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Time Scales Ostrowski and Grüss Type Inequalities Involving Three Functions

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Abstract: In this paper, we present time scales versions of Ostrowski and Grüss type inequalities containing three functions. We assume that the second derivatives of these functions are bounded. Our results are new also for the discrete case.

Keywords: *Ostrowski–Grüss inequality; Ostrowski-like inequality; Montgomery identity; time scales.*

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1 Introduction

Motivated by a recent paper by B. G. Pachpatte [18], our purpose is to obtain time scales versions of some Ostrowski and Grüss type inequalities including three functions, whose second derivatives are bounded. In detail, we will prove time scales analogues of the following three theorems presented in [18].

Theorem 1.1 [See [18, Theorem 1]] *Let $f, g, h : [a, b] \to \mathbb{R}$ be twice differentiable functions on (a, b) such that $f'', g'', h'' : (a, b) \to \mathbb{R}$ are bounded, i.e.,*

$$\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty, \quad \|g''\|_\infty < \infty, \quad \|h''\|_\infty < \infty.$$

Moreover, let

$$A[f, g, h] := gh \int_a^b f(s)\mathrm{d}s + fh \int_a^b g(s)\mathrm{d}s + fg \int_a^b h(s)\mathrm{d}s$$

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and

$$B[f,g,h] := |gh|\,\|f''\|_\infty + |fh|\,\|g''\|_\infty + |fg|\,\|h''\|_\infty .$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right)(fgh)'(t) \right| \leq \frac{1}{6}\left\{ \left(t - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \right\} B[f,g,h](t).$$

Theorem 1.2 [See [18, Theorem 2]] *In addition to the notation and assumptions of Theorem 1.1, let*

$$L[f,g,h] := gh\frac{f(a)+f(b)}{2} + fh\frac{g(a)+g(b)}{2} + fg\frac{h(a)+h(b)}{2}.$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{2}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right)(fgh)'(t) + \frac{1}{3}L[f,g,h](t) \right| \leq \frac{1}{3(b-a)}B[f,g,h](t)\int_a^b \left| p(t,s)\left(s - \frac{a+b}{2}\right)\right| \mathrm{d}s,$$

where $p(t,s) = s-a$ for $a \leq s < t$ and $p(t,s) = s-b$ for $t \leq s \leq b$.

Theorem 1.3 [See [18, Theorem 3]] *In addition to the notation and assumptions of Theorem 1.1, let*

$$M[f,g,h] := gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a}.$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right)M[f,g,h](t) \right| \leq \frac{1}{3(b-a)^2}B[f,g,h](t)\int_a^b \int_a^b |p(t,\tau)p(\tau,s)|\,\mathrm{d}s\mathrm{d}\tau,$$

where p is defined as in Theorem 1.2.

Our time scales versions of Theorems 1.1–1.3 will contain Theorems 1.1–1.3 as special cases when the time scale is equal to the set of all real numbers, and they will yield new discrete inequalities when the time scale is equal to the set of all integer numbers. Special cases of our results are contained in [2–5, 12, 15, 20] for the general time scales case, in [8–10, 16] for the continuous case and in [1, 17] for the discrete case. One can also use our results for any other arbitrary time scale to obtain new inequalities, e.g., for the quantum case. For further recent results on time scales calculus published in *Nonlinear Dynamics and Systems Theory*, we refer to [11, 13, 14, 19].

The set up of this paper is as follows. In the next section, we give some necessary details of the time scales calculus. Section 3 contains some auxiliary results as well as the assumptions and notation used in this paper. Finally, in Sections 4–6, we prove time scales analogues of Theorems 1.1–1.3. Each result is followed by several examples and remarks. We would like to point out here that our results are new also for the discrete case.

2 Preliminaries

Now we briefly introduce some necessary time scales elements and refer the reader to the books [6, 7] for further details.

Definition 2.1 A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, *left-dense*, and *left-scattered* provided $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. The set $\mathbb{T}^\kappa$ is defined to be equal to the set $\mathbb{T}$ without its left-scattered maximum (if it exists). A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* and we write $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ if it is continuous at all right-dense points and its left-sided limits exist and are finite at all left-dense points, and f is called *delta differentiable* at $t \in \mathbb{T}^\kappa$, with *delta derivative* $f^\Delta(t) \in \mathbb{R}$, provided given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon\, |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

If f is differentiable such that f^Δ is rd-continuous, then we write $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. The set $\mathrm{C}^2_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is defined similarly. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the *delta integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where} \quad a, b \in \mathbb{T}.$$

Example 2.1 If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $f^\Delta(t) = f'(t)$ for all $t \in \mathbb{R}$ and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t \quad \text{for all} \quad a, b \in \mathbb{R},$$

and if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $f^\Delta(t) = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$ and

$$\int_0^n f(t)\Delta t = \sum_{t=0}^{n-1} f(t) \quad \text{for all} \quad n \in \mathbb{N}.$$

Some results about integrals, that will be used in this paper, are contained in [6, Section 1.4] and collected as follows.

Theorem 2.1 *If a function is rd-continuous, then it possesses a delta antiderivative. For $f, g \in \mathrm{C}_{\mathrm{rd}}([a, b], \mathbb{R})$ and $a, b, c \in \mathbb{T}$, we have*

$$\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$$

$$\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$$

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\,\Delta t,$$

and, if additionally $f, g \in \mathrm{C}^1_{\mathrm{rd}}([a,b],\mathbb{R})$,

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t.$$

We also need the time scales monomials (see [6, Section 1.6]) defined as follows.

Definition 2.2 Define for all $t, s \in \mathbb{T}$

$$g_2(t,s) := \int_s^t (\sigma(\tau) - s)\Delta\tau, \quad h_2(t,s) := \int_s^t (\tau - s)\Delta\tau,$$
$$g_3(t,s) := \int_s^t g_2(\sigma(\tau), s)\Delta\tau, \quad h_3(t,s) := \int_s^t h_2(\tau, s)\Delta\tau.$$

It is known that $g_2(t,s), g_3(t,s), h_2(t,s), h_3(t,s)$ are nonnegative for $t \geq s$ and that $g_2(t,s) = h_2(s,t)$ and $g_3(t,s) = -h_3(s,t)$. Moreover, the following formulas are used in this paper.

Lemma 2.1 *The time scales monomials satisfy the following formulas:*

$$g_2(t,a) - g_2(t,b) = g_2(b,a) + (t-b)(b-a), \tag{1}$$
$$g_2(a,b) + g_2(b,a) = (b-a)^2, \tag{2}$$
$$g_3(t,a) - g_3(t,b) = g_3(b,a) + (t-b)g_2(b,a) + (b-a)g_2(t,b). \tag{3}$$

Proof. The function F defined by $F(t) := g_2(t,a) - g_2(t,b) - g_2(b,a) - (t-b)(b-a)$ satisfies $F^{\Delta}(t) = \sigma(t) - a - (\sigma(t) - b) - (b-a) = 0$ and $F(b) = 0$. Hence $F = 0$ and so (1) holds. Next, (2) follows by letting $t = a$ in (1). Moreover, the function G defined by $G(t) := g_3(t,a) - g_3(t,b) - g_3(b,a) - (t-b)g_2(b,a) - (b-a)g_2(t,b)$ satisfies $G^{\Delta}(t) = g_2(\sigma(t),a) - g_2(\sigma(t),b) - g_2(b,a) - (b-a)(\sigma(t)-b) = F(\sigma(t)) = 0$ and $G(b) = 0$. Hence $G = 0$ and so (3) holds.

3 Auxiliary Results and Assumptions

Throughout this paper we assume that $\mathbb{T}$ is a time scale and that $a, b \in \mathbb{T}$ such that $a < b$. Moreover, when writing $[a,b]$, we mean the time scales interval $[a,b] \cap \mathbb{T}$. The following two Montgomery-type results are used in the proofs of our three main results.

Theorem 3.1 *Suppose* $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. *Let* $t \in [a,b]$ *and* $u_1, u_2 \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. *If*

$$u(\sigma(s)) = \begin{cases} u_1(\sigma(s)) & \text{for} \quad a \leq s < t, \\ u_2(\sigma(s)) & \text{for} \quad t \leq s \leq b, \end{cases} \tag{4}$$

then

$$\begin{aligned} \int_a^b u(\sigma(s))f^{\Delta}(s)\Delta s &= (u_1(t) - u_2(t))f(t) - u_1(a)f(a) + u_2(b)f(b) \\ &\quad - \int_a^t u_1^{\Delta}(s)f(s)\Delta s - \int_t^b u_2^{\Delta}(s)f(s)\Delta s. \end{aligned} \tag{5}$$

Proof. We use Theorem 2.1 to split the integral into two parts, each of which is evaluated by applying the integration of parts formula, i.e.,

$$\begin{aligned}\int_a^b u(\sigma(s))f^\Delta(s)\Delta s &= \int_a^t u_1(\sigma(s))f^\Delta(s)\Delta s + \int_t^b u_2(\sigma(s))f^\Delta(s)\Delta s\\ &= u_1(t)f(t) - u_1(a)f(a) - \int_a^t u_1^\Delta(s)f(s)\Delta s\\ &\quad + u_2(b)f(b) - u_2(t)f(t) - \int_t^b u_2^\Delta(s)f(s)\Delta s,\end{aligned}$$

from which (5) follows.

Theorem 3.2 *Suppose $f \in \mathrm{C}^2_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. Let $t \in [a,b]$ and $u_i, v_i \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ be such that $u_i^\Delta(s) = v_i(\sigma(s))$ for all $s \in [a,b]$, where $i \in \{1,2\}$. If u satisfies (4), then*

$$\begin{aligned}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s &= (u_1(t)-u_2(t))f^\Delta(t) - (v_1(t)-v_2(t))f(t)\\ &\quad - u_1(a)f^\Delta(a) + v_1(a)f(a) + u_2(b)f^\Delta(b) - v_2(b)f(b)\\ &\quad + \int_a^t v_1^\Delta(s)f(s)\Delta s + \int_t^b v_2^\Delta(s)f(s)\Delta s.\end{aligned} \tag{6}$$

Proof. Using (5) with f^Δ replaced by $f^{\Delta\Delta}$ and subsequently applying integration by parts twice, we obtain

$$\begin{aligned}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \int_a^t u_1^\Delta(s)f^\Delta(s)\Delta s - \int_t^b u_2^\Delta(s)f^\Delta(s)\Delta s\\ &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \int_a^t v_1(\sigma(s))f^\Delta(s)\Delta s - \int_t^b v_2(\sigma(s))f^\Delta(s)\Delta s\\ &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \left\{v_1(t)f(t) - v_1(a)f(a) - \int_a^t v_1^\Delta(s)f(s)\Delta s\right\}\\ &\quad - \left\{v_2(b)f(b) - v_2(t)f(t) - \int_t^b v_2^\Delta(s)f(s)\Delta s\right\},\end{aligned}$$

from which (6) follows.

Assumption (H) For the remaining three sections of this paper, we assume that $\mathbb{T}$ is a time scale and that $a, b \in \mathbb{T}$ such that $a < b$. We assume that $f, g, h \in \mathrm{C}^2_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ are such that

$$\left\|f^{\Delta\Delta}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{\Delta\Delta}(t)\right| < \infty, \quad \left\|g^{\Delta\Delta}\right\|_\infty < \infty, \quad \left\|h^{\Delta\Delta}\right\|_\infty < \infty \tag{7}$$

and define

$$\begin{aligned}
A[f,g,h] &:= gh\int_a^b f(s)\Delta s + fh\int_a^b g(s)\Delta s + fg\int_a^b h(s)\Delta s,\\
B[f,g,h] &:= |gh|\left\|f^{\Delta\Delta}\right\|_\infty + |fh|\left\|g^{\Delta\Delta}\right\|_\infty + |fg|\left\|h^{\Delta\Delta}\right\|_\infty,\\
C[f,g,h] &:= ghf^\Delta + fhg^\Delta + fgh^\Delta,\\
D[f,g,h] &:= \left(\int_a^b g(s)h(s)\Delta s\right)\left(\int_a^b f(s)\Delta s\right) + \left(\int_a^b f(s)h(s)\Delta s\right)\left(\int_a^b g(s)\Delta s\right)\\
&\quad + \left(\int_a^b f(s)g(s)\Delta s\right)\left(\int_a^b h(s)\Delta s\right),\\
L[f,g,h] &:= gh\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} + fh\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{(b-a)^2}\\
&\quad + fg\frac{g_2(b,a)h(a)+h_2(b,a)h(b)}{(b-a)^2},\\
M[f,g,h] &:= gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a}.
\end{aligned}$$

4 Time Scales Version of Theorem 1.1

Theorem 4.1 *Assume* (H). *Then, for all $t\in[a,b]$, we have*

$$\left|f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\right| \le \frac{1}{3}\left(h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\right)B[f,g,h](t) \quad (8)$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{1}{3(b-a)^2}D[f,g,h] - \frac{1}{3(b-a)}\int_a^b\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Delta t\right| \le \frac{1}{3(b-a)}\int_a^b\left(h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\right)B[f,g,h](t)\Delta t. \quad (9)$$

Proof. Fix $t\in[a,b]$ and define u by (4), where

$$u_1(s) = g_2(s,a), \quad u_2(s) = h_2(b,s).$$

With the notation as in Theorem 3.2, using Definition 2.2, we have

$$v_1(s) = s-a, \quad v_2(s) = s-b, \quad v_1^\Delta(s) = v_2^\Delta(s) = 1$$

and $u_1(a) = v_1(a) = u_2(b) = v_2(b) = 0$. Moreover, we have

$$u_1(t) - u_2(t) \overset{(1)}{=} (t-b)(b-a) + g_2(b,a), \quad v_1(t) - v_2(t) = b-a.$$

By (6), we therefore obtain

$$\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s = ((t-b)(b-a)+g_2(b,a))f^{\Delta}(t)-(b-a)f(t)+\int_a^b f(s)\Delta s$$

and thus

$$f(t)=\frac{1}{b-a}\int_a^b f(s)\Delta s+\left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\Delta}(t)-\frac{1}{b-a}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s. \quad (10)$$

Similarly, we get

$$g(t)=\frac{1}{b-a}\int_a^b g(s)\Delta s+\left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\Delta}(t)-\frac{1}{b-a}\int_a^b u(\sigma(s))g^{\Delta\Delta}(s)\Delta s \quad (11)$$

and

$$h(t)=\frac{1}{b-a}\int_a^b h(s)\Delta s+\left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\Delta}(t)-\frac{1}{b-a}\int_a^b u(\sigma(s))h^{\Delta\Delta}(s)\Delta s. \quad (12)$$

Multiplying (10), (11) and (12) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$\begin{aligned} f(t)g(t)h(t)-\frac{1}{3(b-a)}A[f,g,h](t)-\frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)& \\ =-\frac{1}{3(b-a)}\int_a^b u(\sigma(s))\tilde{B}[f,g,h](t,s)\Delta s,& \quad (13)\end{aligned}$$

where

$$\begin{cases}\tilde{B}[f,g,h](t,s):=g(t)h(t)f^{\Delta\Delta}(s)+f(t)h(t)g^{\Delta\Delta}(s)+f(t)g(t)h^{\Delta\Delta}(s)\\ \quad\text{so that}\quad \left|\tilde{B}[f,g,h](t,s)\right|\le B[f,g,h](t).\end{cases} \quad (14)$$

By taking absolute values in (13) and using (7) and

$$\begin{aligned}\int_a^b |u(\sigma(s))|\Delta s &= \int_a^t g_2(\sigma(s),a)\Delta s+\int_t^b h_2(b,\sigma(s))\Delta s \quad (15)\\ &= g_3(t,a)-g_3(t,b)\\ &\overset{(3)}{=} g_3(b,a)+(t-b)g_2(b,a)+(b-a)h_2(b,t),\end{aligned}$$

we obtain (8). Integrating (13) with respect to t from a to b, dividing by $b-a$, noting that

$$\int_a^b A[f,g,h](s)\Delta s = D[f,g,h], \quad (16)$$

taking absolute values and using (7) and (15), we obtain (9).

Example 4.1 If we let $\mathbb{T}=\mathbb{R}$ in Theorem 4.1, then, since $C[f,g,h]=(fgh)'$,

$$b-\frac{g_2(b,a)}{b-a}=b-\frac{(b-a)^2}{2(b-a)}=b-\frac{b-a}{2}=\frac{a+b}{2}$$

and

$$\begin{aligned}h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a} &= \frac{1}{2}\left\{(t-b)^2+(t-b)(b-a)+\frac{(b-a)^2}{3}\right\}\\ &= \frac{1}{2}\left\{\left(t-b+\frac{b-a}{2}\right)^2-\frac{(b-a)^2}{4}+\frac{(b-a)^2}{3}\right\}\\ &= \frac{1}{2}\left\{\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{12}\right\},\end{aligned}$$

we obtain [18, Theorem 1], in particular, Theorem 1.1.

Example 4.2 If we let $\mathbb{T}=\mathbb{Z}$ and $a=0$, $b=n\in\mathbb{N}$ in Theorem 4.1, then, since

$$b-\frac{g_2(b,a)}{b-a}=b-\frac{(b-a)(b-a+1)}{2(b-a)}=b-\frac{b-a+1}{2}=\frac{a+b-1}{2}=\frac{n-1}{2}$$

and

$$\begin{aligned}&h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\\ &\quad= \frac{1}{2}\left\{(b-t)(b-t-1)+(t-b)(b-a+1)+\frac{(b-a+1)(b-a+2)}{3}\right\}\\ &\quad= \frac{1}{2}\left\{\left(t-b+\frac{b-a+2}{2}\right)^2-\frac{(b-a+2)^2}{4}+\frac{(b-a+1)(b-a+2)}{3}\right\}\\ &\quad= \frac{1}{2}\left\{\left(t+1-\frac{a+b}{2}\right)^2+\frac{(b-a+2)(b-a-2)}{12}\right\}\\ &\quad= \frac{1}{2}\left\{\left(t+1-\frac{n}{2}\right)^2+\frac{n^2-4}{12}\right\},\end{aligned}$$

we obtain

$$\begin{aligned}&\left|f(t)g(t)h(t)-\frac{1}{3n}A[f,g,h](t)-\frac{1}{3}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right|\\ &\qquad\qquad\le\frac{1}{6}\left\{\left(t+1-\frac{n}{2}\right)^2+\frac{n^2-4}{12}\right\}B[f,g,h](t)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{n}\sum_{t=0}^{n-1}f(t)g(t)h(t)-\frac{1}{3n^2}D[f,g,h]-\frac{1}{3n}\sum_{t=0}^{n-1}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right|\\ &\qquad\qquad\le\frac{1}{6n}\sum_{t=0}^{n-1}\left\{\left(t+1-\frac{n}{2}\right)^2+\frac{n^2-4}{12}\right\}B[f,g,h](t),\end{aligned}$$

where

$$
\begin{aligned}
A[f,g,h] &= gh\sum_{s=0}^{n-1} f(s) + fh\sum_{s=0}^{n-1} g(s) + fg\sum_{s=0}^{n-1} h(s),\\
B[f,g,h] &= |gh| \max_{1\le s\le n-1}\left|\Delta^2 f(s)\right| + |fh| \max_{1\le s\le n-1}\left|\Delta^2 g(s)\right| + |fg| \max_{1\le s\le n-1}\left|\Delta^2 h(s)\right|,\\
C[f,g,h] &= gh\Delta f + fh\Delta g + fg\Delta h,\\
D[f,g,h] &= \left(\sum_{s=0}^{n-1} g(s)h(s)\right)\left(\sum_{s=0}^{n-1} f(s)\right) + \left(\sum_{s=0}^{n-1} f(s)h(s)\right)\left(\sum_{s=0}^{n-1} g(s)\right)\\
&\quad + \left(\sum_{s=0}^{n-1} f(s)g(s)\right)\left(\sum_{s=0}^{n-1} h(s)\right).
\end{aligned}
$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 4.1 If we let $h(t) \equiv 1$ in Theorem 4.1, then (8) becomes

$$
\begin{aligned}
&\Bigg| f(t)g(t) - \frac{1}{2(b-a)}\left\{ g(t)\int_a^b f(s)\Delta s + f(t)\int_a^b g(s)\Delta s \right\}\\
&\qquad -\frac{1}{2}\left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t)\right\}\Bigg|\\
&\quad \le \frac{1}{2}\left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\}
\end{aligned}
$$

and (9) turns into

$$
\begin{aligned}
&\Bigg| \frac{1}{b-a}\int_a^b f(t)g(t)\Delta t - \frac{1}{(b-a)^2}\left(\int_a^b f(t)\Delta t\right)\left(\int_a^b g(t)\Delta t\right)\\
&\qquad -\frac{1}{2(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t)\right\}\Delta t\Bigg|\\
&\quad \le \frac{1}{2(b-a)}\int_a^b \left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\cdot\\
&\qquad\qquad \cdot\left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\}\Delta t.
\end{aligned}
$$

If, moreover, we let $g(t) \equiv 1$, then (8) becomes

$$
\begin{aligned}
&\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\Delta s - \left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\Delta}(t)\right|\\
&\qquad \le \left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\left\|f^{\Delta\Delta}\right\|_\infty.
\end{aligned}
$$

From these inequalities, special cases such as discrete inequalities can be obtained.

5 Time Scales Version of Theorem 1.2

Theorem 5.1 *Assume* (H). *Then, for all* $t \in [a,b]$, *we have*

$$\Bigg| f(t)g(t)h(t) - \frac{2}{3(b-a)}A[f,g,h](t) + \frac{1}{3}L[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Bigg| \le \frac{1}{3(b-a)}B[f,g,h](t)I(t) \quad (17)$$

and

$$\Bigg| \frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{2}{3(b-a)^2}D[f,g,h] + \frac{1}{3(b-a)}\int_a^b L[f,g,h](t)\Delta t - \frac{1}{3(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Delta t\Bigg| \le \frac{1}{3(b-a)^2}\int_a^b B[f,g,h](t)I(t)\Delta t, \quad (18)$$

where

$$I(t) := \frac{1}{b-a}\int_a^t |2(b-a)g_2(\sigma(s),a) - (\sigma(s)-a)g_2(b,a)|\,\Delta s + \frac{1}{b-a}\int_t^b |2(b-a)h_2(b,\sigma(s)) - (b-\sigma(s))h_2(b,a)|\,\Delta s.$$

Proof. Fix $t \in [a,b]$ and define u by (4), where

$$u_1(s) = 2(b-a)g_2(s,a) - (s-a)g_2(b,a), \quad u_2(s) = 2(b-a)h_2(b,s) - (b-s)h_2(b,a).$$

With the notation as in Theorem 3.2, using Definition 2.2, we have

$$v_1(s) = 2(b-a)(s-a) - g_2(b,a), \quad v_2(s) = 2(b-a)(s-b) + h_2(b,a),$$
$$v_1^\Delta(s) = v_2^\Delta(s) = 2(b-a)$$

and $u_1(a) = u_2(b) = 0$, $v_1(a) = -g_2(b,a)$, $v_2(b) = h_2(b,a)$. Moreover, we have

$$\begin{aligned} u_1(t) - u_2(t) &= 2(b-a)(g_2(t,a) - h_2(b,t)) - (t-a)g_2(b,a) + (b-t)h_2(b,a) \\ &\overset{(1),(2)}{=} 2(b-a)(g_2(b,a) + (t-b)(b-a)) \\ &\quad -(t-a)g_2(b,a) + (b-t)\left((b-a)^2 - g_2(b,a)\right) \\ &\overset{(2)}{=} (b-a)g_2(b,a) + (t-b)(b-a)^2, \\ v_1(t) - v_2(t) &= 2(b-a)^2 - g_2(b,a) - h_2(b,a) \\ &\overset{(2)}{=} 2(b-a)^2 - (b-a)^2 = (b-a)^2. \end{aligned}$$

By (6), we therefore obtain

$$\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s = (b-a)\left(g_2(b,a) + (t-b)(b-a)\right)f^\Delta(t) - (b-a)^2 f(t) - g_2(b,a)f(a) - h_2(b,a)f(b) + 2(b-a)\int_a^b f(s)\Delta s$$

and thus

$$f(t)=\frac{2}{b-a}\int_a^b f(s)\Delta s-\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} \\ +\left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\Delta}(t)-\frac{1}{(b-a)^2}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s. \quad (19)$$

Similarly, we get

$$g(t)=\frac{2}{b-a}\int_a^b g(s)\Delta s-\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{(b-a)^2} \\ +\left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\Delta}(t)-\frac{1}{(b-a)^2}\int_a^b u(\sigma(s))g^{\Delta\Delta}(s)\Delta s \quad (20)$$

and

$$h(t)=\frac{2}{b-a}\int_a^b h(s)\Delta s-\frac{g_2(b,a)h(a)+h_2(b,a)h(b)}{(b-a)^2} \\ +\left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\Delta}(t)-\frac{1}{(b-a)^2}\int_a^b u(\sigma(s))h^{\Delta\Delta}(s)\Delta s. \quad (21)$$

Multiplying (19), (20) and (21) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$f(t)g(t)h(t)-\frac{2}{3(b-a)}A[f,g,h](t)+\frac{1}{3}L[f,g,h](t) \\ -\frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)=-\frac{1}{3(b-a)^2}\int_a^b u(\sigma(s))\tilde{B}[f,g,h](t,s)\Delta s \quad (22)$$

with $\tilde{B}$ as in (14). By taking absolute values in (22) and using (7) and

$$\frac{1}{b-a}\int_a^b |u(\sigma(s))|\Delta s=I(t), \quad (23)$$

we obtain (17). Integrating (22) with respect to t from a to b, dividing by $b-a$, noting (16), taking absolute values and using (7) and (23), we obtain (18).

Example 5.1 If we let $\mathbb{T}=\mathbb{R}$ in Theorem 5.1, then, since $C[f,g,h]=(fgh)'$,

$$b-\frac{g_2(b,a)}{b-a}=\frac{a+b}{2}$$

and (with p as defined in Theorem 1.2)

$$\begin{aligned} I(t)&=\frac{1}{b-a}\int_a^t\left|(b-a)(s-a)^2-(s-a)\frac{(b-a)^2}{2}\right|\mathrm{d}s \\ &\quad+\frac{1}{b-a}\int_t^b\left|(b-a)(s-b)^2-(b-s)\frac{(b-a)^2}{2}\right|\mathrm{d}s \\ &=\int_a^t\left|(s-a)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s+\int_t^b\left|(s-b)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s \\ &=\int_a^b\left|p(t,s)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s, \end{aligned}$$

we obtain [18, Theorem 2], in particular, Theorem 1.2.

Example 5.2 If we let $\mathbb{T}=\mathbb{Z}$ and $a=0$, $b=n\in\mathbb{N}$ in Theorem 5.1, then, since

$$b-\frac{g_2(b,a)}{b-a}=\frac{n-1}{2}$$

and

$$\begin{aligned} I(t) &= \frac{1}{b-a}\sum_{s=a}^{t-1}\left|(b-a)(s+1-a)(s+2-a)-(s+1-a)\frac{(b-a)(b-a+1)}{2}\right| \\ &\quad+\frac{1}{b-a}\sum_{s=t}^{b-1}\left|(b-a)(b-s-1)(b-s-2)-(b-s-1)\frac{(b-a)(b-a-1)}{2}\right| \\ &= \sum_{s=a}^{t-1}\left|(s+1-a)\left(s+1-\frac{a+b-1}{2}\right)\right| \\ &\quad+\sum_{s=t}^{b-1}\left|(s+1-b)\left(s+1-\frac{a+b-1}{2}\right)\right| \\ &= \sum_{s=0}^{t-1}\left|(s+1)\left(s+1-\frac{n-1}{2}\right)\right|+\sum_{s=t}^{n-1}\left|(s+1-n)\left(s+1-\frac{n-1}{2}\right)\right|, \end{aligned}$$

we have

$$\begin{aligned} &\left|f(t)g(t)h(t)-\frac{2}{3n}A[f,g,h](t)+\frac{1}{3}L[f,g,h](t)-\frac{1}{3}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad\le\frac{1}{3n}B[f,g,h](t)\left\{\sum_{s=1}^{t}s\left|s-\frac{n-1}{2}\right|+\sum_{s=t+1}^{n}(n-s)\left|s-\frac{n-1}{2}\right|\right\} \end{aligned}$$

and

$$\begin{aligned} &\left|\frac{1}{n}\sum_{t=0}^{n-1}f(t)g(t)h(t)-\frac{2}{3n^2}D[f,g,h]\right. \\ &\qquad\left.+\frac{1}{3n}\sum_{t=0}^{n-1}L[f,g,h](t)-\frac{1}{3n}\sum_{t=0}^{n-1}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad\le\frac{1}{3n^2}\sum_{t=0}^{n-1}B[f,g,h](t)\left\{\sum_{s=1}^{t}s\left|s-\frac{n-1}{2}\right|+\sum_{s=t+1}^{n}(n-s)\left|s-\frac{n-1}{2}\right|\right\}, \end{aligned}$$

where in addition to A, B, C, D defined in Example 4.2,

$$\begin{aligned} L[f,g,h] &= gh\frac{(n+1)f(a)+(n-1)f(b)}{2n}+fh\frac{(n+1)g(a)+(n-1)g(b)}{2n} \\ &\quad+fg\frac{(n+1)h(a)+(n-1)h(b)}{2n}. \end{aligned}$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 5.1 If we let $h(t) \equiv 1$ in Theorem 5.1, then (17) becomes

$$\left| f(t)g(t) - \frac{1}{b-a}\left\{ g(t)\int_a^b f(s)\Delta s + f(t)\int_a^b g(s)\Delta s \right\} + g(t)\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{2(b-a)^2} + f(t)\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{2(b-a)^2} - \frac{1}{2}\left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t)+f(t)g^{\Delta}(t)\right\} \right| \le \frac{1}{2(b-a)}\left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\} I(t),$$

(observe (2) when calculating L) and (18) turns into

$$\left| \frac{1}{b-a}\int_a^b f(t)g(t)\Delta t - \frac{2}{(b-a)^2}\left(\int_a^b f(t)\Delta t\right)\left(\int_a^b g(t)\Delta t\right) + \frac{1}{b-a}\int_a^b \left\{ g(t)\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{2(b-a)^2} + f(t)\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{2(b-a)^2} \right\}\Delta t - \frac{1}{2(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t)+f(t)g^{\Delta}(t)\right\}\Delta t \right| \le \frac{1}{2(b-a)^2}\int_a^b \left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\} I(t)\Delta t.$$

If, moreover, we let $g(t) \equiv 1$, then (17) becomes

$$\left| f(t) - \frac{2}{b-a}\int_a^b f(s)\Delta s + \frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} - \left(t-b+\frac{g_2(b,a)}{b-a}\right) f^{\Delta}(t) \right| \le \frac{1}{b-a}\left\|f^{\Delta\Delta}\right\|_\infty I(t).$$

From these inequalities, special cases such as discrete inequalities can be obtained.

6 Time Scales Version of Theorem 1.3

Theorem 6.1 *Assume* (H). *Then, for all* $t \in [a,b]$, *we have*

$$\left| f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right) M[f,g,h](t) \right| \le \frac{1}{3(b-a)^2} B[f,g,h](t)H(t) \quad (24)$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{1}{3(b-a)^2}D[f,g,h](t)\right.$$
$$\left. -\frac{1}{3(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right)M[f,g,h](t)\Delta t\right| \le \frac{1}{3(b-a)^3}\int_a^b B[f,g,h](t)H(t)\Delta t, \tag{25}$$

where

$$H(t) := \int_a^b \int_a^b |p(t,\tau)p(\tau,s)|\,\Delta s\Delta\tau$$

and

$$p(t,s) := \begin{cases} \sigma(s)-a & \textit{for} \quad a \le s < t, \\ \sigma(s)-b & \textit{for} \quad t \le s \le b. \end{cases}$$

Proof. Fix $t \in [a,b]$. We use Theorem 3.1 three times to obtain

$$\begin{aligned}
&\int_a^b \int_a^b p(t,\tau)p(\tau,s)f^{\Delta\Delta}(s)\Delta s\Delta\tau = \int_a^b p(t,\tau)\left\{\int_a^b p(\tau,s)f^{\Delta\Delta}(s)\Delta s\right\}\Delta\tau \\
&= \int_a^b p(t,\tau)\left\{(b-a)f^{\Delta}(\tau) - \int_a^b f^{\Delta}(s)\Delta s\right\}\Delta\tau \\
&= (b-a)\int_a^b p(t,s)f^{\Delta}(s)\Delta s + (f(a)-f(b))\int_a^b p(t,s)\Delta s \\
&= (b-a)\left\{(b-a)f(t) - \int_a^b f(s)\Delta s\right\} + (f(a)-f(b))\left\{(b-a)t - \int_a^b s\Delta s\right\} \\
&= (b-a)^2 f(t) - (b-a)\int_a^b f(s)\Delta s + (f(a)-f(b))\int_b^a (s-t)\Delta s \\
&= (b-a)^2 f(t) - (b-a)\int_a^b f(s)\Delta s + (g_2(t,a)-h_2(b,t))(f(a)-f(b))
\end{aligned}$$

and thus (by using (1))

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{f(b)-f(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)f^{\Delta\Delta}(s)\Delta s\Delta\tau. \tag{26}$$

Similarly, we get

$$g(t) = \frac{1}{b-a}\int_a^b g(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{g(b)-g(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)g^{\Delta\Delta}(s)\Delta s\Delta\tau. \tag{27}$$

and

$$h(t) = \frac{1}{b-a}\int_a^b h(s)\Delta s + \left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{h(b)-h(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)h^{\Delta\Delta}(s)\Delta s\Delta\tau. \quad (28)$$

Multiplying (26), (27) and (28) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - b + \frac{g_2(b,a)}{b-a}\right)M[f,g,h](t) = \frac{1}{3(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)\tilde{B}[f,g,h](t,s)\Delta s\Delta\tau \quad (29)$$

with $\tilde{B}$ as in (14). By taking absolute values in (29) and using (7) and the definition of H, we obtain (24). Integrating (29) with respect to t from a to b, dividing by $b-a$, noting (16), taking absolute values and using (7) and the definition of H, we obtain (25).

Example 6.1 If we let $\mathbb{T} = \mathbb{R}$ in Theorem 6.1, then, by the same calculations as in Example 4.1, we obtain [18, Theorem 3], in particular, Theorem 1.3.

Example 6.2 If we let $\mathbb{T} = \mathbb{Z}$ and $a = 0$, $b = n \in \mathbb{N}$ in Theorem 6.1, then, by the same calculations as in Example 4.2, we obtain

$$\left|f(t)g(t)h(t) - \frac{1}{3n}A[f,g,h](t) - \frac{1}{3}\left(t - \frac{n-1}{2}\right)M[f,g,h](t)\right| \le \frac{1}{3n^2}B[f,g,h](t)H(t)$$

and

$$\left|\frac{1}{n}\sum_{t=0}^{n-1} f(t)g(t)h(t) - \frac{1}{3n^2}D[f,g,h] - \frac{1}{3n}\sum_{t=0}^{n-1}\left(t - \frac{n-1}{2}\right)M[f,g,h](t)\right| \le \frac{1}{3n^3}\sum_{t=0}^{n-1} B[f,g,h](t)H(t),$$

where in addition to A, B, D defined in Example 4.2,

$$M[f,g,h] = gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a},$$

$$H(t) = \sum_{\tau=0}^{n-1}\sum_{s=0}^{n-1}|p(t,\tau)p(\tau,s)|,$$

$$p(t,s) = \begin{cases} s+1, & \text{if } 0 \le s < t, \\ s+1-n, & \text{if } t \le s \le n. \end{cases}$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 6.1 If we let $h(t) \equiv 1$ in Theorem 6.1, then (24) becomes

$$\left| f(t)g(t) - \frac{1}{2(b-a)} \left\{ g(t) \int_a^b f(s)\Delta s + f(t) \int_a^b g(s)\Delta s \right\} \right.$$
$$\left. -\frac{1}{2}\left(t - b + \frac{g_2(b,a)}{(b-a)}\right) \left\{ g(t)\frac{f(b)-f(a)}{b-a} + f(t)\frac{g(b)-g(a)}{b-a} \right\} \right|$$
$$\leq \frac{1}{2(b-a)^2} \left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} H(t)$$

and (25) turns into

$$\left| \frac{1}{b-a} \int_a^b f(t)g(t)\Delta t - \frac{1}{(b-a)^2} \left(\int_a^b f(t)\Delta t \right) \left(\int_a^b g(t)\Delta t \right) \right.$$
$$\left. -\frac{1}{2(b-a)} \int_a^b \left(t - b + \frac{g_2(b,a)}{b-a}\right) \left\{ g(t)\frac{f(b)-f(a)}{b-a} + f(t)\frac{g(b)-g(a)}{b-a} \right\} \Delta t \right|$$
$$\leq \frac{1}{2(b-a)^3} \int_a^b \left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} H(t)\Delta t.$$

If, moreover, we let $g(t) \equiv 1$, then (24) becomes

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(s)\Delta s - \left(t - b + \frac{g_2(b,a)}{(b-a)}\right) \frac{f(b)-f(a)}{b-a} \right|$$
$$\leq \frac{1}{(b-a)^2} \left\| f^{\Delta\Delta} \right\|_\infty H(t).$$

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