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On the Existence of a Common Lyapunov Function for a Family of Nonlinear Mechanical Systems with One Degree of Freedom

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Abstract: Certain classes of essentially nonlinear switched mechanical systems with one degree of freedom are investigated. The conditions are obtained under which, for the families of subsystems corresponding to switched systems, there exist common Lyapunov functions of the prescribed form. The fulfilment of these conditions provides the asymptotic stability of equilibrium positions of switched systems for any switching law.

Keywords: *switched mechanical systems; nonlinear forces; asymptotic stability; common Lyapunov function.*

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1 Introduction

Stability analysis and synthesis of switched systems are fundamental and challenging research problems, see, for example, [4, 7, 11]. In some cases it is required to design a control system in such a way that it remains stable for any admissible switching law [7, 11]. These cases are natural, when switching signal is either unknown, or too complicated to be explicitly taken into account.

A general approach to the above problem is based on the computation of a common Lyapunov function (CLF) for a family of subsystems corresponding to the switched system. This approach has been effectively used in many papers, see [4, 7–9, 11]. However, the conditions of the existence of a CLF are not completely investigated even for the case of families of linear time-invariant systems [7–9].

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This problem is especially complicated for mechanical systems with switching force fields. Motion of mechanical systems is described usually by differential equations of the second order, that results in the appearance of some special properties. In [2], it was mentioned that the known conditions of the existence of CLFs obtained for systems of general form might be ineffective or even nonapplicable for switched mechanical systems. The specific character of mechanical systems leads to the necessity of the separate investigation of such systems as a special subclass of hybrid systems. This subclass possesses certain theoretical features and is of undoubted practical interest [3–5, 11].

In the present paper, certain types of switched nonlinear mechanical systems with one degree of freedom are studied. The conditions of the existence of CLFs for families of subsystems corresponding to switched systems are obtained. The fulfilment of these conditions provides that the equilibrium positions of the considered systems are asymptotically stable for arbitrary switching law.

2 Statement of the Problem

First, consider the linear switched mechanical system with one degree of freedom

$$\ddot{x} + a_\sigma \dot{x} + b_\sigma x = 0. \tag{1}$$

Here scalar variable $x(t)$ is the state of the system; $\sigma = \sigma(t)$ is the piecewise constant function defining the switching law, $\sigma(t) : [0, +\infty) \to Q = \{1, \dots, N\}$. In the present paper, we assume that on every bounded time interval the switching function has a finite number of discontinuities, which are called switching instants of time, and takes a constant value on every interval between two consecutive switching instants. This kind of switching law is called admissible one.

Thus, at each time instant, the behaviour of (1) is described by one of the subsystems

$$\ddot{x} + a_s \dot{x} + b_s x = 0, \quad s = 1, \dots, N, \tag{2}$$

where a_s and b_s are constant coefficients.

Let the inequalities $a_s > 0$, $b_s > 0$, $s = 1, \dots, N$, be fulfilled. Then, for every subsystem from the family (2), the equilibrium position $x = \dot{x} = 0$ is asymptotically stable. In spite of this fact, it is well known [4, 7] that there exist parameters a_s and b_s values and switching laws under which the equilibrium position $x = \dot{x} = 0$ of the corresponding switched system (1) is unstable. It is worthy of note that instability can take place even in the case where family (2) consists of two subsystems ($N = 2$), and switching occurs only in the positional forces ($a_1 = a_2 = \text{const} > 0$).

In the present paper, we consider the nonlinear switched system

$$\ddot{x} + a_\sigma \dot{x} + b_\sigma x^\mu = 0 \tag{3}$$

and the corresponding family of subsystems

$$\ddot{x} + a_s \dot{x} + b_s x^\mu = 0, \quad s = 1, \dots, N. \tag{4}$$

Here the switching function $\sigma(t)$ possesses the same properties as in (1); a_s and b_s are positive constants; μ is a rational number with odd numerator and denominator, $\mu > 1$. Thus, subsystems from the family (4) are subjected to linear dissipative forces and essentially nonlinear potential forces. It is known [10] that the equilibrium position $x = \dot{x} = 0$ of each subsystem is asymptotically stable.

We will look for the conditions providing the asymptotic stability of the equilibrium position $x = \dot{x} = 0$ of (3) for any admissible switching law. To solve the problem, we consider the Lyapunov function of a special form and determine the region of the parameters a_s and b_s values under which CLF of the prescribed form can be constructed for the family of subsystems (4).

Furthermore, we extend the obtained results to the case of switched mechanical system with nonlinear dissipative and potential forces.

3 Conditions of the Existence of a CLF

Consider the Lyapunov function

$$V(x,\dot{x}) = \frac{\dot{x}^2}{2} + c\frac{x^{\mu+1}}{\mu+1} + \gamma x^{\beta}\dot{x}. \tag{5}$$

Here c and γ are positive constants, and β is a rational number with odd numerator and denominator, $\beta \geq 1$.

Differentiating $V(x,\dot{x})$ with respect to the sth subsystem from family (4), we obtain

$$\dot{V} = -a_s\dot{x}^2 - \gamma b_s x^{2\mu} + (c - b_s)x^{\mu}\dot{x} - a_s\gamma x^{\beta}\dot{x} + \gamma\beta x^{\beta-1}\dot{x}^2 \equiv W_s(x,\dot{x}).$$

By the use of generalized homogeneous functions properties [12], one gets the following necessary condition of the negative definiteness of functions $W_1(x,\dot{x}),\ldots,W_N(x,\dot{x})$:

$$\beta = \mu. \tag{6}$$

For such value of the parameter β, the Lyapunov function (5) is positive definite for any $c > 0$ and $\gamma > 0$, and functions $W_1(x,\dot{x}),\ldots,W_N(x,\dot{x})$ are negative definite if and only if the quadratic forms

$$\omega_s(y_1,y_2) = -a_s y_2^2 - \gamma b_s y_1^2 + (c - b_s - a_s\gamma)y_1y_2, \quad s = 1,\ldots,N, \tag{7}$$

possess the same property.

Applying the Sylvester criterion, we obtain $4a_s b_s\gamma > (c - b_s - a_s\gamma)^2$, $s = 1,\ldots,N$. Hence, the inequalities

$$\left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2 < c < \left(\sqrt{a_s\gamma} + \sqrt{b_s}\right)^2, \quad s = 1,\ldots,N,$$

should be valid. It means that, for the existence of the required value of the parameter c, it is necessary and sufficient the fulfilment of the conditions

$$\left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2 < \left(\sqrt{a_j\gamma} + \sqrt{b_j}\right)^2, \quad s,j = 1,\ldots,N. \tag{8}$$

Conditions (8) can be rewritten in the form

$$\sqrt{\gamma}(\sqrt{a_s} + \sqrt{a_j}) > \sqrt{b_s} - \sqrt{b_j}, \qquad \sqrt{\gamma}(\sqrt{a_s} - \sqrt{a_j}) < \sqrt{b_s} + \sqrt{b_j}, \qquad s,j = 1,\ldots,N.$$

Denote

$$A = \max_{s,j=1,\ldots,N} \frac{\sqrt{b_s} - \sqrt{b_j}}{\sqrt{a_s} + \sqrt{a_j}},$$

$B = +\infty$ if $a_s = a_j$ for all $s, j = 1, \ldots, N$, and

$$B = \min_{s,j:\ a_s > a_j} \frac{\sqrt{b_s} + \sqrt{b_j}}{\sqrt{a_s} - \sqrt{a_j}}$$

otherwise.

Finally, we arrive at

Theorem 3.1 *Family* (4) *admits a CLF of the form* (5) *satisfying the assumptions of the Lyapunov asymptotic stability theorem if and only if the inequality*

$$A < B \tag{9}$$

holds.

Remark 3.1 Theorem 3.1 gives us the constructive algorithm for finding the CLF for family (4). If inequality (9) is fulfilled, then the value of parameter β is determined by formula (6), while $\gamma \in (A, B)$, and, for the value of γ chosen from this interval, $c \in (\underline{c}(\gamma), \overline{c}(\gamma))$, where

$$\underline{c}(\gamma) = \max_{s=1,\ldots,N} \left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2, \qquad \overline{c}(\gamma) = \min_{s=1,\ldots,N} \left(\sqrt{a_s\gamma} + \sqrt{b_s}\right)^2.$$

Although we have obtained the necessary and sufficient conditions of the existence of a CLF for family (4), however only for the Lyapunov function of the special form (5). Nevertheless, these conditions permit us to deduce the following interesting and important conclusions about stability of the equilibrium position $x = \dot{x} = 0$ of switched system (3).

Corollary 3.1 *Let the switching take place in the velocity forces only ($b_s = b = \mathrm{const} > 0$, $s = 1, \ldots, N$). Then the equilibrium position $x = \dot{x} = 0$ of system* (3) *is asymptotically stable for any admissible switching law.*

Corollary 3.2 *Let the switching take place in the potential forces only ($a_s = a = \mathrm{const} > 0$, $s = 1, \ldots, N$). Then the equilibrium position $x = \dot{x} = 0$ of system* (3) *is asymptotically stable for any admissible switching law.*

Corollary 3.3 *Let family* (4) *consist of two subsystems ($N = 2$), and the switching take place both in the velocity forces and in the potential forces ($a_1 \neq a_2$, $b_1 \neq b_2$). Then the equilibrium position $x = \dot{x} = 0$ of system* (3) *is asymptotically stable for any admissible switching law.*

Remark 3.2 As it was mentioned in Section 2, the statements of Corollaries 3.2 and 3.3 are not true for the linear case ($\mu = 1$). Thus, in comparison with linear systems, nonlinear ones are "more stable" with respect to the switching of parameters values.

4 Systems with Nonlinear Dissipative and Potential Forces

Consider now the switched system

$$\ddot{x} + a_\sigma x^\nu \dot{x} + b_\sigma x^\mu = 0. \tag{10}$$

The corresponding family of subsystems is described as follows

$$\ddot{x} + a_s x^\nu \dot{x} + b_s x^\mu = 0, \quad s = 1, \dots, N. \tag{11}$$

Here a_s and b_s are positive constants; μ is a rational number with odd numerator and denominator, $\mu > 1$; ν is a positive rational number with even numerator and odd denominator. In this case, considered subsystems are subjected to essentially nonlinear dissipative and potential forces. Equations of such type are called the Lienard ones [6, 10]. It is known [10] that the equilibrium position $x = \dot{x} = 0$ of each subsystem from (11) is asymptotically stable.

To obtain the conditions providing the asymptotic stability of the equilibrium position of (10) for any admissible switching law, construct a CLF for the family (11) in the form

$$V(x, \dot{x}) = \frac{\dot{x}^2}{2} + c\frac{x^{\mu+1}}{\mu+1} + \gamma x^\beta \dot{x} + \varepsilon x \dot{x}^\lambda, \tag{12}$$

where $c > 0$, $\gamma > 0$, $\varepsilon < 0$, while β and λ are rational numbers with odd numerators and denominators, $\beta \geq 1$, $\lambda \geq 1$.

Differentiating $V(x, \dot{x})$ with respect to the sth subsystem from (11), one gets

$$\dot{V} = \varepsilon \dot{x}^{\lambda+1} - a_s x^\nu \dot{x}^2 - \gamma b_s x^{\mu+\beta} + (c - b_s) x^\mu \dot{x} - a_s \gamma x^{\beta+\nu} \dot{x}$$

$$+ \gamma\beta x^{\beta-1} \dot{x}^2 - \varepsilon\lambda a_s x^{\nu+1} \dot{x}^\lambda - \varepsilon\lambda b_s x^{\mu+1} \dot{x}^{\lambda-1} \equiv W_s(x, \dot{x}).$$

By the use of generalized homogeneous functions properties [12] and Lemma 2 from [1], it is easy to obtain the following necessary conditions of the negative definiteness of functions $W_1(x, \dot{x}), \dots, W_N(x, \dot{x})$:

(i) if $\mu > 2\nu + 1$, then

$$\beta = \mu - \nu; \tag{13}$$

(ii) if $\mu \leq 2\nu + 1$, then $\lambda = 1 + 2(\beta - 1)/(\mu + 1)$.

It is worthy of note that, in the case where $\mu = 2\nu + 1$, systems

$$\dot{x} = y, \quad \dot{y} = -a_s x^\nu y - b_s x^\mu, \qquad s = 1, \dots, N,$$

corresponding to equations from (11) are generalized homogeneous.

In what follows, we consider the only case where $\mu > 2\nu + 1$. Under the condition (13), we have

$$W_s(x, \dot{x}) = x^\nu \left(-a_s \dot{x}^2 - \gamma b_s x^{2(\mu-\nu)} + (c - b_s - a_s\gamma) x^{\mu-\nu} \dot{x} \right) + \varepsilon \dot{x}^{\lambda+1}$$

$$+ \gamma\beta x^{\mu-\nu-1} \dot{x}^2 - \varepsilon\lambda a_s x^{\nu+1} \dot{x}^\lambda - \varepsilon\lambda b_s x^{\mu+1} \dot{x}^{\lambda-1}, \quad s = 1, \dots, N.$$

Let

$$\lambda > \frac{2\mu - 2\nu - 1}{\mu - \nu}. \tag{14}$$

Then the Lyapunov function (12) is positive definite, and for the negative definiteness of functions $W_1(x, \dot{x}), \dots, W_N(x, \dot{x})$ it is sufficient the negative definiteness of quadratic forms (7).

With the numbers A and B defined in a similar way as in Section 3, we claim the following result

Theorem 4.1 *Let $\mu > 2\nu + 1$. If inequality* (9) *holds, then for family* (11) *there exists a CLF of the form* (12) *satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

Remark 4.1 In contrast to Theorem 3.1, the conditions of Theorem 4.1 are only sufficient ones for the existence of a CLF of the given form for the considered family.

Remark 4.2 Under the conditions of Theorem 4.1, we obtain the following constructive algorithm for the finding of a CLF for family (11). The Lyapunov function can be chosen in the form (12), where β is defined by the formula (13), λ satisfies inequality (14), ε is an arbitrary negative number, while the values of parameters γ and c are defined in a similar way as in Remark 3.1.

Corollary 4.1 *Let $\mu > 2\nu+1$. If the switching takes place in the velocity forces only ($b_s = b = \text{const} > 0$, $s = 1, \dots, N$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

Corollary 4.2 *Let $\mu > 2\nu + 1$. If the switching takes place in the potential forces only ($a_s = a = \text{const} > 0$, $s = 1, \dots, N$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

Corollary 4.3 *Let $\mu > 2\nu+1$. If family* (11) *consists of two subsystems ($N = 2$), and the switching takes place both in the velocity forces and in the potential forces ($a_1 \neq a_2$, $b_1 \neq b_2$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

5 Conclusion

In the present paper, for certain classes of families of nonlinear mechanical systems with one degree of freedom the conditions of the existence of CLFs of the given form are obtained. The fulfilment of these conditions provides the asymptotic stability of equilibrium positions of corresponding switched systems for any switching law. It is proved that, for considered families of essentially nonlinear systems, we can guarantee the existence of CLFs under weaker assumptions than for linear ones. Thus, in comparison with linear systems, nonlinear ones are "more stable" with respect to the switching of parameters values. Theorems 3.1 and 4.1 can be used for the design of stabilizing controls for mechanical systems. A challenging direction for further research is the extention of the obtained results to the switched nonlinear mechanical systems with several degrees of freedom.

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