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A Common Fixed Point Theorem for a Sequence of Self Maps in Cone Metric Spaces

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Abstract: In this paper, we obtain a new common fixed point theorem by using a new contractive condition in cone metric spaces. Our result generalizes and extends well known result in complete metric spaces.

Keywords: *cone metric spaces; common fixed point; sequence; normal.*

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1 Introduction

The study of fixed points of functions satisfying certain contractive conditions has been at the center of vigorous research activity, for example see [1]–[5] and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, see [6]–[10]. Recently, Huang and Zhang [11] have replaced the real numbers by ordering Banach space and define cone metric space. They have proved some fixed point theorems of contractive mappings on cone metric spaces. The study of fixed point theorems in such spaces is followed by some other mathematicians, see [12]–[16]. Choudhury [17] introduced mutually contractive sequence of self maps and proved a fixed point theorem. The purpose of this paper is to obtain a new common fixed point theorem by using a new contractive condition in cone metric spaces. Our result generalizes and extends many known results in metric spaces.

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Consistent with Huang and Zhang [11], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{\theta\}$;

(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \| x \| \leq K \| y \|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).

Definition 1.1 [11] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.1 [11] Let $E = R^2, P = \{(x, y) \in E | x, y \geq 0\} \subset R^2, X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.2 [11] Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:

(e) a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n, m > N, d(x_n, x_m) \ll c$;

(f) a Convergent sequence if for every $c \in E$ with $\theta \ll c$, there is an N such that for all $n > N, d(x_n, x) \ll c$ for some fixed $x \in X$.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is know that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. The limit of a convergent sequence is unique provided that P is a normal cone with normal constant K[11].

Lemma 1.1 [*11*] *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta (n, m \to \infty)$.*

Lemma 1.2 [*11*] *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.*

Definition 1.3 Let (X, d) be a cone metric space. A sequence $\{T_i\}_{i=1}^{\infty}$ of self-mappings on a complete cone metric space is said to be mutually contractive if for all $i, j = 1, 2, \cdots,$ with $i \neq j$,

$$d(T_i x, T_j y) \leq k d(x, y) \quad \text{for all} \quad x, y \in X \quad \text{with} \quad x \neq y,$$

where $k \in (0, 1)$ is a constant.

2 Main Result

Theorem 2.1 *Let (X,d) be a complete cone metric space. P be a normal cone with normal constant K. $\{T_i\}_{i=1}^{\infty}$ be a sequence of self-mappings on X such that*
(1) T_i is continuous for all $i,j=1,2,\cdots$;
(2) $\{T_i\}_{i=1}^{\infty}$ is mutually contractive;
(3) $T_iT_j=T_jT_i$ for all $i,j=1,2,\cdots$.
Then the sequence $\{T_n\}_n$ has a unique common fixed point in X.

Proof Let x_0 be an arbitrary point in X. We construct a sequence $\{x_n\}\subset X$ as follows:

$$x_1=T_1x_0, x_2=T_2x_1,\cdots,x_n=T_nx_{n-1},\cdots$$

Then the following cases may arise:.

Case I: If no terms of $\{x_n\}$ are equal. Then, using (2), we get:

$$d(x_n,x_{n+1})=d(T_nx_{n-1},T_{n+1}x_n)\le kd(x_{n-1},x_n).$$

By repeated application of above inequalities, we get

$$d(x_n,x_{n+1})=d(T_nx_{n-1},T_{n+1}x_n)\le k^nd(x_0,x_1).$$

So for $n>m$, we have

$$\begin{aligned}d(x_n,x_m)&\le d(x_n,x_{n-1})+\cdots+d(x_{m+1},x_m)\\&\le (k^{n-1}+\cdots+k^m)d(x_0,x_1)\le \frac{k^m}{1-k}d(x_0,x_1)\end{aligned}$$

We get $\|d(x_n,x_m)\|\le \frac{k^m}{1-k}K\|d(x_0,x_1)\|$. This implies $d(x_n,x_m)\to\theta(n,m\to\infty)$. Hence x_n is a Cauchy sequence by Lemma 1.1. By the completeness of X, there is $x^*\in X$ such that $x_n\to x^*(n\to\infty)$. Now, we prove that x^* is a fixed point of T_i.

Since two consecutive terms of $\{x_n\}$ are unequal, for an arbitrary integer $i>0$ and $c\gg\theta$, we can find n such that $x^*\ne x_{n-1}, n>i$,

$$d(x^*,x_n)<c, \quad\text{and}\quad d(x^*,x_{n-1})<c.$$

Then, we get

$$\begin{aligned}d(x^*,T_ix^*)&\le d(x^*,x_n)+d(x_n,T_ix^*)\\&=d(x^*,x_n)+d(T_nx_{n-1},T_ix^*)\\&\le d(x^*,x_n)+kd(x_{n-1},x^*).\end{aligned}$$

Thus, $\|d(x^*,T_ix^*)\|\le K(\|d(x^*,x_n)\|+k\|d(x_{n-1},x^*)\|)\to 0$ since $c\gg\theta$ is arbitrary. Hence $\|d(x^*,T_ix^*)\|=0$. This implies $x^*=T_ix^*$. So, x^* is a fixed point of T_i.

Case II: If $x_i=x_{i-1}$ for some positive integer i. Then $x_{i-1}=T_ix_{i-1}$. Let $x^*=x_{i-1}$, that is, $x^*=T_ix^*$, $x^*\ne T_jx^*$ and further assume that $x^*\ne T_j^nx^*$ for all $n=1,2,\cdots$.

Thus, we get

$$d(x^*,T_j^2x^*)=d(T_ix^*,T_j(T_jx^*))\le kd(x^*,T_jx^*).$$

Similarly,

$$d(x^*, T_j^3 x^*) \leq k^2 d(x^*, T_j x^*).$$

Consequently,

$$d(x^*, T_j^n x^*) \leq k^{n-1} d(x^*, T_j x^*) \text{ for all } n = 2, 3, \cdots.$$

We get $\|d(x^*, T_j^n x^*)\| \leq k^{n-1} K \|d(x^*, T_j x^*)\|$. This implies $d(x^*, T_j^n x^*) \to \theta$ as $n \to \infty$, that is

$$T_j^n x^* \to x^* \text{ as } n \to \infty.$$

Since T_i is continuous, we get

$$T_j(T_j^n x^*) = T_j^{n+1} x^* \to T_j x^* \text{ as } n \to \infty.$$

In the view of Lemma 1.2, we have $x^* = T_j x^*, j = 1, 2, \cdots$. This is a contradiction, so $x^* = T_j^l x^*$ for some l.

Let l be the smallest integer with this property. Then, we get

$$x^* \neq T_j^m x^* \text{ for some } m = 1, 2, \cdots, l-1.$$

Thus,

$$\begin{aligned} d(x^*, T_j^{l-1} x^*) &= d(T_i x^*, T_j(T_j^{l-2} x^*)) \leq k d(x^*, T_j^{l-2} x^*) \\ &= k d(T_i x^*, T_j(T_j^{l-3} x^*)) \leq k^2 d(x^*, T_j^{l-3} x^*) \leq \cdots \leq k^{l-2} d(x^*, T_j x^*), \end{aligned}$$

hence $x^*, T_j x^*, T_j^2 x^*, \cdots, T_j^{k-1} x^*$ are all distinct. Therefore,

$$\begin{aligned} d(x^*, T_j x^*) &= d(T_j^l x^*, T_j(T_i x^*)) = d(T_j(T_j^{l-1} x^*), T_i(T_j x^*)) \\ &\leq k d(T_j^{l-1} x^*, T_j x^*) = k d(T_j(T_j^{l-2} x^*), T_i(T_j x^*)) \\ &\leq k^2 d(T_j^{l-2} x^*, T_j x^*) \leq \cdots \leq k^{l-2} d(T_j^2 x^*, T_j x^*) \\ &= k^{l-2} d(T_j^2(T_i x^*), T_j x^*) = k^{l-2} d(T_i(T_j^2 x^*), T_j x^*) \\ &\leq k^{l-1} d(T_j^2 x^*, x^*) = k^{l-1} d(T_j(T_j x^*), T_i x^*) \leq k^l d(T_j x^*, x^*). \end{aligned}$$

Hence $\|d(x^*, T_j x^*)\| = 0$ and $x^* = T_j x^*$ for all $j = 1, 2, \cdots$.

To show uniqueness, assume y^* is another common fixed point of T_i, then

$$d(x^*, y^*) = d(T_i(x^*), T_j(y^*)) \leq k d(x^*, y^*).$$

Hence $\|d(x^*, y^*)\| = 0$ and $x^* = y^*$, that is, x^* is a unique common fixed point of the sequence $\{T_n\}_n$. $\square$

Remark 2.1 Let us remark that in Theorem 2.1, setting $E = R, P = [0, +\infty), \|x\| = |x|, x \in E$, we get the well know result in complete metric space.

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