysis. *Differential Integral Equations* **2** (1989) 91–110.

[2] Anderson, D. R. and Avery, R. I. A fourth-order four-point right focal boundary value problem. *Rocky Mountain J. Math.* **36**(2) (2006) 367–380.

[3] Bai, Z. and Wang, H. On positive solutions of some nonlinear fourth-order beam equations. *J. Math. Anal. Appl.* **270**(2) (2002) 357–368.

[4] Dalmasso, R. Uniqueness of positive solutions for some nonlinear fourth-order equations. *J. Math. Anal. Appl.* **201**(1) (1996) 152–168.

[5] Graef, J. R. and Yang, B. Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems. *Appl. Anal.* **74**(1–2) (2000) 201–214.

[6] Gupta, C. P. A nonlinear boundary value problem associated with the static equilibrium of an elastic beam supported by sliding clamps. *Internat. J. Math. Math. Sci.* **12**(4) (1989) 697–711.

[7] Krasnosel'skii, M. A. *Positive Solutions of Operator Equations.* Noordhoff, Groningen, 1964.

[8] Ma, R. Existence and uniqueness theorems for some fourth-order nonlinear boundary value problems. *Internat. J. Math. Math. Sci.* **23**(11) (2000) 783–788.

[9] Yang, B. Positive solutions for a fourth order boundary value problem. *Electronic J. of Qualitative Theory of Differential Equations* **2005**(3) 1–17.