Nonlinear Dynamics and Systems Theory, **11** (4) (2011) 383–396

Optical Soliton in Nonlinear Dynamics and Its Graphical Representation

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Received: January 17, 2011 ; Revised: September 22, 2011

Abstract: The soliton arising from a robust balance between dispersion and nonlinearity is the solitary wave that maintains its shape while it travels at constant speed. The fiber Optical soliton in media and communication with quadratic nonlinearity and frequency dispersion are theoretically analyzed. The behavior of soliton solutions in the form of KdV partial differential equation have been investigated in the fiber optics solitons theory in communication engineering. In this study optical soliton is studied with illustrated graphical representation.

Keywords: *soliton solution, Korteweg-de Vries equation, Gaussian white noise, stochastic KdV equation, Fourier transform, nonlinear dynamics.*

Mathematics Subject Classification (2000): 35C08, 37K40, 35Q51.

1 Introduction

In recent years there have been important and tremendous developments in the study of nonlinear waves and a class of nonlinear wave equations which arise frequently in many engineering applications. The wide interest in this field comes from the understanding of special waves called *solitons* and the associated development of a method of solution to a class of nonlinear wave equations termed as the nonlinear Korteweg and de Vries (KdV) equation. A soliton phenomenon is an attractive field of present day research not only in nonlinear physics and mathematics but also in nonlinear dynamics and system engineering, specially in fiber optics and communication engineering. The soliton phenomenon was first pioneered by John Scott Russel in 1884, while he was conducting experiments on the Union Canal (near Edinburgh) to measure the relationship between the speed of a boat and its propelling force. Russel demonstrated the following findings as an independent dynamic entity moving with constant shape and speed:

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- (i) Solitary waves have the shape $h \sec h^2[k(x - vt)]$;
- (ii) A sufficiently large initial mass of water produces two or more independent solitary waves;
- (iii) Solitary waves cross each other without change of any kind;
- (iv) A wave of height h and travelling in a channel of depth d has a velocity given by the expression $v = \sqrt{g(d + h)}$ (where g is the acceleration of gravity) implying that a large amplitude solitary wave travels faster than one of low amplitude.

In 1895, Korteweg and de Vries published a theory of shallow water waves that reduced Russell's problem to its essential features (see [10] for details). However, the paper by Korteweg and de Vries was one of the first theoretical treatment in the soliton solution and thus a very important milestone in the history of the development of soliton theory. Another development of the 1960s was Toda's discovery of exact two-soliton interactions on a nonlinear spring-mass system (see for example [23]). The brief discussion of mathematical representation of soliton begins with the Wadati's paper published in 1983 [24]. Russel L. Herman a famous Mathematician improved soliton theory and found some improved results which represent a final solution of soliton [8]. Basically Wadati and Herman both used a non-linear third order partial differential equation known as Korteweg-de Vries equation, they started from this equation and finally gave mathematical assumption of soliton with graphical representation. We refer readers to [5, 7, 9, 10, 13, 17, 20, 23, 25] and the references therein for the detail studies about the history of soliton theory and KdV partial differential equation which is the basic foundation of solitons.

One of the active area of applications of solitons is fiber optics. Much experimentation has been done using solitons in fiber optics applications. In 1973, Robin Bullough [4] showed that solitons could exist in optical fibers while he was presenting the first mathematical report of the existence of optical solitons. He also proposed the idea of a soliton-based transmission system to increase performance of optical telecommunications. Now soliton is an essential tool in communication engineering. Recently the fiber optical soliton is dominating to the global telecommunication research by super performance data transmission in a long distances. See for examples [14,18,19] for more studies on solitons in communication systems. There are varieties of nonlinear equations representing the solitons in the nonlinear domain such as, general equal width wave equation (GEWE), general regularized long wave equation (GRLW), general Kortewegde Vries equation (GKdV), general improved Korteweg-de Vries equation (GIKdV), and Coupled equal width wave equations (CEWE), which are the important soliton equations. See for examples [1, 2] for more details about different varieties. Our aim is to investigate some aspects of Kortewegde Vries equation in soliton physics specially for the case of optical soliton in nonlinear dynamical systems in mathematical physics. We also provide an illustration with some graphical representations.

2 Analysis and Formulation of Soliton Solution

We consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where $u = u(x,t)$ which describes the elongation of the wave at the place x at time t. This equation is known as the KdV equation first derived in 1885 by Korteweg and de Vries to describe long-wave propagation on shallow water. But until recently its properties were not well understood [13]. However, the nonlinear shallow water wave equation can be written in the form

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\eta\frac{\partial \eta}{\partial x} + \frac{2}{3}\frac{\partial \eta}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3 \eta}{\partial x^3}}, \tag{2}$$

where $\sigma = \frac{h^3}{3} - \frac{Th}{g\rho}$, h is the channel height, T is the surface tension, g is the gravitational acceleration and ρ is the density.The solutions to (1) are called *Solitons* or *Solitary waves.*

The nondispersive nature of the soliton solutions to the KdV equation arises not because the effects of dispersion are absent but because they are balanced by nonlinearities in the system. The presence of both phenomena can be appreciated by considering simplified versions of the KdV equation which can be calculated by eliminating the *nonlinear term* $u\frac{\partial u}{\partial x}$ as

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

The equation (3) is now a linear version and the most elementary wave solution of this equation, called the harmonic wave is given by

$$u(x,t) = A\exp|i(kx + \omega t)|, \tag{4}$$

where k is the wave number and ω is the angular frequency. In order for the displacement $u(x,t)$ presented in equation (4) to be a solution of equation (3), ω and k must satisfy the relation

$$\omega = k^3. \tag{5}$$

The relation (5) is known as dispersion relation and it contains all the characteristics of the original differential equation. Two important concepts connected with the dispersion relation are called the *phase velocity* $v_p = \frac{\omega}{k}$ and the *group velocity* $v_g = \frac{\partial \omega}{\partial k}$. The phase velocity measures how fast a point of constant phase is moving, while the group velocity measures how fast the energy of the wave moves. The waves described by equation (3) are said to be dispersive because a wave with large k will have larger phase and group velocities than a wave with small k.

Now eliminating the *dispersive term* $\frac{\partial^3 u}{\partial x^3}$, we obtain the simple nonlinear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{6}$$

admitting the wave solution in the form $u(x,t) = f(x - ut)$, where the function f is arbitrary. For such kind of waves, the important thing to note is that the velocity of a point of constant displacement u is equal to that displacement. As a result, the wave breaks; that is, portions of the wave undergoing greater displacements move faster than, and therefore overtake, those undergoing smaller displacements. This multivaluedness is a result of the nonlinearity and, like dispersion, leads to a change in form as the wave propagates. We refer readers to [6, 11, 15, 16, 21, 22] for further readings as well as recent developments on nonlinear dynamics and stability analysis in the system theory.

2.1 Mathematical derivation of KdV

We recall that KdV equation is the basic foundation of soliton solution. So we present here the brief sketch of calculation for the derivation of KdV equation in the form (1). In order to derive (1), we consider another nonlinear partial differential equation, called *Kadomtsev-Petviashvili equation* (or simply the KP equation) in two spatial and one temporal coordinate which describes the evolution of nonlinear, long waves of small amplitude with slow dependence on the transverse coordinate (see for details [3]).The normalized form of the equation is as follows:

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) \pm u_{yy} = 0, \tag{7}$$

where u_t, u_x, u_{xxx} and u_{yy} stand for the partial derivatives $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^3 u}{\partial x^3}$ and $\frac{\partial^2 u}{\partial y^2}$ respectively. The equation (7) can be calculated as

$$u_{xt} = u_{xxx} + 3u_{yy} - 6u_y u_{xx} - 6{u_x}^2 u_{xx}. \tag{8}$$

Thus the equation (8), after making a detailed calculation can be simplified as

$$u_t + 6uu_x + u_{xxx} = \xi(t). \tag{9}$$

Here $\xi(t)$ represents a time dependent *Gaussian white noise* . The stochastic process is called Gaussian white noise if its statistical average is zero i.e.; $\langle \xi(t) \rangle = 0$. See [7] for more details about Gaussian white noise.

Now a relation between two covariance functions in terms of Gaussian white noise is given by

$$\xi(t)\xi(T + t) = \sigma^2 \delta(t). \tag{10}$$

For the Fourier transformation of stationary two times covariance function we obtain

$$\begin{aligned} F(\omega) &= \int dt \langle \xi(t)\xi(T + t) \rangle e^{i\omega T} \\ \Longrightarrow F(\omega) &= \sigma^2 \int dt \delta(t) e^{i\omega T} \\ \Longrightarrow F(\omega) &= \sigma^2. \end{aligned} \tag{11}$$

In other words, it is clear from the above that it does not depend upon ω because there is no co-relation in time. This is why it is called white noise.

Now for simplicity, let us assume a one-dimensional stochastic differential equation with additive noise,

$$\frac{dx(t)}{dt} = a(x(t), t) + \eta(t). \tag{12}$$

Here $a(x(t), t)$ is a Langiven (see for example [12]) equation which can be interpreted as a deterministic or average drift term perturbed by a noisy diffusion term $\xi(t)$. For the increase dx during a time step dt, we get

$$\begin{aligned} dx(t) &= a(x(t), t)dt + d\omega(t), \\ d\omega(t) &= \int_t^{t+dt} \eta(t')dt' \end{aligned} \tag{13}$$

and also we assume that

$$\begin{aligned} d\omega(t^2) &= \int_t^{t+dt} dt_1 \int_t^{t+dt} dt_2 \langle \eta(t_1)\eta(t_2)\rangle \\ &= \int_t^{t+dt} dt_1 \int_t^{t+dt} dt_2 \sigma^2 \delta(t-t') \\ &= \sigma^2 dt. \end{aligned} \tag{14}$$

Thus in the intervals $[t, t+dt]$ and $[t', t'+dt']$ which is a true successive step, we get

$$\langle d\omega\omega(t) d\omega'\rangle = 2\sigma\sigma(t-t'). \tag{15}$$

If $\delta = \varepsilon$ then in general we can write

$$\langle \xi(t)\xi(t')\rangle = 2\sigma\sigma(t-t'). \tag{16}$$

For such time dependent noise the stochastic equation can be transformed into unperturbed KdV equation [10] in the form

$$U_T + 6UU_X + U_{XXX} = 0. \tag{17}$$

Let us now introduce the Galilean transformation

$$\begin{cases} u(x,t) = U(X,T) + \omega(T), \\ X = x + m(t), \\ T = t, \\ m(t) = -\sigma \displaystyle\int_0^t \omega(t')dt'. \end{cases} \tag{18}$$

Under the above transformation we have from calculus that the derivatives transform as

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} = \frac{\partial}{\partial X}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T} = -\sigma\omega(T)\frac{\partial}{\partial X} + \frac{\partial}{\partial T}. \tag{19}$$

Now using this transformation we have

$$\begin{aligned} \varepsilon(t) &= u_t + 6uu_x + u_{xxx} \\ &= (U+\omega)_T - 6\omega U_X + 6(U+\omega)U_X + U_{XXX} \\ &= U_T + 6UU_X + U_{XXX} + \omega_T. \end{aligned} \tag{20}$$

Let us now define

$$\xi = \omega_T \quad \text{or} \quad \omega(t) = \int_0^t \xi(t')dt'$$

which leads to the KdV equation.

A remarkable property of the KdV equation is that dispersion and nonlinearity balance each other and allow wave solutions that propagate without changing its form. An example of such a solution is one-soliton solution.

We next consider one-soliton solution. In such case, let us consider

$$U(X,T) = 2\eta\eta \sec^2(\eta(X - 4\eta^2 T - X_0)). \tag{21}$$

Then the above mentioned transformation leads directly to an exact solution of stochastic KdV equation

$$\begin{aligned} u(x,t) &= 2\eta^2 \sec h^2(\eta(x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt')) + \omega(t) \\ \Longrightarrow \langle u(x,t)\rangle &= 2\eta^2 \langle \sec h^2(\eta\eta(x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'))\rangle. \end{aligned} \tag{22}$$

Formally we can write

$$\sec h^2 z = \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2}. \tag{23}$$

Then according to [24], it can be presented by computing

$$\begin{aligned} \langle u(x,t)\rangle &= 8\eta^2 \textstyle\sum_{n=1}^{\infty}(-1)^{n+1}\langle \exp[2n\eta(x - 4\eta^2 t - x_0 - 6\displaystyle\int_0^t \omega(t')dt'))\rangle \\ &= -2\frac{d}{dz}\frac{1}{1 + e^{2z}} \\ &= -2\frac{d}{dz}\Big(\sum_{n=0}^{\infty}(e^{2z})^n\Big) = 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{2nz}. \end{aligned} \tag{24}$$

In order to complete this composition some following useful relations (we are omitting the details) are needed. We have

- $\langle \omega(t)\rangle = 0$,
- $\langle \omega(t_1)\omega(t_2)\rangle = 2\varepsilon \min(t_1, t_2)$,
- $\langle \exp(c\omega(t))\rangle = \exp\Big(\frac{1}{2}c^2\langle \omega^2(t)\rangle\Big)$.

Applying this result, we get

$$\begin{aligned} \langle \exp\Big(\pm 12n\eta\int_0^t \omega(t')dt'\Big)\rangle &= \exp\Big(72n^2\eta^2\int_0^t\int_0^t \langle \omega(t_1)\omega(t_2)dt_1dt_2\rangle\Big) \\ &= \exp(48n^2\eta^2\varepsilon t^3). \end{aligned} \tag{25}$$

This leads to the following form

$$\begin{aligned} &\langle u(x,t)\rangle = 8\eta^2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{na + nb^2}, \\ &a = 2\eta(x - x_0 - 2\eta^2 t), \quad b = 48\eta^2\varepsilon t^3. \end{aligned} \tag{26}$$

In principle this result should be sufficient but we will go further to find analytically that gives an expression to this result.

Now differentiating the series with respect to a and b we obtain the partial differential equation

$$w_b = w_{aa}, \quad \text{where} \quad w(a,b) = \langle u(x,t)\rangle.$$

Furthermore we have $w(a,0) = 2\eta^2 \sec h^2 \frac{a}{2}$.

This is an initial value problem for the heat or diffusion equation on the real line. To solve this problem we use Fourier transformation. The Fourier transform is defined by

$$\tilde{w}(k,b) = \int_{-\infty}^{\infty} w(a,b) e^{-iak} da,$$

and the inverse transform is

$$w(a,b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}(k,b) e^{iak} dk.$$

The heat equation leads to the simple initial value problem

$$\tilde{w}_b = -k^2 \tilde{w} \quad \text{where} \quad \tilde{w}(k,0) = 2\eta^2 \int_{-\infty}^{\infty} \sec h^2 \frac{a}{2} e^{-iak} da = 8\eta^2 \frac{\pi k}{\sinh \pi k}.$$

Therefore, we obtain

$$\tilde{w}(k,b) = 8\eta^2 \frac{\pi k}{\sinh \pi k} e^{-bk^2}. \tag{27}$$

Thus the solution is found out from inverse Fourier transform as

$$u(x,t) = \frac{4\eta^2}{\pi} \int_{-\infty}^{\infty} \frac{\pi k}{\sinh \pi k} e^{iak - bk^2} dk. \tag{28}$$

However, the technique followed in [24] asserts that this simply can be calculated using the convolution theorem. Namely we note that

$$\tilde{w}(k,b) = \tilde{f}(k)\tilde{g}(k,b) \quad \text{for} \quad \tilde{f}(k) = 8\eta^2 \frac{\pi k}{\sinh \pi k} \quad \text{and} \quad \tilde{g}(k,b) = e^{-bk^2}.$$

The inverse transforms for these expressions are given by

$$f(a) = 2\eta^2 \sec h^2 \frac{a}{2} \quad \text{and} \quad g(a,b) = \frac{1}{\sqrt{4\pi b}} e^{\frac{a^2}{4b}}.$$

The last expression is just the statement for Fourier transformation of a Gaussian. Now from the Gaussian Convolution of the functions, we have

$$\begin{aligned} \langle u(x,t) \rangle = w(a,b) = (f * g)(a) &= \int_{-\infty}^{\infty} f(s) g(a-s) ds \\ &= \int_{-\infty}^{\infty} \Big(2\eta^2 \sec h^2 \frac{s}{2}\Big)\Big(\frac{1}{\sqrt{4\pi b}} e^{\frac{-(a-s)^2}{4b}}\Big) ds \\ &= \frac{\eta^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} e^{\frac{-(a-s)^2}{4b}} \sec h^2 \frac{s}{2} ds. \end{aligned} \tag{29}$$

This is the exact solution of stochastic KdV equation which we will now compare to any simulation results. The result of simulation solution of stochastic KdV equation is given in Figure 1.

Most of the focuses of any simulations are with respect to the asymptotic results that Wadati derived from the above solution. We will discuss the graphical representations of the behavior of soliton solution in two cases, namely, for small times and for large times.

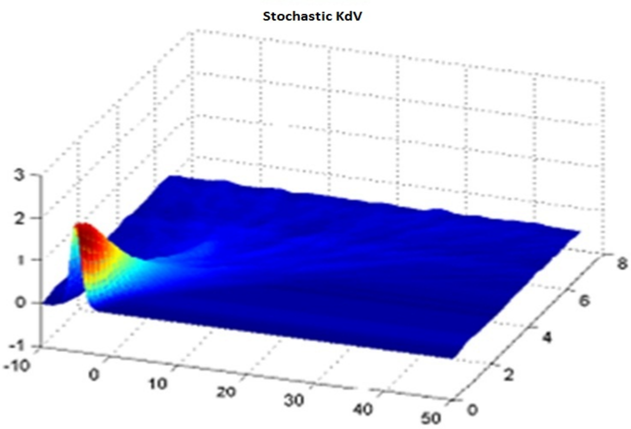

Figure 1: This graph represents the solution generated by doing a simulation of the stochastic KdV equation.

Case I: *For small times* $(e.g., b = 48\eta^2\varepsilon t^2 < 1)$.
In this case, it is a simple matter to show that

$$\langle u(x,t)\rangle = 2\eta^2 \sum_{0}^{\infty} \frac{b^n}{n!}\frac{\partial^{2n}}{\partial a^{2n}} \sec h^2 \frac{a}{2}. \tag{30}$$

Now using the equation(30) the numerically illustrated exact solution is given below in Figure 2.

We are now in a position to present here a comparison result between the simulation result of the stochastic KdV equation shown in Figure 1 and the result of exact solution shown in Figure 2. The comparison result is shown in the following Figure 3.

Case II: *For large times* $(e.g., b = 48\eta^2\varepsilon t^2 > 1)$.
In this case, the solution can be calculated as

$$\langle u(x,t)\rangle = \frac{4\eta^2}{\sqrt{\pi}}\Big(1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2)bB_n\pi^{2n}}{(2n)!}\frac{\partial^n}{\partial b^n}\Big)\frac{e^{\frac{-a^2}{4b}}}{\sqrt{b}} \tag{31}$$

and also the numerical simulation for this case is presented in Figure 4.

Now we will focus on the case when $t \longrightarrow \infty$ and the result can be approximated as

$$\langle u(x,t)\rangle \approx \frac{\eta}{\sqrt{3\pi\varepsilon}}\frac{1}{\sqrt[2]{t^3}} \exp\Big(-\frac{(x-x_0-4\eta^2 t)^2}{48\varepsilon t^3}\Big) \tag{32}$$

and the simulation result of equation (32) is presented in Figure 5.

Once again we present in Figure 6, the amplitude of the two solutions shown in the Figure 4 and in Figure 5.

Thus we end this section providing an extensive graphical illustrations about the shape and behavior of soliton solutions derived from the KdV equation. We also present

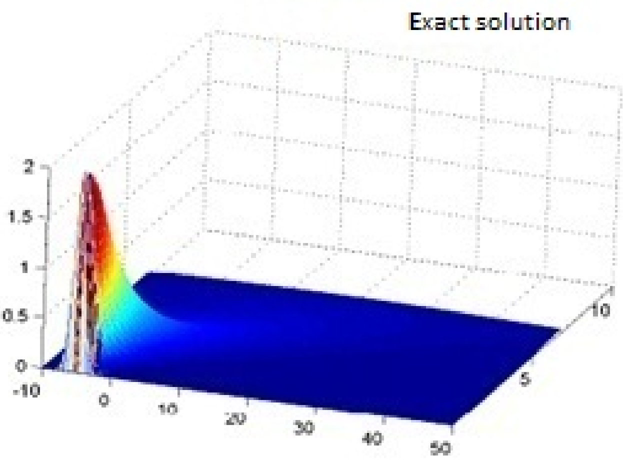

Figure 2: This graph represents the exact solution of the equation (30).

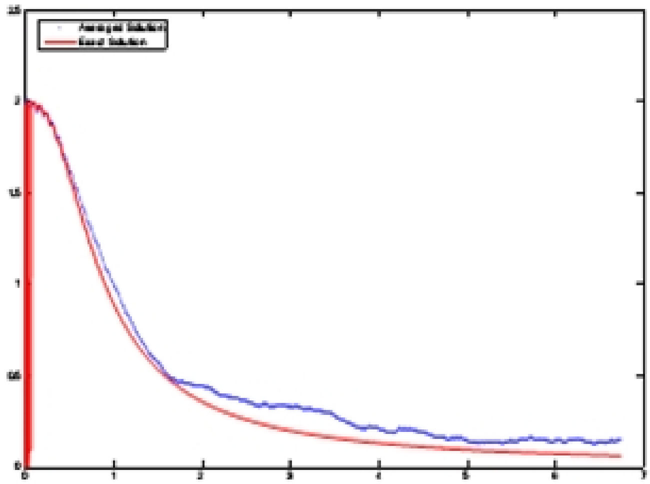

Figure 3: The physical graph of comparison of the amplitudes from the exact solution and a simulation solution.

the comparison result derived from the exact solution of stochastic KdV equation as well as the simulation result.

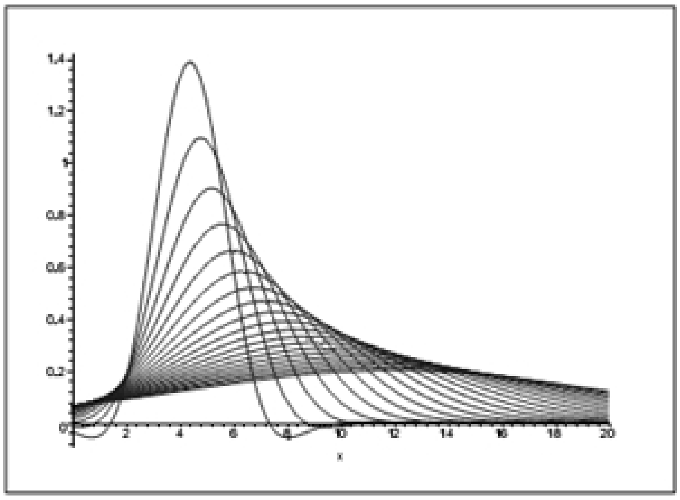

Figure 4: The graph for large times based on the solution using equation (31).

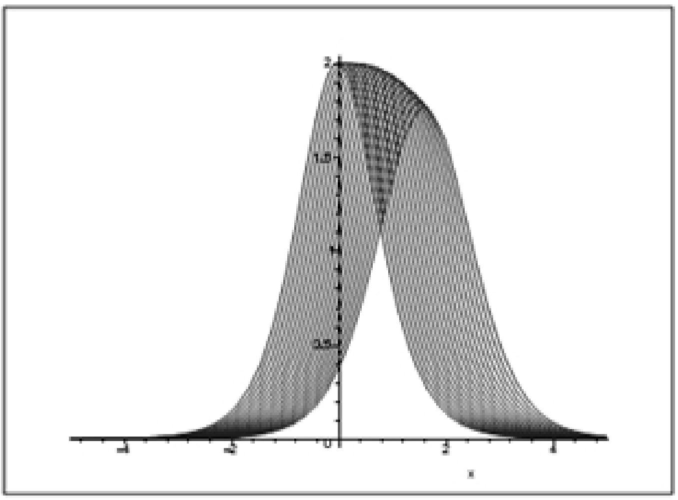

Figure 5: The solution for large times based upon equation (32).

3 A Particular Problem and Solution

In this section we will discuss the solution of a particular problem.There are several problems with Wadati's derivation. These also appear elsewhere in the literature references to Wadati's paper [24]. Here we discuss the alternatives of several shortcomings to Wadati's derivation by this particular problem.

First, we note that the series expansion for the $\sec h^2 z$ is not quite right. We should

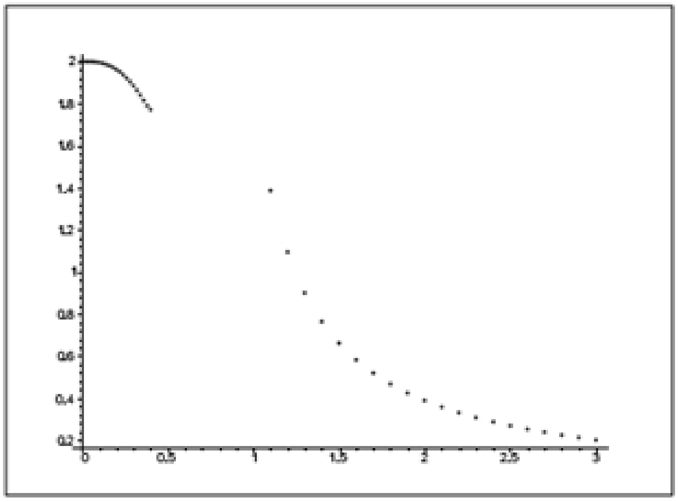

Figure 6: The amplitude of the solutions provided in the Figure 4 and Figure 5.

instead have derived it as follows (for $z \neq 0$)

$$\begin{aligned} \sec h^2 z &= \frac{4}{(e^z + e^{-z})^2} = \frac{4e^{-2|z|}}{(1 + e^{-2|z|})^2} \\ &= 2 \text{ sgn } \frac{d}{dz}\Big(\sum_{n=0}^{\infty}(-e^{-2|z|})^n\Big) = 2\sum_{n=1}^{\infty}(-1)^{n+1} n e^{-2n|z|} \end{aligned} \tag{33}$$

This accounts for the convergence of the geometric series used in the derivation. Namely, in the original derivation, one should have noted that $|e^{2z}| < 1$ or $z < 0$.

This new derivation accounts for the case $z > 0$. Konotop and Vazquez [9] used this in their review of Wadati's derivation. They presented the infinite series result as

$$\langle u(x,t)\rangle = 8\eta^2 \sum_{n=0}^{\infty}(-1)^{n+1} n e^{-n|a|+n^2 b}. \tag{34}$$

There also appeared to be a problem with the derivation of the average, where Wadati should actually have computed

$$\langle u(x,t)\rangle = 8\eta^2 \sum_{n=0}^{\infty}(-1)^{n+1} n \Big\langle \exp\Big(- 2n\eta|x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'|\Big)\Big\rangle. \tag{35}$$

One could get around this problem by computing the average for space-time regions where $x - 4\eta^2 t - x_0 - 6\int_0^t \omega(t')dt'$ is definitely of one sign. Another approach would instead be directly expanded as

$$u(x,t) = 2\eta^2 \sec h^2\Big(\eta(x - 4\eta^2 t - x_0 - 4\eta^2 t) - 6\eta\int_0^t \omega(t')dt'\Big) = 2\eta^2 \sec h^2(\theta + \sigma). \tag{36}$$

In the case when $\sigma = 0$ for $\sigma = -6\eta \int_0^t \omega(t')dt'$, we have

$$2\eta^2 \sec h^2(\theta + \sigma) = 2\eta^2 \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^n}{\partial \theta^n} \sec h^2 \theta. \tag{37}$$

The average can now be computed as

$$\langle u(x,t)\rangle = 2\eta^2 \sum_{n=0}^{\infty} \frac{\langle \sigma^n \rangle}{n!} \frac{\partial^n}{\partial \theta^n} \sec h^2 \theta \tag{38}$$

provided that we can compute $\langle \sigma^n \rangle$ as

$$\langle \sigma^n \rangle = \Big\langle \Big(-6\eta \int_0^t \omega(t')dt' \Big)^n \Big\rangle. \tag{39}$$

Herman [8] showed that such averages can be computed based upon the nature of the Gaussian noise as

$$\langle \sigma^n \rangle = \begin{cases} 0, & \text{when n is odd,} \\ (2l-1)!\langle \sigma^2 \rangle, & \text{when n=2l is even.} \end{cases}$$

Thus, we just need to compute $\langle \sigma^2 \rangle$ which can be completed by the following calculation

$$\begin{aligned} \langle \sigma^2 \rangle &= \Big\langle 36\eta^2 \int_0^t \omega(t_1)dt_1 \int_0^t \omega(t_2)dt_2 \Big\rangle = 72\varepsilon\eta^2 \int_0^t \int_0^t \min(t_1,t_2)dt_1dt_2 \\ &= 72\varepsilon\eta^2 \int_0^t \Big(\int_0^{t_2} \min(t_1,t_2)dt_1 + \int_{t_2}^t \min(t_1,t_2)dt_1 \Big) dt_2 \\ &= 72\varepsilon\eta^2 \int_0^t \Big(\frac{{t_2}^2}{2} + t_2(t-t_2) \Big) dt_2 = 24\varepsilon\eta^2 t^3. \end{aligned} \tag{40}$$

Now inserting this result in equation (38) we obtain

$$\langle u(x,t)\rangle = 2\eta^2 \sum_{t=0}^{\infty} \frac{\langle 12\varepsilon\eta^2 t^3 \rangle}{l!} \frac{\partial^{2l}}{\partial \theta^{2l}} \sec h^2 \theta. \tag{41}$$

In order to see the agreement with Wadati's result for small $b = 48\varepsilon\eta^2 t^3$, we need to set $\theta = \frac{a}{2}$. We also note that $\frac{\partial^{2l}}{\partial \theta^{2l}} = 2^{2l} \frac{\partial^{2l}}{\partial a^{2l}}$. Thus we obtain

$$\langle u(x,t)\rangle = 2\eta^2 \sum_{t=0}^{\infty} \frac{\langle 48\varepsilon\eta^2 t^3 \rangle}{l!} \frac{\partial^{2l}}{\partial a^{2l}} \sec h^2 \frac{a}{2} = 2\eta^2 \sum_{t=0}^{\infty} \frac{b^l}{l!} \frac{\partial^{2l}}{\partial a^{2l}} \sec .h^2 \frac{a}{2} \tag{42}$$

We further note that this solution again satisfies the heat equation and that for $b = 0$ this solution reduces to the soliton initial condition. Thus, we have seemingly bypassed any problem with the computing the average with an absolute value. However, this series is divergent for $b > 1$.

4 Conclusion

Soliton theory has been a challenging area of research over the years, especially since the mid-1970s, the soliton concept has become established in several areas of applied science and its applications in the diverse fields of science and engineering such as nonlinear analysis, water waves, relativistic and quantum field theory, control and system theory as well as electrical and communication engineering have made this theory more attractive. In this study the soliton solution and some of its large scale applications are studied with simulations. The mathematical derivation of soliton is shown by using the Korteweg-de Vries equation and Kadomtsev–Petviashvili equation in the form of nonlinear partial differential equation. It is necessary to mention, however, that not all nonlinear partial differential equations have soliton solutions. Those that do are generic and belong to a class for which the general initial-value problem can be solved by a technique called the inverse scattering transform, a brilliant scheme developed by Kruskal and his coworkers in 1965. We also investigate the result of the solution which is generated by doing a simulation of the stochastic KdV equation and the exact solution and a comparison between them is shown graphically. Also the solutions based on small and large times are represented graphically.

Acknowledgment

The authors would like to thank the reviewers for the careful reading of our paper and their constructive suggestions which help us in the further modifications of this paper. The logistic support of Mathematics Discipline of Khulna University, Bangladesh is also acknowledged.

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