$\mathcal{H}_\infty$ Filtering for Discrete-time Nonlinear Singularly-Perturbed Systems

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Abstract: In this paper, we consider the $\mathcal{H}_\infty$ filtering problem for discrete-time singularly-perturbed (two time-scale) nonlinear systems. Two types of filters, namely, (i) decomposition; and (ii) aggregate, are discussed, and sufficient conditions for the approximate solvability of the problem in terms of discrete-time Hamilton–Jacobi–Isaacs equations (DHJIEs) are presented. In addition, for each type of filter above, reduced-order filters are also derived in each case. The results are also specialized to linear systems, in which case the HJIEs reduce to a system of linear-matrix-inequalities (LMIs) which are computationally efficient. An example is also given to demonstrate the approach.

Keywords: *discrete-time nonlinear filtering; $\mathcal{H}_\infty$-norm; discrete-time singularly-perturbed nonlinear system; decomposition filters; aggregate filters; discrete-time Hamilton–Jacobi–Issacs equations (DHJIEs).*

Mathematics Subject Classification (2010): 93C10, 93E10, 93E11, 93B36.

1 Introduction

The optimal control problem for linear and nonlinear discrete-time singularly-perturbed systems has been considered by several authors [8–10], [16, 18]. On the other hand, the filtering problem for linear singularly-perturbed systems has received little attention [5, 18, 22]. Kalman filtering techniques have generally been considered, and various types of filters have been proposed, including composite and reduced-order filters. However, to the best of our knowledge, the nonlinear filtering problem and in particular the problem for affine nonlinear singularly-perturbed systems has not been considered by any authors.

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Moreover, recently the authors have discussed the Kalman filtering problem for this class of systems and it is therefore our aim in this paper to discuss the nonlinear $\mathcal{H}_\infty$ filtering problem for discrete-time singularly-perturbed systems.

Singularly-perturbed systems are those class of systems that are characterized by a discontinuous dependence of the system properties on a small perturbation parameter ϵ. They arise in many physical systems such as electrical power systems and electrical machines (eg. an asynchronous generator, a dc motor, electrical converters), electronic systems (e.g. oscillators) mechanical systems (eg. fighter aircrafts), biological systems (eg. bacterial-yeast-virus cultures, heart) and also economic systems with various competing sectors. This class of systems has two time-scales, namely, a "fast" and a "slow" dynamics. This makes their analysis and control more complicated than regular systems. Nevertheless, they have been studied extensively [15, 17].

Furthermore, statistical discrete-time nonlinear filtering techniques developed using minimum-variance, Bayesian and maximum-likelihood criteria [6, 19, 21] are too complicated, and approximations [14, 20] are still computationally intensive to implement. On the other hand, the nonlinear $\mathcal{H}_\infty$ filter is easy to derive using a deterministic approach and relies on finding a smooth solution to a discrete-time Hamilton–Jacobi–Isaac's (DHJI) partial-differential-equation (PDE) or DHJIE in short, which can be found using polynomial approximations or other methods. Therefore, $\mathcal{H}_\infty$ filtering techniques for nonlinear discrete-time systems have been considered by several authors [24–26] including the authors [2, 3]. As is well-known, the $\mathcal{H}_\infty$ filter has several advantages over the extended-Kalman filter [4], among which are robustness towards $\mathcal{L}_2$-bounded disturbances and uncertainties, as well as the fact that it is derived from a completely deterministic setting.

A solution to the discrete-time (sub-optimal) nonlinear $\mathcal{H}_\infty$ filtering problem is given in [24] under the simplifying assumption that the solution to the DHJIE is quadratic in the estimation error. This approach is very useful for practical applications, but a complete solution to the problem is also desirable in its own right. Hence recently, the authors have presented exact and approximate solutions to the problem [2, 3]. Moreover, the authors have proposed two-degree-of-freedom (2-DOF) proportional-derivative (PD) and proportional-integral (PI)-filters, and the advantages of these approaches over the 1-DOF filters have also been demonstrated. Thus, in this paper, we extend some of these results to discrete-time singularly-perturbed nonlinear systems which hitherto have not been considered by any authors.

In this paper, we propose to discuss the $\mathcal{H}_\infty$ filtering problem for discrete-time affine nonlinear singularly-perturbed systems. Two types of filters, namely, (i) decomposition, and (ii) aggregate filters will be considered, and sufficient conditions for the solvability of the problem in terms of Hamilton–Jacobi–Isaacs equations (HJIEs) will be presented. The paper is organized as follows. In the remainder of this section, we introduce notations. Then in Section 2, we define the problem and give also some other preliminary definitions. In Section 3, we present a solution to the filtering problem using decomposition filters. This is followed in Section 4 by an alternative solution using aggregate filters. An example is then presented in Section 5, and finally in Section 6, we give conclusions.

The notation is standard, except where otherwise stated. Moreover, $\|(.)\|$ will denote the standard Euclidean vector norm on $\Re^n$, the spaces $\ell_2([k_0, \infty), \Re^n)$ $\ell_\infty([k_0, \infty), \Re^n)$ are the time-domain standard Lebesgue spaces of square-summable and essentially bounded vector-valued sequences. While $\mathcal{H}_\infty(j\Re)$ is the corresponding frequency-domain subspace of the counterpart frenquency-domain space of $\ell_\infty([k_0, \infty), \Re^n)$ of vector functions that

are analytic on the open right-hand complex plane $\mathbf{C}_+$. We shall only use this notation to refer to stable input-output maps and when there is no confusion. The norm on the above ℓ_2, and ℓ_∞-spaces are defined accordingly as $\|(.)\|_2^2 \triangleq \sum_{k_0}^\infty \|(.)\|^2$, $\|(.)\|_\infty \triangleq \sup_k \|(.)\|$. Other notations will be defined accordingly.

2   Problem Definition and Preliminaries

The general set-up for studying $\mathcal{H}_\infty$ filtering problems is shown in Figure 1, where $\mathbf{P}_k$ is the plant, while $\mathbf{F}_k$ is the filter. The noise signal $w \in \mathcal{P}'$ is in general a bounded power signal (e.g. a Gaussian white-noise signal) which belongs to the set $\mathcal{P}'$ of bounded spectral signals, and similarly $\tilde{z} \in \mathcal{P}'$, is also a bounded power signal or ℓ_2 signal. Thus, the induced norm from w to $\tilde{z}$ (the penalty variable to be defined later) is the ℓ_∞-norm of the interconnected system $\mathbf{F}_k \circ \mathbf{P}_k$, i.e.

$$\|\mathbf{F}_k \circ \mathbf{P}_k\|_{\ell_\infty} \triangleq \sup_{0 \neq w \in \mathcal{S}'} \frac{\|\tilde{z}\|_{\mathcal{P}'}}{\|w\|_{\mathcal{P}'}}, \tag{1}$$

where

$$\mathcal{P}' \triangleq \{w : w \in \ell_\infty, R_{ww}(k), S_{ww}(j\omega) \text{ exist for all } k \text{ and all } \omega \text{ resp.}, \|w\|_{\mathcal{P}'} < \infty\},$$

$$\|z\|_{\mathcal{P}'}^2 \triangleq \lim_{K \to \infty} \frac{1}{2K} \sum_{k=-K}^K \|z_k\|^2$$

and $R_{ww}, S_{ww}(j\omega)$ are the autocorrelation and power spectral density matrices of w. Notice also that, $\|(.)\|_{\mathcal{P}'}$ is a seminorm. In addition, if the plant is stable, we replace the induced ℓ_∞-norm above by the equivalent $\mathcal{H}_\infty$ subspace norms.

At the outset, we consider the following singularly-perturbed affine nonlinear causal discrete-time state-space model of the plant which is defined on $\mathcal{X} \subseteq \Re^{n_1+n_2}$ with zero control input:

$$\mathbf{P}_{sp}^{da} : \begin{cases} x_{1,k+1} &= f_1(x_{1,k}, x_{2,k}) + g_{11}(x_{1,k}, x_{2,k})w_k; \ x_1(k_0, \varepsilon) = x^{10}, \\ \varepsilon x_{2,k+1} &= f_2(x_{1,k}, x_{2,k}, \varepsilon) + g_{21}(x_{1,k}, x_{2,k})w_k; \ x_2(k_0, \varepsilon) = x^{20}, \\ y_k &= h_{21}(x_{1,k}) + h_{22}(x_{2,k}) + k_{21}(x_{1,k}, x_{2,k})w_k, \end{cases} \tag{2}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X}$ is the state vector with x_1 the slow state which is n_1-dimensional and x_2 the fast, which is n_2-dimensional; $w \in \mathcal{W} \subseteq \Re^r$ is an unknown disturbance (or noise) signal, which belongs to the set $\mathcal{W} \subset \ell_2[k_0, \infty) \subset \mathcal{P}'$ of admissible exogenous inputs; $y \in \mathcal{Y} \subset \Re^m$ is the measured output (or observation) of the system, and belongs to $\mathcal{Y}$, the set of admissible measured-outputs; while ε is a small perturbation parameter.

The functions $f_1 : \mathcal{X} \to \Re^{n_1}$, $\mathcal{X} \subset \Re^{n_1+n_2}$, $f_2 : \mathcal{X} \times \Re \to \Re^{n_2}$, $g_{11} : \mathcal{X} \to \mathcal{M}^{n_1 \times r}(\mathcal{X})$, $g_{21} : \mathcal{X} \to \mathcal{M}^{n_2 \times r}(\mathcal{X})$, where $\mathcal{M}^{i \times j}$ is the ring of $i \times j$ smooth matrices over $\mathcal{X}$, $h_{21}, h_{22} : \mathcal{X} \to \Re^m$, and $k_{21} : \mathcal{X} \to \mathcal{M}^{m \times r}(\mathcal{X})$ are real C^∞ functions of x. More specifically, f_2 is of the form $f_2(x_{1,k}, x_{2,k}, \varepsilon) = (\varepsilon x_{2,k} + \bar{f}_2(x_{1,k}, x_{2,k})$ for some smooth function $\bar{f}_2 : \mathcal{X} \to \Re^{n_2}$. Furthermore, we assume without any loss of generality that the system (2) has an isolated equilibrium-point at $(x_1^T, x_2^T) = (0, 0)$ such that $f_1(0, 0) = 0$, $f_2(0, 0) = 0$, $h_{21}(0, 0) = h_{22}(0, 0) = 0$. We also assume that there exists a unique solution $x(k, k_0, x_0, w, \varepsilon) \ \forall k \in \mathbf{Z}$ for the system, for all initial conditions $x(k_0) \triangleq x^0 = (x^{10^T}, x^{20^T})^T$, for all $w \in \mathcal{W}$, and all $\varepsilon \in \Re$.

The suboptimal $\mathcal{H}_\infty$ local filtering/state estimation problem is defined as follows.
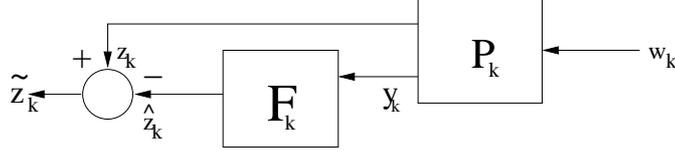
Figure 1: Set-up for discrete-time $\mathcal{H}_\infty$ filtering.

Definition 2.1 (Sub-optimal $\mathcal{H}_\infty$ Local State Estimation (Filtering) Problem). Find a filter, $\mathbf{F}_k$, for estimating the state x_k or a function of it, $z_k = h_1(x_k)$, from observations $\mathsf{Y}_k \overset{\Delta}{=} \{y_i : i \leq k\}$ of y_i up to time k, to obtain the estimate

$$\hat{x}_k = \mathbf{F}_k(\mathsf{Y}_k),$$

such that, the $\mathcal{H}_\infty$-norm from the input $w \in \mathcal{W}$ to some suitable penalty function z is locally rendered less than or equal to a given number γ for all initial conditions $x^0 \in \mathcal{O} \subset \mathcal{X}$, for all $w \in \mathcal{W}$. Moreover, if the filter solves the problem for all $x^0 \in \mathcal{X}$, we say the problem is solved globally.

In the above definition, the condition that the $\mathcal{H}_\infty$-norm is less than or equal to γ, is more correctly referred to as the ℓ_2-gain condition

$$\sum_{k_0}^{\infty} \|z_k\|^2 \leq \gamma^2 \sum_{k_0}^{\infty} \|w_k\|^2, \quad x^0 \in \mathcal{O} \subset \mathcal{X}, \ \forall w \in \mathcal{W}. \tag{3}$$

We shall adopt the following definition of observability.

Definition 2.2 For the nonlinear system $\mathbf{P}^a_{sp}$, we say that, it is locally zero-input observable, if for all states $x_1, x_2 \in U \subset \mathcal{X}$ and input $w(.) = 0$,

$$y(k; x_1, w) \equiv y(k; x_2, w) \implies x_1 = x_2,$$

where $y(., x_i, w), i = 1, 2$ is the output of the system with the initial condition $x_{k_0} = x_i$. Moreover, the system is said to be zero-input observable, if it is locally observable at each $x^0 \in \mathcal{X}$ or $U = \mathcal{X}$.

3   Solution to the $\mathcal{H}_\infty$ Filtering Problem Using Decomposition Filters

In this section, we present a decomposition approach to the $\mathcal{H}_\infty$ estimation problem defined in the previous section, while in the next section, we present an aggregate approach.

We construct two time-scale filters corresponding to the decomposition of the system into a *"fast"* and *"slow"* subsystems. As in the linear case [5,12,16,18,22], we first assume that there exists locally a smooth invertible coordinate transformation (a diffeomorphism) $\varphi : x \mapsto \xi$, i.e.

$$\xi_1 = \varphi_1(x), \quad \varphi_1(0) = 0, \quad \xi_2 = \varphi_2(x), \quad \varphi_2(0) = 0, \ \ \xi_1 \in \Re^{n_1}, \xi_2 \in \Re^{n_2}, \tag{4}$$

such that the system (2) is locally decomposed into the form

$$\tilde{\mathbf{P}}^{\mathbf{da}}_{sp} : \begin{cases} \xi_{1,k+1} &= \tilde{f}_1(\xi_{1,k}, \varepsilon) + \tilde{g}_{11}(\xi_k, \varepsilon)w_k, \quad \xi_1(k_0) = \varphi_1(x^0, \varepsilon), \\ \varepsilon\xi_{2,k+1} &= \tilde{f}_2(\xi_{2,k}, \varepsilon) + \tilde{g}_{21}(\xi_k, \varepsilon)w_k; \quad \xi_2(k_0) = \varphi_2(x^0, \varepsilon), \\ y_k &= \tilde{h}_{21}(\xi_{1,k}, \xi_{2,k}, \varepsilon) + \tilde{h}_{22}(\xi_{1,k}, \xi_{2,k}, \varepsilon) + \tilde{k}_{21}(\xi_k, \varepsilon)w. \end{cases} \tag{5}$$

Remark 3.1 It is virtually impossible to find a coordinate transformation such that $\tilde{h}_{2j} = \tilde{h}_{2j}(\xi_j), j = 1, 2$. Thus, we have made the more practical assumption that $\tilde{h}_{2j} = \tilde{h}_{2j}(\xi_1, \xi_2), j = 1, 2$.

Necessary conditions that such a transformation must satisfy are given in [1]. The filter is then designed based on this transformed model as follows

$$
\mathbf{F}_{1c}^{da} : \begin{cases}
\hat{\xi}_{1,k+1} &= \tilde{f}_1(\hat{\xi}_{1,k}, \varepsilon) + \tilde{g}_{11}(\hat{\xi}_k, \varepsilon)w_k^\star + L_1(\hat{\xi}_k, y_k, \varepsilon)[y_k - \tilde{h}_{21}(\hat{\xi}_k, \varepsilon) - \\
& \quad \tilde{h}_{22}(\hat{\xi}_k, \varepsilon)]; \\
& \quad \hat{\xi}_1(k_0, \varepsilon) = 0, \\
\varepsilon\hat{\xi}_{2,k+1} &= \tilde{f}_2(\hat{\xi}_{2,k}, \varepsilon) + \tilde{g}_{21}(\hat{\xi}_k, \varepsilon)w_k^\star + L_2(\hat{\xi}_k, y_k, \varepsilon)[y_k - \tilde{h}_{21}(\hat{\xi}_k, \varepsilon) - \\
& \quad \tilde{h}_{22}(\hat{\xi}_k, \varepsilon)]; \\
& \quad \hat{\xi}_2(k_0, \varepsilon) = 0,
\end{cases}
\tag{6}
$$

where $\hat{\xi} \in \mathcal{X}$ is the filter state, $L_1 \in \Re^{n_1 \times m}$, $L_2 \in \Re^{n_2 \times m}$ are the filter gains, and $w^\star$ is the worst-case noise, while all the other variables have their corresponding previous meanings and dimensions. We can then define the penalty variable or estimation error at each instant k as

$$
z_k = y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k).
\tag{7}
$$

The problem can then be formulated as a dynamic optimization problem with the following cost functional

$$
\begin{cases}
\min_{L_1, L_2 \in \Re^{n \times m}} \sup_{w \in \mathcal{W}} J_1(L_1, L_2, w) = \frac{1}{2} \sum_{k=k_0}^{\infty} \left\{ \|z_k\|^2 - \gamma^2 \|w_k\|^2 \right\}, \\
s.t. \quad (6) \text{ and with } w = 0 \quad \lim_{k \to \infty} \{\hat{\xi}_k - \xi_k\} = 0.
\end{cases}
\tag{8}
$$

To solve the problem, we form the Hamiltonian function $H : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \Re^{n_1 \times m} \times \Re^{n_2 \times m} \times \Re \to \Re$:

$$
H(\hat{\xi}, w, y, L_1, L_2, V, \varepsilon) = V\Big(\tilde{f}_1(\hat{\xi}_1, \varepsilon) + \tilde{g}_{11}(\hat{\xi}, \varepsilon)w + L_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}_1, \varepsilon) -
$$

$$
h_{22}(\hat{\xi}_2\varepsilon)), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon) + \tilde{g}_{21}(\hat{\xi}, \varepsilon)w + \frac{1}{\varepsilon}L_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - \tilde{h}_{22}(\hat{\xi}, \varepsilon)), y\Big) -
$$

$$
V(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|\tilde{z}\|^2 - \gamma^2 \|w\|^2)
\tag{9}
$$

for some C^1 positive-definite function $V : \mathcal{X} \times \mathcal{Y} \to \Re_+$ and where $\hat{\xi}_1 = \hat{\xi}_{1,k}$, $\hat{\xi}_2 = \hat{\xi}_{2,k}$ $y = y_k$, $z = \{z_k\}$, $w = \{w_k\}$. We then determine the worst-case noise $w^\star$ and the optimal gains $\hat{L}_1^\star$ and $\hat{L}_2^\star$ by maximizing and minimizing H with respect to w and L_1, L_2 respectively in the above expression (9), as

$$
w^\star = \arg\sup_w H(\hat{\xi}, w, y, L_1, L_2, V, \varepsilon),
\tag{10}
$$

$$
[L_1^\star, L_2^\star] = \arg\min_{L_1, L_2} H(\hat{\xi}, w^\star, y, L_1, L_2, V, \varepsilon).
\tag{11}
$$

However, because the Hamiltonian function (9) is not a linear or quadratic function of w and L_1, L_2, only implicit solutions may be obtained [1]. Thus, the only way to obtain an explicit solution is to use an approximate scheme. In [1] we have used a second-order Taylor series approximationn of the Hamiltonian about $(\tilde{f}_1(\hat{\xi}_1), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2), y)$ in the direction of the state vectors $(\hat{\xi}_1, \hat{\xi}_2)$. It is believed that, this would capture most,

if not all, of the system dynamics. However, for the $\mathcal{H}_\infty$ problem at hand, such an approximation becomes too messy and the solution becomes more involved. Therefore, instead we would rather use a first-order Taylor approximation which is given by

$$
\begin{aligned}
\widehat{H}(\hat{\xi}, \hat{w}, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &= \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\
&\quad \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)[\tilde{g}_{11}(\hat{\xi}, \varepsilon)\hat{w} + \\
&\quad \hat{L}_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)] + \\
&\quad \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2, \varepsilon}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)[\tilde{g}_{21}(\hat{\xi}, \varepsilon)\hat{w} + \\
&\quad \hat{L}_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)] + \\
&\quad \frac{1}{2}(\|\tilde{z}\|^2 - \gamma^2\|\hat{w}\|^2) + O(\|\hat{\xi}\|^2),
\end{aligned} \tag{12}
$$

where $\hat{V}$, $\hat{w}$, $\hat{L}_1$, $\hat{L}_2$ are the corresponding approximate functions, and $\hat{V}_{\hat{\xi}_1}$, $\hat{V}_{\hat{\xi}_2}$ are the row vectors of first-partial derivatives of $\hat{V}$ with respect to $\hat{\xi}_1$, $\hat{\xi}_2$ respectively. We can now obtain $w^\star$ as

$$
\hat{w}^\star = \frac{1}{\gamma^2}[\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \frac{1}{\varepsilon}\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)). \tag{13}
$$

Then substituting $\hat{w} = \hat{w}^\star$ in (12), we have

$$
\begin{aligned}
\widehat{H}(\hat{\xi}, \hat{w}^\star, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &\approx \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\
&\quad \frac{1}{2\gamma^2}\Big[\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \\
&\quad \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2, \varepsilon}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\Big] + \\
&\quad \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\hat{L}_1(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) + \\
&\quad \frac{1}{2\gamma^2}\Big[\frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2), y)\tilde{g}_{21}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + \\
&\quad \frac{1}{\varepsilon^2}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{21}(\hat{\xi})\tilde{g}_{21}^T(\hat{\xi})\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\Big] + \\
&\quad \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\hat{L}_2(\hat{\xi}, y, \varepsilon)(y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon)) + \frac{1}{2}\|\tilde{z}\|^2. \tag{14}
\end{aligned}
$$

Completing the squares now for $\hat{L}_1(\hat{\xi}, y)$ and $\hat{L}_2(\hat{\xi}, y)$ in (14), we get

$$
\begin{aligned}
\widehat{H}(\hat{\xi}, \hat{w}^\star, y, \hat{L}_1, \hat{L}_2, \hat{V}, \varepsilon) &\approx \hat{V}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) - \hat{V}(\hat{\xi}, y_{k-1}) + \\
&\quad \frac{1}{2\gamma^2}\Big[\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{11}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) \\
&\quad + \frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\tilde{g}_{11}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\hat{\xi}, \varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y)\Big] \\
&\quad + \frac{1}{2}\left\|\hat{L}_1^T(\hat{\xi}, y)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1, \varepsilon), \frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2, \varepsilon), y) + (y - \tilde{h}_{21}(\hat{\xi}, \varepsilon) - h_{22}(\hat{\xi}, \varepsilon))\right\|^2 +
\end{aligned}
$$

$$\frac{1}{2\gamma^2}\Big[\frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\tilde{g}_{21}(\hat{\xi},\varepsilon)\tilde{g}_{11}^T(\hat{\xi},\varepsilon)\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)$$

$$+\frac{1}{\varepsilon^2}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2),y)\tilde{g}_{21}(\hat{\xi})\tilde{g}_{21}^T(\hat{\xi})\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\Big]-$$

$$\frac{1}{2}\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2),y)\hat{L}_1(\hat{\xi},y,\varepsilon)\hat{L}_1^T(\hat{\xi},y,\varepsilon)\hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)-$$

$$\frac{1}{2\varepsilon^2}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\hat{L}_2(\hat{\xi},y,\varepsilon)\hat{L}_2^T(\hat{\xi},y,\varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)$$

$$+\frac{1}{2}\left\|\frac{1}{\varepsilon}\hat{L}_2^T(\hat{\xi},y,\varepsilon)\hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)+(y-\tilde{h}_{21}(\hat{\xi},\varepsilon)-h_{22}(\hat{\xi},\varepsilon))\right\|^2-$$

$$\frac{1}{2}\|z\|^2.$$

Hence, setting the optimal gains as

$$\hat{V}_{\hat{\xi}_1,\varepsilon}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\hat{L}_1^\star(\hat{\xi},y,\varepsilon) = -(y-\tilde{h}_{21}(\hat{\xi},\varepsilon)-h_{22}(\hat{\xi},\varepsilon))^T, \quad (15)$$

$$\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\hat{L}_2^\star(\hat{\xi},y,\varepsilon) = -\varepsilon(y-\tilde{h}_{21}(\hat{\xi},\varepsilon)-h_{22}(\hat{\xi},\varepsilon))^T, \quad (16)$$

minimizes the Hamiltonian $\widehat{H}(.,.,\hat{L}_1,\hat{L}_2,.,.)$ and guarantees that the saddle-point condition [7]

$$\widehat{H}(.,\hat{w}^\star,\hat{L}_1^\star,\hat{L}_2^\star,.,.) \le \widehat{H}(.,\hat{w}^\star,\hat{L}_1,\hat{L}_2,.,.) \quad \forall \hat{L}_1 \in \Re^{n_1\times m}, \hat{L}_2 \in \Re^{n_2\times m} \quad (17)$$

is satisfied. Finally, substituting the above optimal gains in (12) and setting

$$\widehat{H}(\hat{\xi},w^\star,y,\hat{L}_1^\star,\hat{L}_2^\star,\hat{V},\varepsilon)=0,$$

results in the following discrete Hamilton-Jacobi-Isaacs equation (DHJIE)

$$\hat{V}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)-\hat{V}(\hat{\xi},y_{k-1})+$$

$$\frac{1}{2\gamma^2}\big[\ \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\quad \hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\ \big]\times$$

$$\begin{bmatrix}\tilde{g}_{11}(\hat{\xi})\tilde{g}_{11}^T(\hat{\xi},\varepsilon) & \frac{1}{\varepsilon}\tilde{g}_{11}(\hat{\xi},\varepsilon)\tilde{g}_{21}^T(\hat{\xi},\varepsilon) \\ \frac{1}{\varepsilon}\tilde{g}_{21}(\hat{\xi},\varepsilon)\tilde{g}_{11}^T(\hat{\xi},\varepsilon) & \frac{1}{\varepsilon^2}\tilde{g}_{21}(\hat{\xi},\varepsilon)\tilde{g}_{21}^T(\hat{\xi},\varepsilon)\end{bmatrix}\begin{bmatrix}\hat{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y) \\ \hat{V}_{\hat{\xi}_2}^T(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)\end{bmatrix}-$$

$$\frac{3}{2}(y-\tilde{h}_{21}(\hat{\xi},\varepsilon)-h_{22}(\hat{\xi},\varepsilon))^T(y-\tilde{h}_{21}(\hat{\xi},\varepsilon)-h_{22}(\hat{\xi},\varepsilon))=0 \quad \hat{V}(0,0,0)=0. (18)$$

We then have the following result.

Proposition 3.1 *Consider the nonlinear discrete system (2) and the $\mathcal{H}_\infty$-filtering problem for this system. Suppose the plant $\mathbf{P}_{\mathbf{sp}}^{\mathbf{da}}$ is locally asymptotically stable about the equilibrium-point $x=0$ and zero-input observable. Further, suppose there exist a local diffeomorphism φ that transforms the system to the partially decoupled form (5), a C^1 positive-semidefinite function $\hat{V}:\hat{N}\times\hat{\Upsilon}\to\Re_+$ locally defined in a neighborhood $\hat{N}\times\hat{\Upsilon}\subset \mathcal{X}\times\mathcal{Y}$ of the origin $(\hat{\xi},y)=(0,0)$, and matrix functions $\hat{L}_i:\hat{N}\times\hat{\Upsilon}\to\Re^{n_i\times m}$, $i=1,2$, satisfying the DHJIE (18) together with the side-conditions (15), (16) for some $\gamma>0$. Then, the filter $\mathbf{F}_{1c}^{da}$ solves the $\mathcal{H}_\infty$ filtering problem for the system locally in $\hat{N}$.*

Proof The optimality of the filter gains $\hat{L}_1^\star$, $\hat{L}_2^\star$ has already been shown above. It remains to show that the saddle-point conditions [7]

$$\widehat{H}(.,\hat{w},\hat{L}_1^\star,\hat{L}_2^\star,.,.) \leq \widehat{H}(.,\hat{w}^\star,\hat{L}_1^\star,\hat{L}_2^\star,.,.) \leq \widehat{H}(.,\hat{w}^\star,\hat{L}_1,\hat{L}_2,.,.),$$
$$\forall \hat{L}_1 \in \Re^{n_1 \times m}, \hat{L}_2 \in \Re^{n_2 \times m}, \forall w \in \ell_2[k_0,\infty). \tag{19}$$

and the ℓ_2-gain condition (3) hold for all $w \in \mathcal{W}$. Moreover, that there is asymptotic convergence of the estimation error vector.

Now, the right-hand-side of the above inequality (19) has already been shown. It remains to show that the left hand side also holds. Accordingly, it can be shown from (12), (18) that

$$\widehat{H}(\hat{\xi},\hat{w},\hat{L}_1^\star,\hat{L}_2^\star,\hat{V},\varepsilon) = \widehat{H}(\hat{\xi},\hat{w}^\star,\hat{L}_1^\star,\hat{L}_2^\star,\hat{V},\varepsilon) - \frac{1}{2}\gamma^2\|\hat{w}-\hat{w}^\star\|^2.$$

Therefore, we also have the left-hand side of (19) satisfied, and the pair $(\hat{w}^\star,[\hat{L}_1^\star,L_2^\star])$ constitute a saddle-point solution to the dynamic game (8), (6).

Next, let $\hat{V} \geq 0$ be a C^1 solution of the DHJIE (18). Then, consider the time-variation of $\hat{V}$ along a trajectory of (6), with $\hat{L}_1 = \hat{L}_1^\star$, $L_2 = \hat{L}_2^\star$, and $w \in \mathcal{W}$, to get

$$\hat{V}(\hat{\xi}_{1,k+1},\hat{\xi}_{2,k+1},y) \approx \hat{V}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y) + \hat{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)$$

$$.[\tilde{g}_{11}(\hat{\xi},\varepsilon)\hat{w} + \hat{L}_1^\star(\hat{\xi},y,\varepsilon)(y - \tilde{h}_{21}(\hat{\xi},\varepsilon) - h_{22}(\hat{\xi},\varepsilon))]$$

$$+\frac{1}{\varepsilon}\hat{V}_{\hat{\xi}_2}(\tilde{f}_1(\hat{\xi}_1,\varepsilon),\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2,\varepsilon),y)[\tilde{g}_{21}(\hat{\xi},\varepsilon)w + \hat{L}_2^\star(\hat{\xi},y,\varepsilon)(y - \tilde{h}_{21}(\hat{\xi},\varepsilon) - h_{22}(\hat{\xi},\varepsilon))]$$

$$= \hat{V}(\hat{\xi},y_{k-1}) - \frac{\gamma^2}{2}\|\hat{w}-\hat{w}^\star\|^2 + \frac{1}{2}(\gamma^2\|\hat{w}\|^2 - \|\tilde{z}\|^2)$$

$$\leq \hat{V}(\hat{\xi},y_{k-1}) + \frac{1}{2}(\gamma^2\|\hat{w}\|^2 - \|\tilde{z}\|^2) \quad \forall \hat{w} \in \mathcal{W}, \tag{20}$$

where we have used the first-order Taylor approximation in the above, and the last inequality follows from using the DHJIE (18). Moreover, the last inequality is the discrete-time dissipation-inequality [?] which also implies that the ℓ_2-gain inequality (3) is satisfied.

In addition, setting $w = 0$ in (20) implies that

$$\hat{V}(\hat{\xi}_{1,k+1},\hat{\xi}_{2,k+1},y) - \hat{V}(\hat{\xi}_{1,k},\hat{\xi}_{2,k},y_{k-1}) = -\frac{1}{2}\|z_k\|^2.$$

Therefore, the filter dynamics is stable, and $V(\hat{\xi},y)$ is non-increasing along a trajectory of (6). Further, the condition that $\hat{V}(\hat{\xi}_{1,k+1},\hat{\xi}_{2,k+1},y) \equiv \hat{V}(\hat{\xi}_{1,k},\hat{\xi}_{2,k},y_{k-1}) \forall k \geq k_s$ (say!) implies that $\tilde{z}_k \equiv 0$, which further implies that $y_k = \tilde{h}_{21}(\hat{\xi}_k) + \tilde{h}_{22}(\hat{\xi}_k) \forall k \geq k_s$. By the zero-input observability of the system, this implies that $\hat{\xi} = \xi$. Finally, since φ is invertible and $\varphi(0) = 0$, $\hat{\xi} = \xi$ implies $\hat{x} = \varphi^{-1}(\hat{\xi}) = \varphi^{-1}(\xi) = x$. □

Next, we consider the limiting behavior of the filter (6) and the corresponding DHJIE (18). Letting $\varepsilon \downarrow 0$, we obtain from (6)

$$0 = \tilde{f}_2(\hat{\xi}_{2,k}) + L_2(\hat{\xi}_k,y_k)(y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k)) \quad \forall k,$$

and since $\tilde{f}_2(.)$ is asymptotically stable, we have $\hat{\xi}_2 \to 0$. Therefore $H(.,.,.,.,.,.)$ in (9) becomes

$$
\begin{aligned}
H_0(\hat{\xi}, w, y, L_1, L_2, V, 0) &= V\left(\tilde{f}_1(\hat{\xi}_1) + \tilde{g}_{11}(\hat{\xi})w + L_1(\hat{\xi}, y)(y - \tilde{h}_{21}(\hat{\xi}_1) - h_{22}(\hat{\xi}_2)), 0, y\right) \\
&\quad - V(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|z\|^2 - \gamma^2\|w\|^2).
\end{aligned}
\tag{21}
$$

A first-order Taylor approximation of this Hamiltonian about $(\tilde{f}_1(\hat{\xi}_1), 0, y)$ similarly yields

$$
\begin{aligned}
\hat{H}_0(\hat{\xi}, \hat{w}, y, \hat{L}_{10}, \bar{V}, 0) &= \bar{V}(\tilde{f}_1(\hat{\xi}_1), 0, y) + \bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\hat{L}_{10}^T(\hat{\xi}, y)(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi})) \\
&\quad + \bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\tilde{g}_{11}(\hat{\xi})w - \bar{V}(\hat{\xi}, y_{k-1}) + \frac{1}{2}(\|z\|^2 - \gamma^2\|\hat{w}\|^2) + \\
&\quad O(\|\hat{\xi}\|^2)
\end{aligned}
\tag{22}
$$

for some corresponding positive-semidefinite function $\bar{V} : \mathcal{X} \times \mathcal{Y} \to \Re$, and gain $\hat{L}_{10}$. Minimizing again this Hamiltonian, we obtain the worst-case noise $w_{10}^\star$ and optimal gain $\hat{L}_{10}^\star$ given by

$$
\hat{w}_{10}^\star = -\tilde{g}_{11}^T(\hat{\xi})\bar{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), 0, y),
\tag{23}
$$

$$
\bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\hat{L}_{10}^\star(\hat{\xi}, y) = -(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi}))^T
\tag{24}
$$

where $\bar{V}$ satisfies the reduced-order DHJIE

$$
\bar{V}(\tilde{f}_1(\hat{\xi}_1), 0, y) + \frac{1}{2\gamma^2}\bar{V}_{\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), 0, y)\tilde{g}_{11}(\hat{\xi})\tilde{g}_{11}^T(\hat{\xi})\bar{V}_{\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), 0, y) - \bar{V}(\hat{\xi}_1, 0, y_{k-1}) -
$$

$$
\frac{3}{2}(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi}))^T)(y - \tilde{h}_{21}(\hat{\xi}) - h_{22}(\hat{\xi})) = 0, \quad \bar{V}(0, 0, 0) = 0.
\tag{25}
$$

The corresponding reduced-order filter is given by

$$
\bar{\mathbf{F}}_{1r}^{da} : \left\{ \quad \dot{\hat{\xi}}_1 = \tilde{f}_1(\hat{\xi}_1) + \hat{L}_{10}^\star(\hat{\xi}_1, y)(y - \tilde{h}_{21}(\hat{\xi}) - \tilde{h}_{22}(\hat{\xi})) + O(\varepsilon). \right.
\tag{26}
$$

Moreover, since the gain $\hat{L}_{10}^\star$ is such that the estimation error $e_k = y_k - \tilde{h}_{21}(\hat{\xi}_k) - \tilde{h}_{22}(\hat{\xi}_k) \to 0$, and the vector-field $\tilde{f}_2(\hat{\xi}_2)$ is locally asymptotically stable, we have $\hat{L}_2^\star(\hat{\xi}_k, y_k) \to 0$ as $\varepsilon \downarrow 0$. Correspondingly, the solution $\bar{V}$ of the DHJIE (25) can be represented as the asymptotic limit of the solution of the DHJIE (18) as $\varepsilon \downarrow 0$, i.e.,

$$
\hat{V}(\hat{\xi}, y) = \bar{V}(\hat{\xi}_1, y) + O(\varepsilon).
$$

We can specialize the result of Proposition 3.1 to the following discrete-time linear singularly-perturbed system (DLSPS) [5, 16, 18, 22] in the slow coordinate:

$$
\mathbf{P}_{dsp}^l : \left\{
\begin{array}{rcl}
x_{1,k+1} &=& A_1 x_{1,k} + A_{12} x_{2,k} + B_{11} w_k; \quad x_1(k_0) = x^{10}, \\
\varepsilon x_{2,k+1} &=& A_{21} x_{1,k} + (\varepsilon I_{n_2} + A_2) x_{2,k} + B_{21} w_k; \quad x_2(k_0) = x^{20}, \\
y_k &=& C_{21} x_{1,k} + C_{22} x_{2,k} + w_k,
\end{array}
\right.
\tag{27}
$$

where $A_1 \in \Re^{n_1 \times n_1}$, $A_{12} \in \Re^{n_1 \times n_2}$, $A_{21} \in \Re^{n_2 \times n_1}$, $A_2 \in \Re^{n_2 \times n_2}$, $B_{11} \in \Re^{n_1 \times s}$, and $B_{21} \in \Re^{n_2 \times s}$, while the other matrices have compatible dimensions. Then, an explicit form of the required transformation φ above is given by the Chang transformation [12]:

$$
\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} - \varepsilon\mathsf{HL} & -\varepsilon\mathsf{H} \\ \mathsf{L} & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},
\tag{28}
$$

where the matrices L and H satisfy the equations

$$
\begin{aligned}
0 &= (\varepsilon I_{n_2} + A_2)\mathsf{L} - A_{21} - \varepsilon\mathsf{L}(A_1 - A_{12}\mathsf{L}), \\
0 &= -\mathsf{H}[(\varepsilon I_{n_2} + A_2) + \varepsilon\mathsf{L}A_{12}] + A_{12} + \varepsilon(A_1 - A_{12}\mathsf{L})\mathsf{H}.
\end{aligned}
$$

The system is then represented in the new coordinates by

$$
\tilde{\mathbf{P}}_{dsp}^{l} : \left\{
\begin{aligned}
\xi_{1,k+1} &= \tilde{A}_1\xi_{1,k} + \tilde{B}_{11}w_k; & \xi_1(k_0) &= \xi^{10}, \\
\varepsilon\xi_{2,k+1} &= \tilde{A}_2\xi_{2,k} + \tilde{B}_{21}w_k; & \xi_2(k_0) &= \xi^{20}, \\
y_k &= \tilde{C}_{21}\xi_{1,k} + \tilde{C}_{22}\xi_{2,k} + w_k,
\end{aligned}
\right. \tag{29}
$$

where

$$
\begin{aligned}
\tilde{A}_1 &= A_1 - A_{12}\mathsf{L} = A_1 - A_{12}(\varepsilon I_{n_2} + A_2)^{-1}A_{21} + O(\varepsilon), \\
\tilde{B}_{11} &= B_{11} - \varepsilon\mathsf{H}\mathsf{L}B_{11} - \mathsf{H}B_{21} = B_{11} - A_{12}A_2^{-1}B_{21} + O(\varepsilon), \\
\tilde{A}_2 &= (\varepsilon I_{n_2} + A_2) + \varepsilon\mathsf{L}A_{12} = A_2 + O(\varepsilon), \\
\tilde{B}_{21} &= B_{21} + \varepsilon\mathsf{L}B_{11} = B_{21} + O(\varepsilon), \\
\tilde{C}_{21} &= C_{21} - C_{22}\mathsf{L} = C_{21} - C_{22}(\varepsilon I_{n_2} + A_2)^{-1}A_{21} + O(\varepsilon), \\
\tilde{C}_{22} &= C_{22} + \varepsilon(C_{21} - C_{22})\mathsf{H} = C_{22} + O(\varepsilon).
\end{aligned}
$$

Adapting the filter (6) to the system (29) yields the following filter

$$
\mathbf{F}_{1c}^{dl} : \left\{
\begin{aligned}
\hat{\xi}_{1,k+1} &= \tilde{A}_1\hat{\xi}_{1,k} + \tilde{B}_{11}w_k^{\star} + \hat{L}_1(y_k - \tilde{C}_{21}\hat{\xi}_{1,k} - \tilde{C}_{22}\hat{\xi}_{2,k}), \\
\varepsilon\hat{\xi}_{2,k} &= \tilde{A}_2\hat{\xi}_{2,k} + \tilde{B}_{21}w_k^{\star} + \hat{L}_2(y_k - \tilde{C}_{21}\hat{\xi}_{1,k} - \tilde{C}_{22}\hat{\xi}_{2,k}).
\end{aligned}
\right. \tag{30}
$$

Taking

$$
\hat{V}(\hat{\xi}, y) = \frac{1}{2}(\hat{\xi}_1^T\hat{P}_1\hat{\xi}_1 + \hat{\xi}_2^T\hat{P}_2\hat{\xi}_2 + y^T\hat{Q}y),
$$

for some symmetric positive-definite matrices $\hat{P}_1$, $\hat{P}_2$, $\hat{Q}$, the DHJIE (18) reduces to the following algebraic equation

$$
\begin{aligned}
&(\hat{\xi}_1^T\tilde{A}_1^T\hat{P}_1\tilde{A}_1\hat{\xi}_1 + \frac{1}{\varepsilon^2}\hat{\xi}_2^T\tilde{A}_2^T\hat{P}_2\tilde{A}_2\hat{\xi}_2 + y^T\hat{Q}y) - (\hat{\xi}_1^T\hat{P}_1\hat{\xi}_1 + \hat{\xi}_2^T\hat{P}_2\hat{\xi}_2 + y_{k-1}^T\hat{Q}y_{k-1}) + \\
&\frac{1}{\gamma^2}\Big[\hat{\xi}_1^T\tilde{A}_1^T\hat{P}_1\tilde{B}_{11}\tilde{B}_{11}^T\hat{P}_1\tilde{A}_1\hat{\xi}_1 + \frac{1}{\varepsilon^2}\hat{\xi}_2^T\tilde{A}_2^T\hat{P}_2\tilde{B}_{21}\tilde{B}_{11}^T\hat{P}_1\tilde{A}_1\hat{\xi}_1 + \frac{1}{\varepsilon^2}\hat{\xi}_1^T\tilde{A}_1^T\hat{P}_1\tilde{B}_{11}\tilde{B}_{21}^T\hat{P}_2\tilde{A}_2\hat{\xi}_2 \\
&+\frac{1}{\varepsilon^4}\hat{\xi}_2^T\tilde{A}_2^T\hat{P}_2\tilde{B}_{21}\tilde{B}_{21}^T\hat{P}_2\tilde{A}_2\hat{\xi}_2\Big] - 3(y^Ty - \hat{\xi}_1^T\tilde{C}_{21}^Ty - y^T\tilde{C}_{21}^T\hat{\xi}_1 - y^T\tilde{C}_{22}^T\hat{\xi}_1 - y^T\tilde{C}_{22}^T\hat{\xi}_2 - \\
&\hat{\xi}_2^T\tilde{C}_{22}^Ty + \hat{\xi}_1^T\tilde{C}_{21}^T\tilde{C}_{21}\hat{\xi}_1 + \hat{\xi}_1^T\tilde{C}_{21}^T\tilde{C}_{22}\hat{\xi}_2 + \hat{\xi}_2^T\tilde{C}_{22}^T\tilde{C}_{21}\hat{\xi}_1 + \hat{\xi}_2^T\tilde{C}_{22}^T\tilde{C}_{22}\hat{\xi}_2) = 0. \tag{31}
\end{aligned}
$$

Subtracting now $\frac{1}{2}y^T\hat{R}y$ for some symmetric matrix $\hat{R} > 0$ from the left-hand side of the above equation (and abosorbing $\hat{R}$ in $\hat{Q}$), we have the following matrix-inequality

$$
\left[
\begin{array}{cccc}
\tilde{A}_1^T\hat{P}_1A_1 - \hat{P}_1 + \frac{1}{\gamma^2}\tilde{A}_1^T\hat{P}_1\tilde{B}_{11}\tilde{B}_{11}^T\hat{P}_1\tilde{A}_1 - 3\tilde{C}_{21}^T\tilde{C}_{21} & \frac{1}{\gamma^2\varepsilon^2}\tilde{A}_1^T\hat{P}_1\tilde{B}_{11}\tilde{B}_{21}^T\hat{P}_2\tilde{A}_2 + 3\tilde{C}_{21}^T\tilde{C}_{22} & 3\tilde{C}_{21}^T & 0 \\
\frac{1}{\gamma^2\varepsilon^2}\tilde{A}_2^T\hat{P}_2\tilde{B}_{21}\tilde{B}_{11}^T\hat{P}_1\tilde{A}_1 + 3\tilde{C}_{22}^T\tilde{C}_{21} & \frac{1}{\varepsilon^2}\tilde{A}_2^T\hat{P}_2\tilde{A}_2 - \hat{P}_2 + \frac{1}{\gamma^2\varepsilon^4}\tilde{A}_2^T\hat{P}_2\tilde{B}_{21}\tilde{B}_{21}^T\hat{P}_2\tilde{A}_2 - 3\tilde{C}_{22}^T\tilde{C}_{22} & 3\tilde{C}_{22}^T & 0 \\
3\tilde{C}_{21} & 3\tilde{C}_{22} & \hat{Q} - 3I & 0 \\
0 & 0 & 0 & -\hat{Q}
\end{array}
\right] \le 0. \tag{32}
$$

While the side conditions (15), (16) reduce to the following LMIs

$$
\begin{bmatrix}
0 & 0 & \frac{1}{2}(\tilde{A}_1^T \hat{P}_1 \hat{L}_1 - \tilde{C}_{21}^T) \\
0 & 0 & -\frac{1}{2}\tilde{C}_{22}^T \\
\frac{1}{2}(\tilde{A}_1^T \hat{P}_1 \hat{L}_1 - \tilde{C}_{21}^T)^T & -\frac{1}{2}\tilde{C}_{22}^T & (1-\delta_1)I
\end{bmatrix} \le 0,
\tag{33}
$$

$$
\begin{bmatrix}
0 & 0 & -\frac{1}{2}\tilde{C}_{21}^T \\
0 & 0 & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \hat{P}_2 \hat{L}_2 - \tilde{C}_{22}^T) \\
-\frac{1}{2}\tilde{C}_{21} & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \hat{P}_2 \hat{L}_2 - \tilde{C}_{22}^T)^T & (1-\delta_2)I
\end{bmatrix} \le 0
\tag{34}
$$

for some numbers δ_1, $\delta_2 \ge 1$. The above matrix inequality (32) can be further simplified using Schur's complements, but cannot be made linear because of the off-diagonal and coupling terms. This is primarily because the assumed transformation φ can only achieve a partial decoupling of the original system, and a complete decoupling of the states will require more stringent assumptions and conditions.

Consequently, we have the following corollary to Proposition 3.1.

Corollary 3.1 *Consider the DLSPS (27) and the $\mathcal{H}_\infty$ filtering problem for this system. Suppose the plant $\mathbf{P}_{sp}^l$ is locally asymptotically stable about the equilibrium-point $x = 0$ and observable. Suppose further, it is transformable to the form (29), and there exist symmetric positive-definite matrices $\hat{P}_1 \in \Re^{n_1 \times n_1}$, $\hat{P}_2 \in \Re^{n_2 \times n_2}$, $\hat{Q} \in \Re^{m \times m}$, and matrices $\hat{L}_1 \in \Re^{n_1 \times m}$, $\hat{L}_2 \in \Re^{n_2 \times m}$, satisfying the matrix inequalities (32), (33), (34) for some numbers $\delta_1, \delta_2 \ge 1$ and $\gamma > 0$. Then, the filter $\mathbf{F}_{1c}^{dl}$ solves the $\mathcal{H}_\infty$ filtering problem for the system.*

Similarly, for the reduced-order filter (26) and the DHJIE (25), we have respectively

$$
\mathbf{F}_{1r}^{dl}: \quad \hat{\xi}_{1,k+1} \;=\; \tilde{A}_1 \hat{\xi}_{1,k} + \hat{L}_{10}^\star (y_k - \tilde{C}_{21} \hat{\xi}_{1,k} - \tilde{C}_{22} \hat{\xi}_{2,k}) \;,
\tag{35}
$$

$$
\begin{bmatrix}
\tilde{A}_1^T \hat{P}_{10} \tilde{A}_1 - \hat{P}_{10} - 3\tilde{C}_{21}^T \tilde{C}_{21} & \tilde{A}_1^T \hat{P}_{10} \tilde{B}_{11} & 3\tilde{C}_{21} & 0 \\
\tilde{B}_{11}^T \hat{P}_{10} \tilde{A}_1 & -\gamma^{-2}I & 0 & 0 \\
3\tilde{C}_{21}^T & 0 & \hat{Q} - 3I & 0 \\
0 & 0 & 0 & \hat{Q}
\end{bmatrix} \le 0,
\tag{36}
$$

$$
\begin{bmatrix}
0 & 0 & \frac{1}{2}(\tilde{A}_1^T \hat{P}_{10} \hat{L}_{10} - \tilde{C}_{21}^T) \\
0 & 0 & -\frac{1}{2}\tilde{C}_{22}^T \\
\frac{1}{2}(\tilde{A}_1^T \hat{P}_{10} \hat{L}_{10} - \tilde{C}_{21}^T)^T & -\frac{1}{2}\tilde{C}_{22}^T & (1-\delta_{10})I
\end{bmatrix} \le 0
\tag{37}
$$

for some symmetric positive-definite matrices $\hat{P}_{10}$, $\hat{Q}_{10}$, gain matrix $\hat{L}_{10}$ and some number $\delta_{10} > 0$.

Proposition 3.1 has not yet exploited the benefit of the coordinate transformation in designing the filter (6) for the system (5). We shall now design separate reduced-order filters for the decomposed subsystems which should be more efficient than the previous one. If we let $\varepsilon \downarrow 0$ in (5), we obtain the following reduced system model:

$$
\widetilde{\mathbf{P}}_{\mathbf{r}}^{\mathbf{a}}: \begin{cases}
\xi_{1,k+1} &= \tilde{f}_1(\xi_1) + \tilde{g}_{11}(\xi)w, \\
0 &= \tilde{f}_2(\xi_2) + \tilde{g}_{21}(\xi)w, \\
y_k &= \tilde{h}_{21}(\xi) + \tilde{h}_{22}(\xi) + \tilde{k}_{21}(\xi)w.
\end{cases}
\tag{38}
$$

Then, we assume the following [15, 17].

Assumption 3.1 The system (2), (38) is in the *"standard form"*, i.e., the equation

$$0 = \tilde{f}_2(\xi_2) + \tilde{g}_{21}(\xi)w \tag{39}$$

has $l \geq 1$ isolated roots, we can denote any one of these solutions by

$$\bar{\xi}_2 = q(\xi_1, w) \tag{40}$$

for some C^1 function $q : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$.

Under Assumption 3.1, we obtain the reduced-order slow subsystem

$$\mathbf{P_r^a} : \begin{cases} \xi_{1,k+1} & = & \tilde{f}_1(\xi_{1,k}) + \tilde{g}_{11}(\xi_{1,k}, q(\xi_{1,k}, w_k))w_k + O(\varepsilon), \\ y_k & = & \tilde{h}_{21}(\xi_{1,k}, q(\xi_{1,k}, w_k)) + \tilde{h}_{22}(\xi_{1,k}, q(\xi_{1,k}, w_k)) + \\ & & \tilde{k}_{21}(\xi_{1,k}, q(\xi_{1,k}, w_k))w_k + O(\varepsilon) \end{cases}$$

and a boundary-layer (or quasi-steady-state) subsystem as

$$\bar{\xi}_{2,m+1} = \tilde{f}_2(\bar{\xi}_{2,m}, \varepsilon) + \tilde{g}_{21}(\xi_{1,m}, \bar{\xi}_{2,m})w_m, \tag{41}$$

where $m = \lfloor k/\varepsilon \rfloor$ is a stretched-time parameter. This subsystem is guaranteed to be asymptotically stable for $0 < \varepsilon < \varepsilon^\star$ (see Theorem 8.2 in Ref. [15]) if the original system (2) is asymptotically stable.

We can then proceed to redesign the filter (6) for the composite system (41), (41) separately as

$$\widetilde{\mathbf{F}}_{2c}^{da} : \begin{cases} \breve{\xi}_{1,k+1} = \tilde{f}_1(\breve{\xi}_{1,k}) + \tilde{g}_{11}(\hat{\xi}_{1,k})\breve{w}_{1,k}^\star + \breve{L}_1(\breve{\xi}_{1,k}, y_k)(y_k - \widetilde{h}_{21}(\breve{\xi}_{1,k}) - \widetilde{h}_{22}(\breve{\xi}_{1,k})), \\ \varepsilon\breve{\xi}_{2,k+1} = \tilde{f}_2(\breve{\xi}_{2,k}, \varepsilon) + \tilde{g}_{21}(\hat{\xi}_k)\breve{w}_{2,k}^\star + \breve{L}_2(\breve{\xi}_{2,k}, y_k)(y_k - \tilde{h}_{21}(\breve{\xi}_k) - \tilde{h}_{22}(\breve{\xi}_k)), \end{cases} \tag{42}$$

where

$$\widetilde{h}_{21}(\breve{\xi}_{1,k}) = \tilde{h}_{21}(\breve{\xi}_{1,k}, q(\breve{\xi}_{1,k}, \hat{w}_{1,k}^\star)), \quad \widetilde{h}_{22}(\breve{\xi}_{1,k}) = \tilde{h}_{21}(\breve{\xi}_{1,k}, q(\breve{\xi}_{1,k}, \hat{w}_{2,k}^\star)).$$

Notice also that, ξ_2 cannot be estimated from (40) since this is a "quasi-steady-state" approximation. Then, using a similar approximation procedure as in Proposition 3.1, we arrive at the following result.

Theorem 3.1 *Consider the nonlinear system (2) and the $\mathcal{H}_\infty$ estimation problem for this system. Suppose the plant $\mathbf{P_{sp}^{da}}$ is locally asymptotically stable about the equilibrium-point $x = 0$ and zero-input observable. Further, suppose there exists a local diffeomorphism φ that transforms the system to the partially decoupled form (5), and Assumption 3.1 holds. In addition, suppose for some $\gamma > 0$, there exist C^1 positive-semidefinite functions $\breve{V}_i : \breve{N}_i \times \breve{\Upsilon}_i \to \Re_+$, $i = 1, 2$, locally defined in neighborhoods $\breve{N}_i \times \breve{\Upsilon}_i \subset \mathcal{X} \times \mathcal{Y}$ of the origin $(\breve{\xi}_i, y) = (0, 0)$ $i = 1, 2$ respectively, and matrix functions $\breve{L}_i : \breve{N}_i \times \tilde{\Upsilon}_i \to \Re^{n_i \times m}$,*

$\breve{\Upsilon}_i \subset \mathcal{Y}$, $i = 1, 2$ *satisfying the pair of DHJIEs:*

$$\breve{V}_1(\tilde{f}_1(\hat{\xi}_1), y) + \frac{1}{2\gamma^2}\breve{V}_{1,\hat{\xi}_1}(\tilde{f}_1(\hat{\xi}_1), y)\tilde{g}_{11}(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star))\tilde{g}_{11}^T(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star))\breve{V}_{1,\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), y) -$$

$$\bar{V}_1(\hat{\xi}_1, y_{k-1}) - \frac{3}{2}(y - \tilde{h}_{21}(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star)) - h_{22}(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star)))^T(y - \tilde{h}_{21}(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star)) -$$

$$h_{22}(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star))) = 0, \quad \breve{V}_1(0, 0) = 0, \tag{43}$$

$$\breve{V}_2(\frac{1}{\varepsilon}\tilde{f}_2(\breve{\xi}_2, \varepsilon), y) + \frac{1}{2\gamma^2}\bar{V}_{2,\breve{\xi}_2}(\frac{1}{\varepsilon}\tilde{f}_2(\breve{\xi}_2, \varepsilon), y)\tilde{g}_{21}(\hat{\xi}, \varepsilon)\tilde{g}_{21}^T(\breve{\xi}, \varepsilon)\breve{V}_{2,\breve{\xi}_2}^T(\frac{1}{\varepsilon}\tilde{f}_2(\breve{\xi}_2, \varepsilon), y) -$$

$$\breve{V}_2(\breve{\xi}_2, y_{k-1}) - \frac{3}{2}(y - \tilde{h}_{21}(\breve{\xi}, \varepsilon) - \tilde{h}_{22}(\breve{\xi}, \varepsilon))^T(y - \tilde{h}_{21}(\breve{\xi}, \varepsilon) - h_{22}(\breve{\xi}, \varepsilon)) = 0,$$

$$\breve{V}_2(0, 0) = 0 \tag{44}$$

together with the side-conditions

$$\breve{w}_1^\star = \frac{1}{\gamma^2}\tilde{g}_{11}^T(\hat{\xi}_1, q(\xi_1, \breve{w}_1^\star))\breve{V}_{1,\hat{\xi}_1}^T(\tilde{f}_1(\hat{\xi}_1), y), \tag{45}$$

$$\breve{w}_2^\star = \frac{1}{\gamma^2}\tilde{g}_{21}^T(\hat{\xi})\breve{V}_{2,\hat{\xi}_2}^T(\frac{1}{\varepsilon}\tilde{f}_2(\hat{\xi}_2), y), \tag{46}$$

$$\hat{V}_{1,\hat{\xi}_1}(\tilde{f}_1(\breve{\xi}_1))\breve{L}_1^\star(\breve{\xi}_1, y) = -(y - \widetilde{h}_{21}(\breve{\xi}_1) - \widetilde{h}_{22}(\hat{\xi}))^T, \tag{47}$$

$$\breve{V}_{2,\breve{\xi}_2}^T(\frac{1}{\varepsilon}\tilde{f}_2(\breve{\xi}_2, \varepsilon), y)\breve{L}_2^\star(\breve{\xi}, y, \varepsilon) = -\varepsilon(y - \tilde{h}_{21}(\breve{\xi}, \varepsilon) - \tilde{h}_{22}(\breve{\xi}))^T. \tag{48}$$

Then the filter $\widetilde{\mathbf{F}}_{\mathbf{2c}}^{\mathbf{da}}$ solves the $\mathcal{H}_\infty$ filtering problem for the system locally in $\cup \breve{N}_i$.

Proof We define separately two Hamiltonian functions $H_i : \mathcal{X} \times \mathcal{W} \times \mathcal{Y} \times \Re^{n_i \times m} \times \Re \to \Re$, $i = 1, 2$ for each of the two separate components of the filter (42). Then the rest of the proof follows along the same lines as Proposition 3.1. □

Remark 3.2 Comparing (43), (47) with (24), (25), we see that the two reduced-order filter approximations are similar. Moreover, notice that $\breve{\xi}_1$ appearing in (48), (44) is not considered as an additional variable, because it is assumed to be known from (42a), (47) respectively, and is therefore regarded as a parameter. In addition, we observe that, the DHJIE (43) is implicit in $\breve{w}_1^\star$, and therefore, some sort of approximation is required in order to obtain an explicit solution.

Remark 3.3 Notice also that, in the determination of $\breve{w}_1^\star$, we assume $\bar{\xi}_2 = q(\xi_1, w)$ is frozen in the Hamiltonian H_2, and therefore the contribution to $\breve{w}_1^\star$ from $\tilde{g}_{11}(.,.)$, $\tilde{h}_{21}(.,.)$ is neglected.

We can similarly specialize the result of Theorem 3.1 to the discrete-time linear system (27) in the following corollary.

Corollary 3.2 *Consider the DLSPS (27) and the $\mathcal{H}_\infty$ filtering problem for this system. Suppose the plant $\mathbf{P}_{sp}^l$ is locally asymptotically stable about the equilibrium-point $x = 0$ and observable. Suppose further, it is transformable to the form (29) and Assumption 3.1 is satisfied, i.e., $\tilde{A}_2$ is nonsingular. In addition, suppose for some $\gamma > 0$*

there exist symmetric positive-definite matrices $\breve{P}_i \in \Re^{n_i \times n_i}$, $\breve{Q}_i \in \Re^{m \times m}$, and matrix $\breve{L}_i \in \Re^{n_i \times m}$, $i = 1, 2$ satisfying the following LMIs

$$
\begin{bmatrix}
\tilde{A}_1^T \breve{P}_1 \tilde{A}_1 - \breve{P}_1 - 3\widetilde{C}_{21}^T \widetilde{C}_{21} & \tilde{A}_1^T \breve{P}_1 \widetilde{B}_{11} & 3\widetilde{C}_{21}^T & 0 \\
\widetilde{B}_{11}^T \breve{P}_1 \tilde{A}_1 & -\gamma^2 I & 0 & 0 \\
3\widetilde{C}_{21} & 0 & \breve{Q}_1 - 3I & 0 \\
0 & 0 & 0 & -\breve{Q}
\end{bmatrix} \leq 0, \quad (49)
$$

$$
\begin{bmatrix}
-3\tilde{C}_{21}^T \tilde{C}_{21} & -3\tilde{C}_{21}^T \tilde{C}_{22} & 0 & 3\tilde{C}_{21}^T & 0 \\
-3\tilde{C}_{22}^T \tilde{C}_{21} & \tilde{A}_2^T \breve{P}_2 \tilde{A}_2 - \breve{P}_2 - 3\tilde{C}_{22}^T \tilde{C}_{22} & \tilde{A}_2^T \breve{P}_2 \tilde{B}_{21} & 3\tilde{C}_{22}^T & 0 \\
0 & \tilde{B}_{21}^T \breve{P}_2 \tilde{A}_2 & \gamma^2 \varepsilon^2 I & 0 & 0 \\
3\tilde{C}_{21} & 3\tilde{C}_{22} & 0 & \breve{Q}_2 - 3I - \breve{R}_2 & 0 \\
0 & 0 & 0 & 0 & -\breve{Q}_2
\end{bmatrix} \leq 0, \quad (50)
$$

$$
\begin{bmatrix}
0 & \frac{1}{2}(\tilde{A}_1^T \breve{P}_1 \breve{L}_1 - \widetilde{C}_{21}^T) \\
\frac{1}{2}(\tilde{A}_1^T \hat{P}_1 \breve{L}_1 - \widetilde{C}_{21}^T)^T & (1 - \delta_3) I
\end{bmatrix} \leq 0, \quad (51)
$$

$$
\begin{bmatrix}
0 & 0 & -\frac{1}{2}\tilde{C}_{21}^T \\
0 & 0 & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \breve{P}_2 \breve{L}_2 - \tilde{C}_{22}^T) \\
-\frac{1}{2}\tilde{C}_{21} & \frac{1}{2\varepsilon^2}(\tilde{A}_2^T \breve{P}_2 \breve{L}_2 - \tilde{C}_{22}^T)^T & (1 - \delta_4) I
\end{bmatrix} \leq 0, \quad (52)
$$

for some numbers $\delta_3, \delta_4 > 0$, where

$$
\widetilde{B}_{11} = \tilde{B}_{11} + \tilde{C}_{22} \tilde{A}_2^{-1} \tilde{B}_{21}, \quad \widetilde{C}_{21} = \tilde{C}_{21} - \frac{1}{\gamma^2} \tilde{C}_{22} \tilde{A}_2^{-1} \tilde{B}_{21} \tilde{B}_{11}^T \breve{P}_1 \tilde{A}_1.
$$

Then, the filter $\mathbf{F}_{2c}^{dl}$ solves the $\mathcal{H}_\infty$ filtering problem for the system.

Proof We take similarly

$$
\breve{V}_1(\hat{\xi}_1, y) = \frac{1}{2}(\breve{\xi}_1^T \breve{P}_1 \breve{\xi}_1 + y^T \breve{Q}_1 y),
$$

$$
\breve{V}_2(\hat{\xi}_2, y) = \frac{1}{2}(\breve{\xi}_2^T \breve{P}_2 \breve{\xi}_2 + y^T \breve{Q}_2 y)
$$

and apply the result of the Theorem. □

4 Aggregate Filters

In the absence of the coordinate transformation, φ discussed in the previous section, a filter has to be designed to solve the problem for the aggregate system (2). We discuss this class of filters in this section. Accordingly, consider the following class of filters $\mathbf{F}_{3ag}^{da}$:

$$
\begin{cases}
\grave{x}_{1,k+1} &= f_1(\grave{x}_k) + g_{11}(\grave{x}_k)\grave{w}_k^\star + \grave{L}_1(\grave{x}_k, y_k, \varepsilon)(y_k - h_{21}(\grave{x}_{1,k}) - h_{22}(\grave{x}_{2,k})); \\
&\quad \grave{x}_1(k_0) = \bar{x}^{10}, \\
\varepsilon \grave{x}_{2,k+1} &= f_2(\grave{x}_k, \varepsilon) + g_{21}(\grave{x}_k)\grave{w}_k^\star + \grave{L}_2(\grave{x}_k, y_k, \varepsilon)(y_k - h_{21}(\grave{x}_{1,k}) - h_{22}(\grave{x}_{2,k})); \quad (53) \\
&\quad \grave{x}_2(k_0) = \bar{x}^{20}, \\
\grave{z}_k &= y_k - h_{21}(\grave{x}_{1,k}) - h_{22}(\grave{x}_{2,k}),
\end{cases}
$$

where $\grave{L}_1, \grave{L}_2 \in \Re^{n \times m}$ are the filter gains, and $\grave{z}$ is the new penalty variable. We can repeat the same kind of derivation above to arrive at the following.

Theorem 4.1 *Consider the nonlinear system (2) and the $\mathcal{H}_2$ estimation problem for this system. Suppose the plant $\mathbf{P}^{\mathbf{da}}_{\mathbf{sp}}$ is locally asymptotically stable about the equilibrium-point $x = 0$, and zero-input observable. Further, suppose there exist a C^1 positive-definite function $\grave{V} : \grave{N} \times \grave{\Upsilon} \to \Re_+$, locally defined in a neighborhood $\grave{N} \times \grave{\Upsilon} \subset \mathcal{X} \times \mathcal{Y}$ of the origin $(\grave{x}_1, \grave{x}_2, y) = (0, 0, 0)$, and matrix functions $\grave{L}_i : \grave{N} \times \grave{\Upsilon} \to \Re^{n_i \times m}$, $i = 1, 2$, satisfying the DHJIE:*

$$\grave{V}(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) - \grave{V}(\grave{x}, y_{k-1}) +$$

$$\frac{1}{2\gamma^2} [\; \grave{V}_{\grave{x}_1}(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(x, \varepsilon), y) \quad \grave{V}_{\grave{x}_2}(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) \;] \times$$

$$\begin{bmatrix} g_{11}(\grave{x}) g_{11}^T(\grave{x}) & \frac{1}{\varepsilon} g_{11}(\grave{x}) g_{21}^T(\grave{x}) \\ \frac{1}{\varepsilon} g_{21}(\grave{x}) g_{11}^T(\grave{x}) & \frac{1}{\varepsilon^2} g_{21}(\grave{x}) g_{21}^T(\grave{x}) \end{bmatrix} \begin{bmatrix} \grave{V}_{\grave{x}_1}^T(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) \\ \grave{V}_{\grave{x}_2}^T(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) \end{bmatrix}$$

$$-\frac{3}{2}(y - h_{21}(\grave{x}_1) - h_{22}(\grave{x}_2))^T (y - h_{21}(\grave{x}_1) - h_{22}(\grave{x}_2)) = 0, \quad \hat{V}(0, 0) = 0, \quad (54)$$

together with the side-conditions

$$\grave{V}_{\grave{x}_1}(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) \grave{L}_1^{\star}(\grave{x}, y) \;\; = \;\; -(y - h_{21}(\grave{x}_1) - h_{22}(\grave{x}_2))^T, \quad (55)$$

$$\grave{V}_{\grave{x}_2}(f_1(\grave{x}), \frac{1}{\varepsilon} f_2(\grave{x}, \varepsilon), y) \grave{L}_2^{\star}(\grave{x}, y) \;\; = \;\; -\varepsilon(y - h_{21}(\grave{x}_1) - h_{22}(\grave{x}_2)). \quad (56)$$

Then, the filter $\mathbf{F}^a_{3ag}$ solves the $\mathcal{H}_\infty$ filtering problem for the system locally in $\grave{N}$.

Proof Proof follows along the same lines as Proposition 3.1. □

For the DLSPS (27), the Chang transformation φ is always available as given by (28). Moreover, the result of Theorem 4.1 specialized to the DLSPS is horrendous, in the sense that, the resulting inequalities are not linear and too involved. Thus, it is more useful to consider the reduced-order filter which will be introduced shortly as a special case of the nonlinear.

Using similar procedure as outlined in the previous section, we can obtain the limiting behavior of the filter $\mathbf{F}^a_{3ag}$ as $\varepsilon \downarrow 0$

$$\bar{\mathbf{F}}^{da}_{5ag} : \begin{cases} \dot{x}_{1,k+1} & = & f_1(\grave{x}_k) + g_{11}(\grave{x}_k)\grave{w}^{\star}_{10,k} + \grave{L}_{10}(\grave{x}_k, y_k)(y_k - h_{21}(\grave{x}_{1,k})); \\ & & \grave{x}_1(k_0) = \bar{x}^{10}, \\ \grave{x}_{2,k} & \to & 0 \end{cases} \quad (57)$$

with

$$\grave{w}^{\star}_{10} = \frac{1}{\gamma^2} g_{11}^T(\grave{x}) \grave{V}_{\grave{x}_1}^T(f_1(\grave{x}))$$

and the DHJIE (54) reduces to the DHJIE

$$\grave{V}(f_1(\grave{x}_1), y) + \frac{1}{2\gamma^2} \bar{V}_{\grave{x}_1}(f_1(\grave{x}_1), y) g_{11}(\grave{x}) g_{11}^T(\grave{x}) \grave{V}_{\grave{x}_1, y}^T(f_1(\grave{x})) - \grave{V}(\grave{x}_1, y) -$$

$$\frac{3}{2}(y - h_{21}(\grave{x}_1))^T(y - h_{21}(\grave{x}_1)) = 0, \quad \grave{V}(0) = 0. \quad (58)$$

together with the side-conditions

$$\grave{V}_{\grave{x}_1}(f_1(\grave{x}_1)) \grave{L}^{\star}_{10}(\grave{x}, y) \;\; = \;\; -(y - h_{21}(\grave{x}_1))^T, \quad (59)$$

$$\grave{L}_2(\grave{x}, y) \;\; \to \;\; 0. \quad (60)$$

Similarly, specializing the above result to the DLSPS (27), we obtain the following reduced-order filter

$$\mathbf{F}_{6agr}^{dl} : \left\{ \quad \grave{x}_{1,k+1} \quad = \quad A_1 \grave{x}_{1,k} + B_{11} \grave{w}_{10,k}^{\star} + \grave{L}_{10}^{\star}(y_k - \tilde{C}_{21} \grave{x}_{1,k}) \right. \tag{61}$$

with

$$\grave{w}_{10}^{\star} = \frac{1}{\gamma^2} B_{11}^T \grave{P}_1 A_1 \grave{x}_1$$

and the DHJIE (58) reduces to the LMI

$$\begin{bmatrix} A_1^T \grave{P}_{10} \tilde{A}_1 - \grave{P}_{10} - 3C_{21}^T C_{21} & A_1^T \grave{P}_{10} B_{11} & 3C_{21}^T & 0 \\ B_{11}^T \grave{P}_{10} A_1 & -\gamma^2 I & 0 & 0 \\ 3C_{21} & 0 & \grave{Q}_1 - 3I & 0 \\ 0 & 0 & 0 & -\grave{Q} \end{bmatrix} \leq 0, \tag{62}$$

$$\begin{bmatrix} 0 & \frac{1}{2}(A_1^T \grave{P}_{10} \grave{L}_{10} - C_{21}^T) \\ \frac{1}{2}(A_1^T \grave{P}_{10} \grave{L}_{10} - C_{21}^T)^T & (1 - \delta_5)I \end{bmatrix} \leq 0 \tag{63}$$

for some symmetric positive-definite matrices $\grave{P}_{10}$, $\grave{Q}_{10}$, gain matrix $\grave{L}_{10}$ and some number $\delta_5 \geq 1$.

Remark 4.1 If the nonlinear system (2) is in the standard form, i.e., the equivalent of Assumption 3.1 is satisfied, and there exists at least one root $\bar{x}_2 = \sigma(x_1, w)$ to the equation

$$0 = f_2(x_1, x_2) + g_{21}(x_1, x_2)w,$$

then reduced-order filters can also be constructed for the system similar to the result of Section 3 and Theorem 3.1. Such filters would take the following form

$$\mathbf{F}_{7agr}^a : \left\{ \begin{array}{rcl} \check{x}_{1,k+1} & = & f_1(\check{x}_{1,k}, \sigma(\check{x}_1, \check{w}_{1,k}^{\star})) + g_{11}(\check{x}_1, \sigma(\check{x}_1, \check{w}_{1,k}^{\star}))\check{w}_{1,k}^{\star} + \\ & & \check{L}_1(\check{x}_{1,k}, y_k, \varepsilon)(y_k - h_{21}(\check{x}_{1,k}) - h_{22}(\sigma(\check{x}_1, \check{w}_{1,k}^{\star})); \quad \check{x}_1(k_0) = \bar{x}_{10}, \\ \varepsilon \check{x}_{2,k+1} & = & f_2(\check{x}_k, \varepsilon) + g_{21}(\check{x}_1, \check{x}_2)\check{w}_{2,k}^{\star} + \check{L}_2(\check{x}_k, y_k, \varepsilon)(y_k - h_{21}(\check{x}_{1,k}) - \\ & & h_{22}(\check{x}_{2,k})); \quad \check{x}_2(k_0) = \bar{x}_{20}, \\ \check{z}_k & = & y_k - h_{21}(\check{x}_{1,k}) - h_{22}(\check{x}_{2,k}). \end{array} \right.$$

However, this filter would fall into the class of decomposition filters, rather than aggregate, and because of this, we shall not discuss it further in this section.

In the next section, we consider an example.

5 Examples

Consider the following singularly-perturbed nonlinear system

$$\begin{array}{rcl} x_{1,k+1} & = & x_{1,k}^{\frac{1}{3}} + x_{2,k}^{\frac{1}{2}} + w, \\ \varepsilon x_{2,k+1} & = & -x_{2,k}^{\frac{1}{2}} - x_{2,k}^{\frac{1}{3}}, \\ y_k & = & x_{1,k} + x_{2,k} + w, \end{array}$$

where $w \in \ell_2[0, \infty)$ is a noise process, $\varepsilon \geq 0$. We construct the aggregate filter $\mathbf{F}_{3ag}^a$ presented in the previous section for the above system. It can be checked that the

system is locally observable, and with $\gamma = 1$, the function $\check{V}(\grave{x}) = \frac{1}{2}(\grave{x}_1^2 + \varepsilon\grave{x}_2^2)$, solves the inequality form of the DHJIE (54) corresponding to system. Subsequently, we calculate the gains of the filter as

$$\grave{L}_1(\grave{x}, y) = -\frac{(y - \grave{x}_1 - \grave{x}_2)}{\grave{x}_1^{\frac{1}{3}} + \grave{x}_2^{\frac{1}{2}}}, \quad \grave{L}_2(\grave{x}, y) = \frac{(y - \grave{x}_1 - \grave{x}_2)}{\grave{x}_2^{\frac{1}{2}} + \grave{x}_2^{\frac{1}{3}}}, \tag{64}$$

where the gains $\grave{L}_1$, $\grave{L}_2$ are set equal to zero if $\|\check{x}\| < \epsilon$ (small) to avoid the singularity at the origin $\grave{x} = 0$.

6   Conclusion

In this paper, we have presented a solution to the $\mathcal{H}_\infty$ filtering problem for discrete-time affine nonlinear singularly-perturbed systems. Two classes of filters, namely, decomposition and aggregate filters, have been discussed, and in each case, first-order approximate filters have been presented. Reduced-order filters have also been derived as limiting cases of the above filters as the singular parameter $\varepsilon \downarrow 0$. Sufficient conditions for the solvability of the problem using each filter have been given in terms of DHJIEs. The results have also been specialized to linear systems, in which case, the sufficient conditions reduce to a system of matrix-inequalities or LMIs which are computationally efficient to solve. In addition, an example has been presented to illustrate the approach.

Future efforts would concentrate in finding an explicit form for the coordinate transformation discussed in Section 3, and developing computationally efficient algorithms for solving the DHJIEs.

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