Nonlinear Dynamics and Systems Theory, **12** (1) (2012) 97–103

Existence of Positive Solutions for the p-Laplacian with Nonlinear Boundary Conditions

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Received: April 24, 2011;   Revised: December 21, 2011

Abstract: In this paper, we consider a class of nonlinear elliptic problem with nonlinear boundary condition. The existence of positive solutions are established by sub-supersolution method and the Mountain Pass Lemma.

Keywords: *p-Laplacian equations; sub-supersolution; Mountain Pass Lemma; nonlinear boundary condition; positive solutions.*

Mathematics Subject Classification (2010): 35J55.

1 Introduction

In this paper, we are concerned with the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(x,u), & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$ and $\frac{\partial}{\partial \nu}$ is the out normal derivative.

Recently, Afrouzi and Alizadeh [1] considered p-Laplacian equations with a nonlinear boundary condition, they developed a quasilinearization method in order to construct an iterative scheme that converges to a solution. They extended the results of [2] with $p \neq 2$. When $p = 2$, Song, Wang and Zhao [3] considered problem (1). By the sub-supersolution method, the existence of a positive solution was established. In [4], they

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presented necessary and sufficient conditions of existence for positive solutions of the system with p-Laplacian. For other nonlinear boundary conditions problems, we cite [5–7]. In [8–11], they offered some applications in physics and engineering.

In this paper, we consider a class of nonlinear elliptic problems with nonlinear boundary condition (1). The existence of positive solutions are established by sub-supersolution method and the Mountain Pass Lemma.

The precise assumptions on the source terms f and g are as follows:

(C_1) For all $s \geq 0$, there exist some nonnegative constants A_1, A_2, B_1 and B_2 such that

$$0 \leq f(x,s) \leq A_1 s^{q_1-1} + A_2, \quad \text{a.e. in}\, \Omega,$$
$$0 \leq g(x,s) \leq B_1 s^{q_2-1} + B_2, \quad \text{a.e. on}\, \partial\Omega,$$

where $2 < p < q_1 < 2^* := \frac{2N}{N-2}$ and $2 < p < q_2 < \overline{2^*} = \frac{2(N-1)}{N-2}$;

(C_2) The function $x \mapsto f(x,0) + g(x,0)$ is not identically zero;

(C_3) For all $s \in \mathbb{R}$, the functions $f(\cdot,s), g(\cdot,s) : \overline{\Omega} \to \mathbb{R}$ are continuous and for every $x \in \overline{\Omega}$, the functions $f(x,\cdot), g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are local Lipschitz continuous.

2 Preliminary Lemmas

Let $W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$ with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}},$$

then $W^{1,p}(\Omega)$ is a Banach space.

Now, we definite the concepts of sub-solution and super-solution. We say that $u \in W^{1,p}(\Omega)$ is a weak sub-solution (weak super-solution) of problem (1) if it satisfies

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv dx \leq (\geq) \int_\Omega f(x,u) v dx, \\ \displaystyle\int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} v \leq (\geq) \int_{\partial\Omega} g(x,u) v d\sigma, \end{cases}$$

for all $v \in W^{1,p}(\Omega)$ with $v \geq 0$.

We give the following lemmas which are similar to [1], so we omit the proof here.

Lemma 2.1 *Assume that $\lambda > 0, \mu > 0$ and $u \in W^{2,p}(\Omega)$ satisfies*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u \geq 0, & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \mu |u|^{p-2}u \geq 0, & \text{on } \partial\Omega. \end{cases}$$

Then $u \geq 0$.

Lemma 2.2 *Assume that $\xi \in L^p(\Omega)$ and $\zeta \in L^p(\partial\Omega)$. Then, for any $\lambda, \mu > 0$ the Robin problem:*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = \xi, & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \mu |u|^{p-2}u = \zeta, & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in W^{2,p}(\Omega)$.

Lemma 2.3 *Let $\lambda, \mu > 0$, $\xi \in L^p(\Omega)$ and $\zeta \in L^p(\partial\Omega)$. Then, there exists a constant C such that if u is a weak solution of*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = \xi, & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \mu |u|^{p-2}u = \zeta, & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|u\|_{W^{1,p}(\Omega)} \leq C \left[\|\xi\|_{L^p(\Omega)} + \|\zeta\|_{L^p(\partial\Omega)} \right].$$

Remark 2.1 By the compactness of the imbedding $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and the result of Lemma 2.3, we know that the operator $T : L^p(\Omega) \times L^p(\partial\Omega) \to W^{1,p}(\Omega)$ given by $F(\xi, \zeta) = u$ is compact.

In order to obtain the super-solution of problem (1), we use the following Mountain Pass Lemma.

Lemma 2.4 [12] *Let X be a Banach space and let $I \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. If the following conditions hold:*

(I) $I(0) = 0$*;*

(II) *there exist constants $r, a > 0$ such that $I(u) \geq a$, if $\|u\| = r$;*

(III) *there exists an element $\theta \in X$ with $\|\theta\| > r$, $I(\theta) \leq 0$.*

Define $\Gamma := \{g \in C([0,1], X); g(0) = 0, g(1) = \theta\}$. Then

$$c := \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I[g(t)]$$

is a critical value of I.

3 Main Results

Our main results are as follows:

Theorem 3.1 *Let conditions (C_1)-(C_3) be satisfied. Then problem (1) has one positive solution u for A_2 and B_2 small enough.*

Proof Firstly, from condition (C_1), we know that 0 is a subsolution of problem (1), and 0 is not a solution of problem (1) by condition (C_2). In order to use sub-supersolution method, we need a positive supersolution which comes from the Mountain Pass Lemma. Now, we consider the following problem:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = A_1 u^{q_1 - 1} + A_2, & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = B_1 u^{q_2 - 1} + B_2, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

the functional associated with the problem (2) is

$$J(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx - \frac{A_1}{q_1}\int_\Omega u^{q_1}dx - A_2\int_\Omega u dx - \frac{B_1}{q_2}\int_{\partial\Omega} u^{q_2}d\sigma - B_2\int_{\partial\Omega} u d\sigma.$$

We claim that J satisfies the $(PS)_c$ condition. In fact, let $\{u_n\}$ be a Palais-Smale sequence in $W^{1,p}(\Omega)$, that is $J(u_n) \to c$ and $J'(u_n) \to 0$, then we have

$$\begin{aligned} J(u_n) &- \frac{1}{q}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{W^{1,p}(\Omega)} - \left(\frac{1}{q_1}-\frac{1}{q}\right)A_1\int_\Omega u_n^{q_1}dx - \left(1-\frac{1}{q}\right)A_2\int_\Omega u_n dx \\ &\quad - \left(\frac{1}{q_2}-\frac{1}{q}\right)B_1\int_{\partial\Omega} u_n^{q_2}d\sigma - \left(1-\frac{1}{q}\right)B_2\int_{\partial\Omega} u_n d\sigma \\ &= c+o(1), \end{aligned}$$

where $q := \min\{q_1, q_2\}$, $A_1, A_2, B_1, B_2 > 0$. By the Sobolev embedding theorem and Sobolev trace embedding theorem, we can choose a constant $\tau > 0$ such that

$$c + 1 + \tau\|u_n\|_{W^{1,p}(\Omega)} \geq \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p_{W^{1,p}(\Omega)}.$$

Hence $\{u_n\}$ is bounded in $W^{1.p}(\Omega)$. So $\{u_n\}$ admits a weakly convergent subsequence. Since all the growths in problem (2) are subcritical, by the standard argument we deduce that $\{u_n\}$ admits a strongly convergence subsequence.

Next, we verify the conditions of Mountain Pass Lemma. By the Hölder's inequality, the Sobolev embedding theorem and Sobolev trace embedding theorem, we have

$$\int_\Omega |u|^{q_1}dx = \|u\|^{q_1}_{L^{q_1}(\Omega)} \leq C_1\|u\|^{q_1}_{W^{1,p}(\Omega)}, \quad \int_{\partial\Omega}|u|^{q_2}d\sigma = \|u\|^{q_2}_{L^{q_2}(\partial\Omega)} \leq C_2\|u\|^{q_2}_{W^{1,p}(\Omega)},$$
$$\int_\Omega |u|dx \leq C_3\|u\|_{W^{1,p}(\Omega)}, \quad \int_{\partial\Omega}|u|d\sigma \leq C_4\|u\|_{W^{1,p}(\Omega)}.$$

Therefore, we have

$$\begin{aligned} J(u) &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - C_1\|u\|^{q_1}_{W^{1,p}(\Omega)} - C_2\|u\|^{q_2}_{W^{1,p}(\Omega)} \\ &\quad - C_3A_2\|u\|_{W^{1,p}(\Omega)} - C_4B_2\|u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Assume that $\|u\|_{W^{1,p}(\Omega)} < 1$, then we have

$$J(u) \geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - C_5\|u\|^q_{W^{1,p}(\Omega)} - C_3A_2\|u\|_{W^{1,p}(\Omega)} - C_4B_2\|u\|_{W^{1,p}(\Omega)}.$$

Consider the function $g(s) := \frac{1}{p}s^p - C_5s^q - C_6\rho s$, if we take $s = s_0 = (2pC_6\rho)^{\frac{1}{p-1}}$ such that $g(s_0) = a = C_7\rho^{\frac{p}{p-1}} - C_8\rho^{\frac{q}{p-1}} > 0$, since $\frac{q}{p-1} > \frac{p}{p-1} > 1$, ρ is small enough. This fact implies that $J(u) \geq a > 0$ for all $\|u\|_{W^{1,p}(\Omega)} = s_0$ and A_2, B_2 small enough.

Let $\psi \in C_0^\infty(\Omega)$ with $\psi > 0$ on Ω. Then for any $t \geq 0$, we have

$$\begin{aligned} J(t\psi) &= \frac{t^p}{p}\int_\Omega(|\nabla\psi|^p + |\psi|^p)dx - \frac{A_1t^{q_1}}{q_1}\int_\Omega\psi^{q_1}dx - A_2t\int_\Omega\psi dx \\ &\quad - \frac{B_1t^{q_2}}{q_2}\int_{\partial\Omega}\psi^{q_2}d\sigma - B_2t\int_{\partial\Omega}\psi d\sigma \to -\infty \quad \text{as} \quad t \to +\infty, \end{aligned}$$

since $p < p_1, p_2$. Then we take $\psi_0 = k\psi$, with k large enough, we have $\|\psi_0\|_{W^{1,p}(\Omega)} > s_0$ and $J(\psi_0) < a$. Thus we have a solution $\beta(x)$ of the problem (1) by the Mountain Pass

Lemma. It is easy to see by using standard elliptic regularity that $\beta(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, and $\beta(x)$ is a positive supersolution of problem (1) by condition (C_1).

Denote $N := \max_{x\in\overline{\Omega}} \beta(x)$, by condition (C_3), there exists a constant $\lambda > 0$ such that $|f(x,s_1) - f(x,s_2)| \leq \lambda |s_1 - s_2|$, for all $(x,s_1),(x,s_2) \in \overline{\Omega} \times [0,N]$. So $f(x,s) + \lambda s$ is increasing on $s \in [0,N]$. We choose μ in the same way, and define the function $Q : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$Q(x,u) = \begin{cases} 0, & \text{if } u < 0, \\ u, & \text{if } 0 \leq u \leq \beta(x), \\ \beta(x), & \text{if } u > \beta(x). \end{cases}$$

Consider the compact operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ given by $Tv = u$, where u is the unique solution of the Robin problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda |u|^{p-2}u = f(x, Q(x,v)) + \lambda Q(x,v), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \mu |u|^{p-2}u = g(x, Q(x,v)) + \mu Q(x,v), & \text{on } \partial\Omega. \end{cases}$$

Let $v \leq u$, since $f(x,s) + \lambda s$ is increasing on $s \in [0,N]$, so we have

$$\begin{aligned} & -\Delta_p(Tu) + |Tu|^{p-2}(Tu) + \lambda |Tu|^{p-2}(Tu) \\ = \; & f(x, Q(x,u)) + \lambda Q(x,u) \geq f(x, Q(x,v)) + \lambda Q(x,v) \\ = \; & -\Delta_p(Tv) + |Tv|^{p-2}(Tv) + \lambda |Tv|^{p-2}(Tv), \quad \text{in } \Omega. \end{aligned}$$

On the other hand, by nonlinear boundary condition, we have

$$\begin{aligned} & |\nabla (Tu)|^{p-2}\frac{\partial (Tu)}{\partial \nu} + \mu |(Tu)|^{p-2}(Tu) \\ = \; & g(x, Q(x,u)) + \mu Q(x,u) \geq g(x, Q(x,v)) + \mu Q(x,v) \\ = \; & |\nabla (Tv)|^{p-2}\frac{\partial (Tv)}{\partial \nu} + \mu |(Tv)|^{p-2}(Tv), \quad \text{on } \partial\Omega. \end{aligned}$$

From the maximum principle, it follows that $Tu \geq Tv$. This fact implies that T is increasing.

We claim that $T : \langle 0, \beta(x) \rangle \to \langle 0, \beta(x) \rangle$, where $\langle 0, \beta(x) \rangle = \{u \in C(\overline{\Omega}) : 0 \leq u(x) \leq \beta(x)\}$, $\beta(x)$ is the supersolution of problem (1). In fact, from the definition of supersolution, we have

$$\begin{aligned} & -\Delta_p \beta + |\beta|^{p-2}\beta + \lambda |\beta|^{p-2}\beta \\ \geq \; & f(x,\beta) + \lambda Q(x,\beta) \geq f(x, Q(x,\beta)) + \lambda Q(x,\beta) \\ = \; & -\Delta_p(T\beta) + |Tv|^{p-2}(T\beta) + \lambda |T\beta|^{p-2}(T\beta), \quad \text{in } \Omega. \end{aligned}$$

In a similar way, we have

$$|\nabla \beta|^{p-2}\frac{\partial \beta}{\partial \nu} + \mu |\beta|^{p-2}\beta \geq |\nabla (T\beta)|^{p-2}\frac{\partial (T\beta)}{\partial \nu} + \mu |(T\beta)|^{p-2}(T\beta), \quad \text{on } \partial\Omega.$$

From the maximum principle, we have $T\beta \leq \beta$. So $T : \langle 0, \beta(x) \rangle \to \langle 0, \beta(x) \rangle$. Notice that the positive cone K of $C(\overline{\Omega})$ is regular and the interior of K is not empty, therefore T has a fixed point u satisfying $0 \leq u \leq \beta(x)$ and hence u is a positive solution of problem (1).

Theorem 3.2 *Assume that $f(x,s), g(x,s)$ are nonnegative continuous functions in $\overline{\Omega} \times \mathbb{R}$. Let condition (C_2) hold and problem (1) have a continuous weak supersolution. Then problem (1) has a positive solution.*

Proof Firstly, we know that 0 is a subsolution of problem (1), let $\beta(x)$ be a supersolution of problem (1). For a variational approach, the functional associated with problem (1) is

$$J(u) = \frac{1}{p}\int_{\Omega}(|\nabla u|^p + |u|^p)dx - \int_{\Omega} F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

where $F(x,u) = \int_0^u f(x,z)dz$, $G(x,u) = \int_0^u g(x,z)d\sigma$ and $d\sigma$ is the surface measure.

Let $w \in W^{1,p}(\Omega)$ and define the function $Q : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$Q(x,w) = \begin{cases} 0, & \text{if } w < 0, \\ w, & \text{if } 0 \le w \le \beta(x), \\ \beta(x), & \text{if } w > \beta(x). \end{cases} \tag{3}$$

Now we consider

$$\begin{aligned} I(w) &= \frac{1}{p}\int_{\Omega}(|\nabla w|^p + |w|^p)dx - \int_{\Omega} F(x,Q(x,w(x)))dx - \int_{\partial\Omega} G(x,Q(x,w(x)))d\sigma \\ &= \frac{1}{p}\|w\|_{W^{1,p}(\Omega)} - \left(\int_{\Omega} F(x,Q(x,w(x)))dx + \int_{\partial\Omega} G(x,Q(x,w(x)))d\sigma\right) \\ &= I_1(w) - I_2(w). \end{aligned}$$

We note that $I_1(w)$ is weakly lower semi-continuous. In the following we prove that $I_2(w)$ is weakly continuous. Let $H(w) := \int_{\Omega} F(x,Q(x,w(x)))dx$ and $w_n \rightharpoonup w$ in $W^{1,p}(\Omega)$, then we have $w_n \to w$ a.e. in Ω and $Q(x,w_n(x)) \to Q(x,w(x))$. Since

$$|F(x,Q(x,w_n(x)))| \le \sup_{0 \le w(x) \le \beta(x)} |F(x,w(x))| = N.$$

So, by the Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} H(w_n) = \lim_{n\to\infty}\int_{\Omega} F(x,Q(x,w_n(x)))dx = \int_{\Omega}\lim_{n\to\infty} F(x,Q(x,w_n(x)))dx = H(w),$$

so $I_2(w)$ is weakly continuous. Thus $I(w)$ is weakly lower semi-continuous. Since $f(x,s), g(x,s)$ are continuous and $\beta(x)$ is bounded in $\overline{\Omega}$, we know that $H(w)$ is bounded and we have that $I(w) \to +\infty$ as $\|w\|_{W^{1,p}(\Omega)} \to \infty$, this implies that $I(w)$ is a coercive functional, therefore there exists $w_0 \in W^{1,p}(\Omega)$ such that $I'(w_0) = 0$. By (3), we have $0 \le w_0 \le \beta(x)$. Thus $I'(w_0) = 0$. Notice that 0 is not a solution of problem (1), so w_0 is a positive solutions of problem (1).

For the special case of problem (1):

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = A_1 u^{q_1 - 1} + A_2, & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

we can also obtain the nonexistence results.

Theorem 3.3 *There exists a positive constant $D = D(A_1, A_2, q_1)$ such that the problem (4) has no positive solution for all $A_2 > D$.*

Proof Let $A := \{A_2 > 0 : \text{the problem (4) has a positive solution}\}$. Theorem 3.1 implies that $A \neq \emptyset$. So we can define $D := \sup A$. We claim that $0 < D < +\infty$. Obviously $D > 0$. Let

$$A^* = \max_{s>0}\{s^{p-1} - A_1 s^{q_1-1}\} < +\infty. \tag{5}$$

If $A_2 \in A$, then we have

$$\int_\Omega u^{p-1}dx = A_1 \int_\Omega u^{q_1-1}dx + A_2|\Omega|.$$

From (5), we have $A_2 \leq A^*$. So $0 < D \leq A^* < +\infty$.

Acknowledgment

The authors are supported by Research Fundation during the 12th Five-Year Plan Period of Department of Education of Jilin Province, China (Grant [2011] No. 196), Natural Science Foundation of Changchun Normal University.

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