Nonlinear Dynamics and Systems Theory, **12** (2) (2012) 145–156

Homoclinic Orbits for a Class of Second Order Hamiltonian Systems

A. Benhassine and M. Timoumi [2]

Dpt of Mathematics, Faculty of Sciences 5000 Monastir. Tunisia

Received: January 15, 2010; Revised: March 19, 2012

Abstract: A new result for existence of homoclinic orbits is obtained for the second order Hamiltonian systems $\ddot{x}(t)+V'(t,x(t)) = f(t)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $V(t,x) = -K(t,x) + W(t,x)$ is $T-$periodic in t, $T > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous bounded function, under an assumption weaker than the so-called Ambrosetti–Rabinowitz-type condition.

Keywords: *homoclinic orbits; Hamiltonian systems; critical point; diagonal method.*

Mathematics Subject Classification (2010): 34C37, 35A15, 37J45.

1 Introduction

In this paper we are concerned with the study of the existence of homoclinic solutions for second order time-dependent Hamiltonian systems of the type

$$\ddot{x}(t) + V'(t,x(t)) = f(t), \tag{HS}$$

where $x = (x_1, ..., x_N)$, $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $V'(t,x) = \frac{\partial V}{\partial x}(t,x)$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous function. Here, as usual, we say that a solution x of (HS) is homoclinic $(to\ 0)$ if $x(t) \to 0$ as $t \to \pm\infty$. In addition x is called nontrivial if $x \not\equiv 0$.

The existence of homoclinic solutions for (HS) has been extensively investigated in many papers via the critical point theory, see [8, 11]. These results were obtained under the fact that the potential V is of the type

$$V(t,x) = -\frac{1}{2}L(t)x.x + W(t,x),$$

where $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$.

* Corresponding author: mailto:m_timoumi@yahoo.com

Recently, in [2], Izydorek and Janczewska have studied the existence of such solutions when the potential V is of the form

$$V(t,x) = -K(t,x) + W(t,x),$$

where $K, W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Precisely, they established the following result.

Theorem 1.1 *Assume that V and f satisfy the conditions*
(V_1) $V(t,x) = -K(t,x) + W(t,x)$, *where* $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *are* C^1*-maps,* $T-$*periodic with respect to* t, $T > 0$,
(V_2) *there are constants* $b_1, b_2 > 0$ *such that* $b_1 |x|^2 \leq K(t,x) \leq b_2 |x|^2$ *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$,
(V_3) *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $K(t,x) \leq K'(t,x).x \leq 2K(t,x)$,
(V_4) $W'(t,x) = o(|x|)$, *as* $|x| \to 0$ *uniformly with respect to* t,
(V_5) *there is a constant* $\mu > 2$ *such that* $0 < \mu W(t,x) \leq W'(t,x).x$ *for every* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^N \setminus \{0\}$,
(V_6) $f : \mathbb{R} \to \mathbb{R}^N$ *is a bounded continuous function,*
(V_7) $\bar{b}_1 = min\{1, 2b_1\} > 2M$ *and* $\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} \leq \frac{\beta}{2C}$, *where* $0 < \beta < \bar{b}_1 - 2M$, $M = \sup\left\{W(t,x)\ t \in [0,T], x \in \mathbb{R}^N, |x| = 1\right\}$ *and* C *is a positive Sobolev constant defined in [2]. Then the system* (HS) *possesses a nontrivial homoclinic solution.*

Here and in the following $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

The so-called Ambrosetti–Rabinowitz-type condition (V_5) appears frequently in the studying of existence of homoclinic solutions for (HS). The goal of this work is to prove that Theorem 1.1 still holds if (V_5) is replaced by a weaker condition. The motivation for the paper comes mainly from a paper by An [14], in which he dealt with the existence of periodic solutions for (HS) with a condition weaker than (V_5).

Definition 1.1 A vector field v defined on $\mathbb{R}^N$ is called positive if $v(x).x > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. We call v a normalized positive vector field if v is positive, linear and satisfies the following condition:

$$v(x).x = x.x,\ \forall\ x \in \mathbb{R}^N. \tag{v_1}$$

Consider the following assumptions:
(V_3') there exists normalized positive vector field v such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$

$$K(t,x) \leq K'(t,x).v(x) \leq 2K(t,x),$$

(V_5') there exists constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$

$$0 < \mu W(t,x) \leq W'(t,x).v(x).$$

The main result of this paper is as follows.

Theorem 1.2 *Assume that V and f satisfy* (V_1), (V_2), (V_3'), (V_4), (V_5'), (V_6), (V_7) *and the following assumption:*

$$W(t,x) \leq M |x|^\mu,\ \forall\ t \in \mathbb{R},\ \forall |x| \leq 1. \tag{V_8}$$

Then the system (HS) *possesses a nontrivial homoclinic solution.*

It is obvious that if $v(x) = x$, then (V_3') becomes (V_3) and (V_5') becomes (V_5). Consider the following examples.

Example 1.1 Let $\theta(x)$ be the argument of $x = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ defined by

$$\theta(x) = \begin{cases} arctan(\frac{\xi_2}{\xi_1}),\ if\ \xi_1 > 0, \xi_2 \geq 0, \\ \frac{\pi}{2},\ if\ \xi_1 = 0, \xi_2 > 0, \\ arctan(\frac{\xi_2}{\xi_1}) + \pi,\ if\ \xi_1 < 0, \\ \frac{3\pi}{2},\ if\ \xi_1 = 0, \xi_2 < 0, \\ arctan(\frac{\xi_2}{\xi_1}) + 2\pi,\ if\ \xi_1 > 0, \xi_2 < 0. \end{cases}$$

Define a function $K \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ as follows:

$$K(t,x) = \begin{cases} \frac{|x|^2}{exp(2\sin 4(\ln|x| + \theta(x)))},\ if x \neq 0, \\ 0,\ if\ x = 0. \end{cases}$$

Define a normalized positive vector field v by $v(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x$. An easy computation shows that K satisfies (V_2) and (V_3').

Example 1.2 For any $\mu > 2$, define a function $W \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ as follows:

$$W(t,x) = \begin{cases} \frac{|x|^\mu}{exp(\mu(2 + \sin 4(\ln|x| + \theta(x))))},\ if\ x \neq 0, \\ 0,\ if\ x = 0. \end{cases}$$

A direct computation (see [14]) shows that W satisfies (V_4), (V_5') and (V_8). Moreover, W does not satisfy (V_5).

In order to obtain homoclinic solution of (HS), we consider a sequence of systems of differential equations:

$$\ddot{x}(t) + V'(t, x(t)) = f_k(t), \qquad (HS_k)$$

where $f_k : \mathbb{R} \to \mathbb{R}^N$ is a $2kT$-periodic extension of f to the interval $[-kT, kT[$, $k \in \mathbb{N}$. We will prove the existence of a homoclinic solution of (HS) as the limit of the $2kT$-periodic solution of (HS_k) as in [2,8].

2 Preliminaries

For each $k \in \mathbb{N}$, let $E_k = W^{1,2}_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote the Hilbert space of $2kT$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{E_k} = \left(\int_{-kT}^{kT} (|\dot{x}(t)|^2 + |x(t)|^2) dt \right)^{1/2},$$

and let $L^2_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote the Hilbert space of $2kT$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |x(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

Furthermore, let $L^\infty_{2kT}(\mathbb{R},\mathbb{R}^N)$ be the space of $2kT$-periodic essentially bounded measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{L^\infty_{2kT}} = ess\sup\{|x(t)| : t\in[-kT,kT]\}.$$

Let $\phi_k : E_k \to \mathbb{R}$ be defined by

$$\phi_k(x) = \int_{-kT}^{kT}\left[\frac{1}{2}|\dot{x}(t)|^2 + K(t,x(t)) - W(t,x(t)) + f_k(t).x(t)\right]dt. \tag{2.1}$$

It is well known that $\phi_k \in C^1(E_k,\mathbb{R})$ and for all $x,y\in E_k$

$$\phi_k'(x)y = \int_{-kT}^{kT}\left[\dot{x}(t).\dot{y}(t) + K'(t,x(t)).y(t) - W'(t,x(t)).y(t) + f_k(t).y(t)\right]dt. \tag{2.2}$$

Moreover, the critical points of ϕ_k in E_k are exactly the classical $2kT$-periodic solution of (HS_k) (see [6,9]). We will obtain a critical point of ϕ_k by using the following Mountain Pass Theorem.

Theorem 2.1 *[8] Let E be a real Banach space and $\phi\in C^1(E,\mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*
(i) $\phi(0)=0$,
(ii) *there exist constants $\rho,\alpha>0$ such that $\phi_{/\partial B_\rho(0)}\geq\alpha$,*
(iii) *there exist $e\in E\backslash\overline{B}_\rho(0)$ such that $\phi(e)\leq 0$.*
Then ϕ possesses a critical value $c\geq\alpha$ given by $c=\inf_{g\in\Gamma}\max_{s\in[0,1]}\phi(g(s))$, where
$\Gamma=\{g\in C([0,1],E) : g(0)=0, g(1)=e\}$.

Lemma 2.1 *[2] Let $x:\mathbb{R}\to\mathbb{R}^N$ be a continuous mapping such that $\dot{x}\in L^2_{loc}(\mathbb{R},\mathbb{R}^N)$. For every $t\in\mathbb{R}$ the following inequality holds:*

$$|x(t)| \leq \sqrt{2}\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}}(|\dot{x}(s)|^2+|x(s)|^2)ds\right)^{1/2},$$

where $L^2_{loc}(\mathbb{R},\mathbb{R}^N)$ denotes the space of locally square integrable functions from $\mathbb{R}$ into $\mathbb{R}^N$.

Lemma 2.2 *[14] Denote by φ_s the flow of the linear vector field v with property (v_1), then*

$$|\varphi_s x| = e^s|x|, \forall s\in\mathbb{R}, \forall x\in\mathbb{R}^N.$$

Lemma 2.3 *There exist $a_1,a_2>0$ such that*

$$W(t,x)\geq a_1|x|^\mu - a_2, \forall t\in\mathbb{R}, \forall x\in\mathbb{R}^N. \tag{2.3}$$

Proof. Denote by S^{N-1} the unit sphere in $\mathbb{R}^N$. For any $x\in\mathbb{R}^N\backslash\{0\}$, since

$$\frac{d}{ds}(|\varphi_s x|^2) = 2\varphi_s x.v(\varphi_s(x)) > 0,$$

$(|\varphi_s x|^2)$ is increasing in s. Hence, there exist $s\in\mathbb{R}$ and $\xi\in S^{N-1}$ such that $x=\varphi_s\xi$ (see[13] for details). Since $|x|=|\varphi_s\xi|=e^s$, by (V_5') we have

$$\frac{d}{ds}[W(t,\varphi_s\xi)] = W'(t,\varphi_s\xi).v(\varphi_s\xi) \geq \mu W(t,\varphi_s\xi) > 0, \forall s,t\in\mathbb{R}. \tag{2.4}$$

Let $R > 0$, integrating (2.4) over $[\ln R, s]$ we obtain

$$\int_{\ln R}^{s} \frac{\frac{d}{dl}[W(t,\varphi_l \xi)]}{W(t,\varphi_l \xi)} dl \geq \mu s - \mu \ln R.$$

By (V_5') the quantity $a_1 = \inf_{t\in\mathbb{R},|x|=R}(W(t,x))R^{-\mu}$ is strictly positive and

$$W(t,x) \geq a_1 |x|^{\mu}, \forall |x| \geq R, \forall t \in \mathbb{R}.$$

Let $a_2 = \sup_{t\in\mathbb{R},|x|\leq R} W(t,x)$, then (2.3) holds. □ Let v be the normalized positive vector field in (V_3') and (V_5') of Theorem 1.2. Then v is an invertible linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $a = \frac{1}{\|v^{-1}\|}, b = \|v\|$, where $\|v\|$ and $\|v^{-1}\|$ are operator norms. For any $x \in \mathbb{R}^N$, one has

$$a|x| \leq |v(x)| \leq b|x|. \tag{2.5}$$

Define a vector field $\tilde{v}$ on E_k by

$$(\tilde{v}(x))(t) = v(x(t)). \tag{2.6}$$

Using condition (v_1) and a direct computation we have the following Lemma.

Lemma 2.4 *For any $x \in E_k$, there hold*

$$\int_{-kT}^{kT} |\dot{x}(t)|^2 dt = \int_{-kT}^{kT} \dot{x}(t).\,\dot{\overbrace{\tilde{v}(x)}}(t)dt. \tag{2.7}$$

$$a\|x\|_{E_k} \leq \|\tilde{v}(x)\|_{E_k} \leq b\|x\|_{E_k}. \tag{2.8}$$

Lemma 2.5 *Let $Y : [0,+\infty[\to [0,+\infty[$ be given as follows*

$$Y(s) = \begin{cases} \max_{t\in[0,T],0<|x|\leq s} \frac{W'(t,x).v(x)}{|x|^2}, & s > 0, \\ 0, & s = 0. \end{cases}$$

Then Y is continuous, nondecreasing, $Y(s) > 0$ for $s > 0$ and $Y(s) \to +\infty$ as $s \to +\infty$.

It is easy to prove this lemma by applying $(V_4), (V_5'), (V_8)$, (2.3) and (2.5).

Remark 2.1 Assumptions $(V_4), (V_5'), (V_8)$ and (2.5) imply that $W(t,x) = o(|x|^2)$ as $x \to 0$ uniformly for $t \in [0,T]$ and $W(t,0) = 0$, $W'(t,0) = 0$. Moreover, from (V_2) and (V_3') we conclude that $K(t,0) = 0$, $K'(t,0) = 0$.

3 Proof of Theorem 1.2

Let $\gamma_k : E_k \to [0,+\infty[$ be given by

$$\gamma_k(x) = \left(\int_{-kT}^{kT} \left[|\dot{x}(t)|^2 + 2K(t,x(t))\right] dt\right)^{1/2}. \tag{3.1}$$

Let $\bar{b}_2 = \max\{1, 2b_2\}$, by (V_2) we have

$$\bar{b}_1 \|x\|_{E_k}^2 \leq \gamma_k^2(x) \leq \bar{b}_2 \|x\|_{E_k}^2. \tag{3.2}$$

By (2.1) and (3.1) we have:

$$\phi_k(x) = \frac{1}{2}\gamma_k^2(x) - \int_{-kT}^{kT} W(t, x(t))dt + \int_{-kT}^{kT} f_k(t).x(t)dt. \tag{3.3}$$

Moreover, using (V_3'), (2.6) and (2.7) we obtain

$$\begin{aligned}\phi_k'(x).\tilde{v}(x) \leq & \int_{-kT}^{kT} \left(|\dot{x}(t)|^2 + 2K(t, x(t))\right) dt \\ & - \int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt + \int_{-kT}^{kT} f_k(t).v(x(t))dt \\ = & \gamma_k^2(x) - \int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt + \int_{-kT}^{kT} f_k(t).v(x(t))dt. \end{aligned}\tag{3.4}$$

Lemma 3.1 *Assume that V and f satisfy (V_1), (V_2), (V_3'), (V_4), (V_5'), and $(V_6)-(V_8)$. Then for every $k \in \mathbb{N}$ the system (HS_k) possesses a $2kT$-periodic solution $x_k \in E_k$.*

Proof. It is clear that $\phi_k(0) = 0$. We show that ϕ_k satisfies the Palais-Smale condition. Assume that $(x_j)_{j\in\mathbb{N}} \subset E_k$ is a sequence such that $(\phi_k(x_j))_{j\in\mathbb{N}}$ is bounded and $\phi_k'(x_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_k > 0$ such that

$$|\phi_k(x_j)| \leq C_k, \quad \|\phi_k'(x_j)\|_{E_k^*} \leq C_k, \tag{3.5}$$

for every $j \in \mathbb{N}$. By (3.3) and (V_5') we have

$$\gamma_k^2(x_j) \leq 2\phi_k(x_j) + \frac{2}{\mu}\int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt - 2\int_{-kT}^{kT} f_k(t).x_j(t)dt. \tag{3.6}$$

From (3.4) and (3.6) we obtain

$$(1-\frac{2}{\mu})\gamma_k^2(x_j) \leq 2\phi_k(x_j) - \frac{2}{\mu}\phi_k'(x_j)\tilde{v}(x_j) - 2\int_{-kT}^{kT} f_k(t).x_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} f_k(t).v(x_j(t))dt. \tag{3.7}$$

By (2.8), (3.2) and (3.7) we have

$$\begin{aligned}(1 - \frac{2}{\mu})\bar{b}_1 \|x_j\|_{E_k}^2 \leq & 2\phi_k(x_j) + \frac{2}{\mu}\|\phi_k'(x_j)\|_{E_k^*} b\|x_j\|_{E_k} + 2\left(\int_{-kT}^{kT} |f_k(t)|^2 dt\right)^{\frac{1}{2}} \|x_j\|_{E_k} \\ & + \frac{2}{\mu}\left(\int_{-kT}^{kT} |f_k(t)|^2 dt\right)^{\frac{1}{2}} b\|x_j\|_{E_k}. \end{aligned}\tag{3.8}$$

From (3.5), (3.8) and (V_7) we obtain

$$(1 - \frac{2}{\mu})\bar{b}_1 \|x_j\|_{E_k}^2 - \frac{2C_k}{\mu} b\|x_j\|_{E_k} - (2 + \frac{2b}{\mu})\frac{\beta}{2C}\|x_j\|_{E_k} - 2C_k \leq 0. \tag{3.9}$$

Since $\mu > 2$, (3.9) shows that $(x_j)_{j\in\mathbb{N}}$ is bounded in E_k. In a similar way to Proposition 4.3 in [6], we can prove that $(x_j)_{j\in\mathbb{N}}$ has a convergent subsequence in E_k. Hence, ϕ_k satisfies the Palais-Smale condition.

Now, let us show that there exist constants $\rho, \alpha > 0$ independent of k such that ϕ_k satisfies the assumption (ii) of Theorem 2.1 with these constants. Let $x \in E_k$ such that $0 < \|x\|_{L^\infty_{2kT}} \leq 1$. By (V_8) we have

$$\int_{-kT}^{kT} W(t, x(t))dt \leq M \int_{-kT}^{kT} |x(t)|^2 \, dt \leq M \|x\|^2_{E_k}. \tag{3.10}$$

From (3.2), (3.10) and (V_7) we have

$$\begin{aligned}\phi_k(x) &\geq \frac{1}{2}\bar{b}_1 \|x\|^2_{E_k} - M \|x\|^2_{E_k} - \|f_k\|_{L^2_{2kT}} \|x\|_{L^2_{2kT}} \\ &\geq \frac{1}{2}\bar{b}_1 \|x\|^2_{E_k} - M \|x\|^2_{E_k} - \frac{\beta}{2C} \|x\|_{L^2_{2kT}} \\ &\geq \frac{1}{2}(\bar{b}_1 - \beta - 2M) \|x\|^2_{E_k} + \frac{\beta}{2} \|x\|^2_{E_k} - \frac{\beta}{2C} \|x\|_{E_k}. \end{aligned} \tag{3.11}$$

Note that (V_7) implies $\bar{b}_1 - \beta - 2M > 0$. Set

$$\rho = \frac{1}{C}, \ \alpha = \frac{\bar{b}_1 - \beta - 2M}{2C^2}.$$

(3.11) shows that $\|x\|_{E_k} = \rho$ implies that $\phi_k(x) \geq \alpha$ for $k \in \mathbb{N}$. Finally, it remains to show that ϕ_k satisfies assumption (iii) of Theorem 2.1. By the use of (3.2), (3.3) and (2.3), for every $r \in \mathbb{R}\setminus\{0\}$ and $x \in E_k\setminus\{0\}$, the following inequality holds:

$$\phi_k(rx) \leq \frac{\bar{b}_2 r^2}{2} \|x\|^2_{E_k} - a_1 |r|^\mu \int_{-kT}^{kT} |x(t)|^\mu \, dt + |r| \, \|f_k\|_{L^2_{2kT}} \|x\|_{L^2_{2kT}} + 2kTa_2. \tag{3.12}$$

Take $X \in E_1$ such that $X(\pm T) = 0$. Since $\mu > 2$ and $a_1 > 0$, (3.12) implies that there exists $r_0 \in \mathbb{R}\setminus\{0\}$ such that $\|r_0 X\|_{E_1} > \rho$ and $\phi_1(r_0 X) < 0$. Set $e_1(t) = r_0 X(t)$ and

$$e_k(t) = \begin{cases} e_1(t), \ |t| \leq T \\ 0, \ T < |t| \leq kT \end{cases} \tag{3.13}$$

for $k > 0$. Then $e_k \in E_k, \|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ and $\phi_k(e_k) = \phi_1(e_1) < 0$ for every $k \in \mathbb{N}$. By Theorem 2.1, ϕ_k possesses a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} \phi_k(g(s)), \tag{3.14}$$

where $\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0, g(1) = e_k\}$. Hence, for every $k \in \mathbb{N}$, there exists $x_k \in E_k$ such that

$$\phi_k(x_k) = c_k, \ \phi'_k(x_k) = 0. \tag{3.15}$$

The function x_k is a desired classical $2kT-$periodic solution of (HS_k) for $k \in \mathbb{N}$. Since $c_k > 0$, x_k is a nontrivial solution even if $f_k(t) = 0$. □

Lemma 3.2 *Let $x_k \in E_k$ be a solution of system (HS_k) satisfying* (3.15). *Then there exists a positive constant M_1 independent of k such that*

$$\|x_k\|_{L^\infty_{2kT}} \leq M_1, \ \forall k \in \mathbb{N}. \tag{3.16}$$

Proof. For $k \in \mathbb{N}$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (3.13). Then $g_k \in \Gamma_k$ and $\phi_k(g_k(s)) = \phi_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (3.14)

$$c_k \leq \max_{s\in[0,1]} \phi_1(g_1(s)) \equiv M_0, \ \forall k \in \mathbb{N}, \tag{3.17}$$

where M_0 is independent of k. Since $\phi_k'(x_k) = 0$, we get from (2.7), (3.3), (V_3') and (V_5')

$$c_k = \phi_k(x_k) - \frac{1}{2}\phi_k'(x_k).\tilde{v}(x_k)$$

$$\geq (\frac{\mu}{2} - 1)\int_{-kT}^{kT} W(t, x_k(t))dt + \int_{-kT}^{kT} f_k(t).x_k(t)dt - \frac{1}{2}\int_{-kT}^{kT} f_k(t).v(x_k(t))dt,$$

and hence

$$\int_{-kT}^{kT} W(t, x_k(t))dt \leq \frac{2}{\mu-2}c_k - \frac{2}{\mu-2}\int_{-kT}^{kT} f_k(t).x_k(t)dt + \frac{1}{\mu-2}\int_{-kT}^{kT} f_k(t).v(x_k(t))dt. \tag{3.18}$$

Combining (3.18) with (2.8), (3.2), (3.17) and (V_7) we obtain

$$\frac{\bar{b}_1}{2}\|x_k\|_{E_k}^2 \leq \frac{\mu M_0}{\mu-2} + \frac{\beta(\mu+b)}{2C(\mu-2)}\|x_k\|_{E_k}. \tag{3.19}$$

Since $\bar{b}_1 > 0$ and all coefficients of (3.19) are independent of k, we see that there exist $M_1' > 0$ independent of k such that

$$\|x_k\|_{E_k} \leq M_1', \ \forall k \in \mathbb{N}, \tag{3.20}$$

which, together with [2, Proposition 1.1] impliy that (3.16) holds. □

Let $C_{loc}^p(\mathbb{R}, \mathbb{R}^N)$, where $p \in \mathbb{N}$, denotes the space of C^p functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the topology of almost uniformly convergence of functions and all derivatives up to the order p.

Lemma 3.3 *Let $x_k \in E_k$ be a solution of system (HS_k) satisfying* (3.16). *Then there exists a subsequence (x_{k_m}) of $(x_k)_{k\in\mathbb{N}}$ convergent to a certain $x_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ in $C_{loc}^1(\mathbb{R}, \mathbb{R}^N)$.*

Proof. By (3.16), we know that $(x_k)_{k\in\mathbb{N}}$ is a uniformly bounded sequence. Next, we will show that $(\dot{x}_k)_{k\in\mathbb{N}}$ and $(\ddot{x}_k)_{k\in\mathbb{N}}$ are also uniformly bounded sequences. Since x_k satisfies (HS_k), if $t \in [-kT, kT[$ we have

$$|\ddot{x}_k(t)| \leq |f_k(t)| + |V'(t, x_k(t))| = |f(t)| + |V'(t, x_k(t))|$$

$$\leq \sup_{t\in\mathbb{R}} |f(t)| + \sup_{(t,x)\in[0,T]\times[-M_1,M_1]} |V'(t, x(t))|, \ t \in [-kT, kT[. \tag{3.21}$$

From (3.16), (3.21), (V_1) and (V_6) there is $M_2 > 0$ independent of k such that

$$\|\ddot{x}_k\|_{L^\infty_{2kT}} \leq M_2, \ \forall k \in \mathbb{N}. \tag{3.22}$$

Let $i = -k, -k+1, ..., k-1$. By the continuity of $\dot{x}_k(t)$, we can choose $t_{k_i} \in [iT, (i+1)T]$, such that

$$\dot{x}_k(t_{k_i}) = \frac{1}{T}\int_{iT}^{(i+1)T} \dot{x}_k(s)ds = \frac{1}{T}\left(x_k((i+1)T) - x_k(iT)\right),$$

it follows that for $t \in [iT, (i+1)T],\ i = -k, -k+1, ..., k-1$

$$|\dot{x}_k(t)| = \left| \int_{t_{k_i}}^{t} \ddot{x}_k(s)ds + \dot{x}_k(t_{k_i}) \right| \leq \int_{iT}^{(i+1)T} |\ddot{x}_k(s)|\, ds + |\dot{x}_k(t_{k_i})|$$

$$\leq M_2T + T^{-1} |x_k((i+1)T) - x_k(iT)| \leq M_2T + 2M_1T^{-1} \equiv M_3.$$

Consequently,

$$\|\dot{x}_k\|_{L^\infty_{2kT}} \leq M_3,\ \forall k \in \mathbb{N}. \tag{3.23}$$

The task is now to show that $(x_k)_{k\in\mathbb{N}}$ and $(\dot{x}_k)_{k\in\mathbb{N}}$ are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of k. Let $k \in \mathbb{N}$ and $t, t_0 \in \mathbb{R}$, we have by (3.23)

$$|x_k(t) - x_k(t_0)| = \left| \int_{t_0}^{t} \dot{x}_k(s)ds \right| \leq \left| \int_{t_0}^{t} |\dot{x}_k(s)|\, ds \right| \leq M_3 |t - t_0|.$$

Analogously, we have by (3.22) $|\dot{x}_k(t) - \dot{x}_k(t_0)| \leq M_2 |t - t_0|$. For each $k \in \mathbb{N}$, set $C^1_k = C^1([-kT, kT], \mathbb{R}^N)$ with the norm defined as follows:

$$\|x\|_{C^1_k} = \max_{t\in[-kT,kT]} (|\dot{x}(t)| + |x(t)|),\ x \in C^1_k.$$

Now, we will show that $(x_k)_{k\in\mathbb{N}}$ possesses a convergent subsequence (x_{k_m}) in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$. First, let $(x_k)_{k\in\mathbb{N}}$ be restricted to $[-T, T]$. It is clear that (x_k) and $(\dot{x}_k)$ are uniformly bounded and equicontinuous. By Arzela-Ascoli theorem, there exist a subsequence (x^1_k) of $(x_k)_{k\in\mathbb{N}\setminus\{1\}}$, $x^1 \in C([-T, T], \mathbb{R}^N)$ and $y^1 \in C([-T, T], \mathbb{R}^N)$ such that

$$\left\|x^1_k - x^1\right\|_{C([-T,T],\mathbb{R}^N)} \to 0,\ \left\|\dot{x}^1_k - y^1\right\|_{C([-T,T],\mathbb{R}^N)} \to 0,\ as\ k \to +\infty. \tag{3.24}$$

Note that for $t \in [-T, T]$

$$x^1_k(t) = x^1_k(-T) + \int_{-T}^{t} \dot{x}^1_k(s)ds,\ k \in \mathbb{N}. \tag{3.25}$$

Let $k \to \infty$ in (3.25) and using (3.24) we obtain

$$x^1(t) = x^1(-T) + \int_{-T}^{t} y^1(s)ds,\ for\ t \in [-T, T] \tag{3.26}$$

which shows that $y^1(t) = \dot{x}^1(t)$ for $t \in [-T, T]$ and $x^1 \in C^1_1$. Moreover, it follows from (3.24) that

$$\left\|x^1_k - x^1\right\|_{C^1_1} \to 0,\ \ as\ k \to +\infty.$$

Secondly, let (x^1_k) be restricted to $[-2T, 2T]$. It is clear that (x^1_k) and $(\dot{x}^1_k)$ are uniformly bounded and equicontinuous. Similarly as above, by Arzela-Ascoli theorem, there exist a subsequence (x^2_k) of (x^1_k) satisfying $x_2 \notin (x^2_k)$ and $x^2 \in C^1_2$ such that

$$\left\|x^2_k - x^2\right\|_{C^1_2} \to 0,\ as\ k \to +\infty.$$

By repeating this procedure for all $k \in \mathbb{N}$, there exist $(x^m_k) \subset (x^{m-1}_k)$, $x_m \notin (x^m_k)$ and $x^m \in C^1_m$ such that

$$\|x^m_k - x^m\|_{C^1_m} \to 0,\ as\ k \to +\infty,\ m = 1, 2, ... \tag{3.27}$$

Moreover, we have

$$\left\|x^{m+1}-x^m\right\|_{C^1_m} \le \left\|x_k^{m+1}-x^{m+1}\right\|_{C^1_m} + \left\|x_k^m - x^m\right\|_{C^1_m} + \left\|x_k^{m+1}-x_k^m\right\|_{C^1_m} \to 0$$

as $k \to +\infty$, which leads to

$$x^{m+1}(t) = x^m(t),\ for\ t \in [-mT, mT],\ m = 1, 2, \dots\ . \tag{3.28}$$

Let

$$x_0(t) = x^m(t),\ for\ t \in [-mT, mT],\ m = 1, 2, \dots\ . \tag{3.29}$$

Then $x_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ and $x^m \to x_0$ as $m \to +\infty$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$. Now take a diagonal sequence (x_{k_m}) consisting of $x_1^1, x_2^2, x_3^3, \dots$ (see [4]). For any $m \in \mathbb{N}$, $(x_i^i)_{i=m}^{\infty}$ is a subsequence of $(x_k^m)_{k\in\mathbb{N}}$, so it follows from (3.27) and (3.29) that

$$\left\|x_i^i - x_0\right\|_{C^1_m} = \left\|x_i^i - x^m\right\|_{C^1_m} \to 0,\ as\ i \to +\infty,\ m = 1, 2, \dots$$

That is

$$x_{k_m} \to x_0,\ as\ m \to +\infty\ in\ C^1_{loc}(\mathbb{R}, \mathbb{R}^N). \tag{3.30}$$

Lemma 3.4 *The function x_0 defined in Lemma 3.3 is the desired homoclinic solution of (HS).*

Proof. Firstly we will show that x_0 satisfies (HS). For every $k \in \mathbb{N}$, and $t \in \mathbb{R}$ we have by Lemma 3.1:

$$\ddot{x}_{k_m}(t) = f_{k_m}(t) - V'(t, x_{k_m}(t)). \tag{3.31}$$

Take $l_1, l_2 \in \mathbb{R}$ such that $l_1 < l_2$. There exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$

$$\ddot{x}_{k_m}(t) = f(t) - V'(t, x_{k_m}(t)),\ \forall t \in [l_1, l_2]. \tag{3.32}$$

Integrating (3.32) from l_1 to $t \in [l_1, l_2]$, we have

$$\dot{x}_{k_m}(t) - \dot{x}_{k_m}(l_1) = \int_{l_1}^{t} [f(s) - V'(s, x_{k_m}(s))]ds. \tag{3.33}$$

Since (3.30) shows that $x_{k_m} \to x_0$ uniformly on $[l_1, l_2]$ and $\dot{x}_{k_m} \to \dot{x}_0$ uniformly on $[l_1, l_2]$ as $m \to +\infty$, then by taking $m \to +\infty$ in (3.33), we get

$$\dot{x}_0(t) - \dot{x}_0(l_1) = \int_{l_1}^{t} [f(s) - V'(s, x_0(s))]ds,\ for\ t \in [l_1, l_2]. \tag{3.34}$$

Since l_1 and l_2 are arbitrary, (3.34) shows that x_0 is a solution of (HS). Secondly, we prove that $x_0(t) \to 0,\ as\ t \to \pm\infty$. We have, from (3.20)

$$\int_{-kT}^{kT} (|\dot{x}_k(t)|^2 + |x_k(t)|^2)dt \le M_1'^2,\ \forall k \in \mathbb{N}. \tag{3.35}$$

For every $l \in \mathbb{N}$, there exists $m_1 \in \mathbb{N}$ such that for $m > m_1$

$$\int_{-lT}^{lT} (|\dot{x}_{k_m}(t)|^2 + |x_{k_m}(t)|^2)dt \le M_1'^2. \tag{3.36}$$

Let $m \to +\infty$ in (3.36) and use (3.30), it follows that for each $l \in \mathbb{N}$,

$$\int_{-lT}^{lT} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \le M_1'^2. \tag{3.37}$$

Letting $l \to +\infty$ in (3.37), we obtain

$$\int_{-\infty}^{+\infty} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \le M_1'^2, \tag{3.38}$$

and so

$$\int_{|t|\ge r} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \to 0, \ as\ t \to \pm\infty. \tag{3.39}$$

Combining (3.39) with Lemma 2.3 we obtain our claim.

Now, we show that $\dot{x}_0(t) \to 0, \ as\ t \to \pm\infty$. To do this, observe that by Lemma 2.3

$$|\dot{x}_0(t)|^2 \le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|x_0(s)|^2 + |\dot{x}_0(s)|^2)ds + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{x}_0(s)|^2 \, ds. \tag{3.40}$$

From (3.39) and (3.40) it suffices to prove that

$$\int_{r}^{r+1} |\ddot{x}_0(s)|^2 \, ds \to 0, \ as\ r \to \pm\infty. \tag{3.41}$$

By (HS) we obtain

$$\int_{r}^{r+1} |\ddot{x}_0(s)|^2 \, ds = \int_{r}^{r+1} (|V'(s, x_0(s))|^2 + |f(s)|^2)ds - 2\int_{r}^{r+1} V'(s, x_0(s)).f(s)ds.$$

Since $V'(t, 0) = 0$ for all $t \in \mathbb{R}$, $x_0 \to 0, \ as\ t \to \pm\infty$ and $\int_r^{r+1} |f(s)|^2 \, ds \to 0$, $as\ r \to \pm\infty$, then (3.41) follows.

Finally, we will show that if $f \equiv 0$ then $x_0 \not\equiv 0$. For this purpose we will use the properties of Y given by (2.9). The definition of Y implies that

$$\int_{-kT}^{kT} W'(t, x_k(t)).v(x_k(t))dt \le Y(\|x_k\|_{L^\infty_{2kT}}) \|x_k\|_{E_k}^2 . \tag{3.42}$$

Since $\phi_k'(x_k).v(x_k) = 0$, then (3.4) gives

$$\int_{-kT}^{kT} W'(t, x_k(t)).v(x_k(t))dt = \int_{-kT}^{kT} |\dot{x}_k(t)|^2 \, dt + \int_{-kT}^{kT} K'(t, x_k(t)).v(x_k(t))dt. \tag{3.43}$$

Substituting (3.43) into (3.42), and applying (V_3') and (V_2) we obtain

$$Y(\|x_k\|_{L^\infty_{2kT}}) \ge \min\{1, b_1\} \|x_k\|_{E_k}^2 ,$$

and hence

$$Y(\|x_k\|_{L^\infty_{2kT}}) \ge \min\{1, b_1\} > 0. \tag{3.44}$$

If $\|x_{k_m}\|_{L^\infty_{2k_mT}} \to 0$, as $m \to +\infty$, we would have $Y(0) \ge \min\{1, b_1\} > 0$, a contradiction. Passing to a subsequence of $(x_{k_m})_{m\in\mathbb{N}}$ if necessary, there is $\eta > 0$ such that

$$\|x_{k_m}\|_{L^\infty_{2k_mT}} \ge \eta. \tag{3.45}$$

Moreover, for all $j \in \mathbb{N}$, $t \mapsto x_{k_m,j}(t) = x_{k_m}(t + jT)$ is also a $2k_mT$-periodic solution of (HS_{k_m}). Hence, if the maximum of $|x_{k_m}|$ occurs in $h_{k_m} \in [-k_mT, k_mT]$ then, the maximum of $|x_{k_m,j}|$ occurs in $s_{k_m,j} = h_{k_m} - jT$. Then there exists a $j_{k_m} \in \mathbb{Z}$ such that $s_{k_m,j_{k_m}} \in [-T, T]$. Consequently,

$$\left\| x_{k_m,j_{k_m}} \right\|_{L^\infty_{2k_mT}} = \max_{t \in [-T,T]} \left| x_{k_m,j_{k_m}}(t) \right| .$$

Suppose, contrary to our claim, that $x_0 = 0$. Then, by Lemma 3.3,

$$\left\| x_{k_m,j_{k_m}} \right\|_{L^\infty_{2k_mT}} = \max_{t \in [-T,T]} \left| x_{k_m,j_{k_m}}(t) \right| \to 0,$$

which contradicts (3.45). □

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