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Positive Solutions to an Nth Order Multi-point Boundary Value Problem on Time Scales

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Abstract: In this paper, we consider an nth order multi-point boundary value problem on time scales. We establish criteria for the existence of at least one or two positive solutions. We shall also obtain criteria which lead to nonexistence of positive solutions. Examples applying our results are also given.

Keywords: *positive solutions; fixed-point theorems; time scales; dynamic equations; cone.*

Mathematics Subject Classification (2000): 39A10.

1 Introduction

We are concerned with the following boundary value problem (BVP) on time scales $\mathbb{T}$:

$$\begin{cases} y^{\Delta^n}(t) + \lambda f(y^\sigma(t))) = 0, \quad t \in [a,b] \subset \mathbb{T}, \\ y^{\Delta^i}(a) = 0, \quad 0 \le i \le n-2, \\ \sum_{i=1}^m \alpha_i y^{\Delta^{n-2}}(\xi_i) = y^{\Delta^{n-2}}(\sigma(b)) \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $f \in \mathcal{C}([0,\infty),[0,\infty)),\ n \ge 3,\ m \ge 1$ are integers, $a < \xi_1 < \xi_2 < ... < \xi_m < b,\ \alpha_i \in (0,+\infty)$ for $1 \le i \le m$ and $\sum_{i=1}^m \alpha_i < 1$.

We assume that $D = \sigma(b) - a - \sum_{i=1}^m \alpha_i(\xi_i - a) > 0$ and $\sigma(b)$ is right dense so that $\sigma^j(b) = \sigma(b)$ for $j \ge 1$.

The study of dynamic equations on time scales goes back to its founder Stefan Hilger [10]. Some preliminary definitions and theorems on time scales can be found in the books [2,3] which are excellent references for the calculus of time scales.

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Recently, existence results for positive solutions of second-order multi-point boundary value problems was studied by some authors [9, 11–16].

A few papers can be found in the literature on higher-order multi-point boundary value problems [4–7].

We were, in particular, motivated by [6, 7]. We study more general problem and we present results which guarantee the existence of at least one or two positive solutions and the nonexistence positive solutions. The methods discussed here are similar to earlier work [1].

This paper is organized as follows. Section 2 introduces some notation and several lemmas which play important roles in this paper. Section 3 gives nonexistence and multiplicity results for positive solutions to the BVP (1.1). In this article, the main tool is the following well-known Krasnosel'skii fixed point theorem in a cone [8].

Theorem 1.1 [8]. *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega_1} \subset \Omega_2$, and let*

$$A : P \cap (\bar{\Omega_2} \setminus \Omega_1) \to P$$

be a completely continuous operator such that, either

(i) $\|Ay\| \leq \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \geq \|y\|$, $y \in P \cap \partial\Omega_2$; *or*

(ii) $\|Ay\| \geq \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ay\| \leq \|y\|$, $u \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega_2} \setminus \Omega_1)$.

2 Preliminaries and Lemmas

Let $G_2(t,s)$ be the Green's function for the boundary value problem

$$\begin{cases} y^{\Delta^2}(t) = 0, & t \in [a,b], \\ y(a) = 0, & \\ \sum_{i=1}^{m} \alpha_i y(\xi_i) = y(\sigma(b)). & \end{cases} \tag{2.1}$$

Then

$$G_2(t,s) = \begin{cases} \frac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-t)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(t-\sigma(s))}{\sigma(b)-a-\sum_{i=1}^{m}\alpha_i(\xi_i-a)}, \\ a \leq t \leq \sigma(b), \quad \xi_{i-1} \leq \sigma(s) \leq \min\{\xi_i, t\}, \quad i = \overline{1, m+1}, \\ \frac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))]}{\sigma(b)-a-\sum_{i=1}^{m}\alpha_i(\xi_i-a)}, \\ a \leq t \leq \sigma(b), \quad \max\{\xi_{i-1}, t\} \leq \sigma(s) \leq \xi_i, \quad i = \overline{1, m+1}. \end{cases} \tag{2.2}$$

Lemma 2.1 *There exist a number $k \in (0,1)$ and a continuous function $\psi : [a,b] \to \mathbb{R}^+$ such that*

$$G_2(t,s) \leq \psi(s), \quad t \in [a, \sigma(b)], \; s \in [a,b],$$

and

$$G_2(t,s) \geq k\psi(s), \quad t \in [\xi_1, \sigma(b)], \; s \in [a,b],$$

where

$$\psi(s)=\tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

$$k=\min_{2\le i\le m}\{\tfrac{1}{\sigma(b)}\textstyle\sum_{j=i}^{m}\alpha_j(\sigma(b)-\xi_j),\ \tfrac{\xi_1-a}{\sigma(s)-a}[1-\sum_{j=i}^{m}\alpha_j]\}. \tag{2.3}$$

Proof Now, we will show that we may take $\psi(s)=\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}$.
Upper bounds:

Case 1. Consider $a\le\sigma(s)\le\xi_1,\ \sigma(s)\le t$. Then

$$G_2(t,s)=\tfrac{\sigma(b)-t-\sum_{j=1}^{m}\alpha_j(\xi_j-t)}{D}(\sigma(s)-a)=\tfrac{\sigma(b)-\sum_{j=1}^{m}\alpha_j\xi_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a).$$

Since $\sum_{j=1}^{m}\alpha_j<1$, the maximum occurs when $t=\sigma(s)$ and then

$$G_2(t,s)\le\tfrac{\sigma(b)-\sigma(s)+\sum_{j=1}^{m}\alpha_j(\sigma(s)-\xi_j)}{D}(\sigma(s)-a)\le\tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

since $\sum_{j=1}^{m}\alpha_j(\sigma(s)-\xi_j)\le 0$ for $\sigma(s)\le\xi_1$ and $\xi_j\in(a,b)$ with $a<\xi_1<\xi_2<...<\xi_{m-2}<b$.

Case 2. For $\xi_{r-1}\le t\le\xi_r,\ 2\le r\le m+1,\ \xi_{i-1}\le\sigma(s)\le\xi_i,\quad 2\le i\le r,\ \sigma(s)\le t$, we have

$$G_2(t,s)=\tfrac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-t)(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-a)(t-\sigma(s))}{D}$$

$$=\tfrac{(\sigma(b)-t)(\sigma(s)-a)-\sum_{j=i}^{m}\alpha_j(\xi_j-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{m}\alpha_j(t-\sigma(s))(\sigma(s)-a)+\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))(t-\sigma(s))}{D}$$

$$\le\tfrac{\sigma(b)-t+\sum_{j=1}^{m}\alpha_j(t-\sigma(s))}{D}(\sigma(s)-a)$$

$$\le\tfrac{\sigma(b)-\sigma(s)\sum_{j=1}^{m}\alpha_j+t(\sum_{j=1}^{m}\alpha_j-1)}{D}(\sigma(s)-a)$$

since $\sum_{j=i}^{m}\alpha_j(\sigma(s)-\xi_j)\le 0$ and $\sum_{j=1}^{i-1}\alpha_j(\xi_j-\sigma(s))\le 0$ for $\xi_{i-1}\le\sigma(s)\le\xi_i,\ 2\le i\le m+1$.

Since $\sum_{j=1}^{m}\alpha_j<1$, the maximum occurs when $t=\sigma(s)$ so

$$G_2(t,s)\le\frac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D}.$$

Case 3. For $\xi_{r-1}\le t\le\xi_r,\ 2\le r\le m,\ \xi_{i-1}\le\sigma(s)\le\xi_i,\quad r\le i\le m,\ t\le\sigma(s)$, we obtain

$$G_2(t,s)=\tfrac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^{m}a_j(\xi_j-\sigma(s))]}{D}\le\tfrac{(\sigma(b)-\sigma(s))(t-a)}{D}\le\tfrac{(\sigma(b)-\sigma(s))(\sigma(s)-a)}{D},$$

since $\sum_{j=i}^m \alpha_j(\xi_j - \sigma(s)) \geq 0$ for $\xi_{i-1} \leq \sigma(s) \leq \xi_i,\ 2 \leq i \leq m$.

Case 4. For $\xi_m \leq \sigma(s) \leq \sigma(b),\ \ t \leq \sigma(s)$, we clearly have

$$G_2(t,s) \leq \frac{(\sigma(b) - \sigma(s))(\sigma(s) - a)}{D}.$$

Lower bounds: We shall show that we may take an arbitrary interval $[\xi_1, \sigma(b)] \subset (a, \sigma(b)]$. We are looking for $\min\{G_2(t,s) : t \in [\xi_1, \sigma(b)]\}$ as a function of s of the same form as the upper bound.

Case 1. Consider $0 \leq \sigma(s) \leq \xi_1,\ \sigma(s) \leq t$, we get

$$G_2(t,s) = \frac{\sigma(b) - t - \sum_{j=1}^m \alpha_j(\xi_j - t)}{D}(\sigma(s) - a) = \frac{\sigma(b) - \sum_{j=1}^m \alpha_j \xi_j + t(\sum_{j=1}^m \alpha_j - 1)}{D}(\sigma(s) - a).$$

Since $\sum_{j=1}^m \alpha_j < 1$, the minimum occurs when $t = \sigma(b)$ and then

$$\begin{aligned} G_2(t,s) &\geq \frac{\sigma(b) - \sum_{j=1}^m \alpha_j \xi_j + \sigma(b)(\sum_{j=1}^m \alpha_j - 1)}{D}(\sigma(s) - a) \\ &> \frac{(\sigma(b) - \sigma(s))(\sigma(s) - a)}{D} \frac{1}{\sigma(b)} \sum_{j=1}^m \alpha_j(\sigma(b) - \xi_j). \end{aligned}$$

Case 2. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m+1,\ \xi_{i-1} \leq \sigma(s) \leq \xi_i,\ \ 2 \leq i \leq r,\ \sigma(s) \leq t$, we have

$$\begin{aligned} &G_2(t,s) \\ &= \frac{(\sigma(b)-t)(\sigma(s)-a) - \sum_{j=i}^m \alpha_j(\xi_j - \sigma(s))(\sigma(s)-a) + \sum_{j=1}^m \alpha_j(t - \sigma(s))(\sigma(s)-a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - \sigma(s))(t - \sigma(s))}{D} \\ &= \frac{1}{D}[t[(\sum_{j=1}^m \alpha_j - 1)(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - \sigma(s))] + [\sigma(b) - \sigma(s)\sum_{j=1}^m \alpha_j \\ &\qquad - \sum_{j=i}^m \alpha_j(\xi_j - \sigma(s))](\sigma(s) - a) - \sigma(s)\sum_{j=1}^{i-1} \alpha_j(\xi_j - \sigma(s))]. \end{aligned}$$

Since $(\sum_{j=1}^m \alpha_j - 1)(\sigma(s) - a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - \sigma(s)) < 0$, the minimum occurs when $t = \sigma(b)$, then

$$\begin{aligned} G_2(t,s) &\geq \frac{-\sum_{j=i}^m \alpha_j(\xi_j - \sigma(b))(\sigma(s)-a) + \sum_{j=1}^{i-1} \alpha_j(\xi_j - a)(\sigma(b) - \sigma(s))}{D} \\ &\geq \frac{1}{D}\sum_{j=i}^m \alpha_j(\sigma(b) - \xi_j)(\sigma(s) - a) \\ &> \frac{(\sigma(b) - \sigma(s))(\sigma(s) - a)}{D} \frac{1}{\sigma(b)} \sum_{j=i}^m \alpha_j(\sigma(b) - \xi_j). \end{aligned}$$

Case 3. For $\xi_{r-1} \leq t \leq \xi_r,\ 2 \leq r \leq m,\ \xi_{i-1} \leq \sigma(s) \leq \xi_i,\ \ r \leq i \leq m,\ t \leq \sigma(s)$, we obtain

$$\begin{aligned} G_2(t,s) &= \tfrac{(t-a)[\sigma(b)-\sigma(s)-\sum_{j=i}^m \alpha_j(\xi_j-\sigma(s))]}{D} \\ &= \tfrac{(t-a)[(\sigma(b)-\sigma(s))(1-\sum_{j=i}^m \alpha_j)-\sum_{j=i}^m \alpha_j(\xi_j-\sigma(b))]}{D} \\ &\geq \tfrac{(t-a)(\sigma(b)-\sigma(s))}{D}[1-\textstyle\sum_{j=i}^m \alpha_j] \\ &\geq \tfrac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D}[1-\textstyle\sum_{j=i}^m \alpha_j] \\ &= \tfrac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\tfrac{\xi_1-a}{\sigma(s)-a}[1-\textstyle\sum_{j=i}^m \alpha_j]. \end{aligned}$$

Case 4. For $\xi_m \leq \sigma(s) \leq \sigma(b), \quad t \leq \sigma(s)$, we have

$$G_2(t,s) = \tfrac{(t-a)(\sigma(b)-\sigma(s))}{D} \geq \tfrac{(\xi_1-a)(\sigma(b)-\sigma(s))}{D} = \tfrac{(\sigma(s)-a)(\sigma(b)-\sigma(s))}{D}\tfrac{\xi_1-a}{\sigma(s)-a}.$$

Thus we can take

$$k = \min_{2\leq i\leq m}\{\tfrac{1}{\sigma(b)}\textstyle\sum_{j=i}^m \alpha_j(\sigma(b)-\xi_j),\ \tfrac{\xi_1-a}{\sigma(s)-a}[1-\textstyle\sum_{j=i}^m \alpha_j]\}. \ \square$$

Lemma 2.2 *If y satisfies the boundary conditions*

$$\begin{cases} y^{\Delta^i}(a) = 0, \quad 0 \leq i \leq n-2, \\ \sum_{i=1}^m \alpha_i y^{\Delta^{n-2}}(\xi_i) = y^{\Delta^{n-2}}(\sigma(b)) \end{cases}$$

and

$$y^{\Delta^n}(t) \leq 0, \quad t \in [a,b],$$

then

$$y^{\Delta^{n-2}}(t) \geq 0.$$

Proof Let $P(t) = y^{\Delta^{n-2}}(t),\ t \in [a,\sigma(b)]$. Then we have

$$P^{\Delta^2}(t) \leq 0, \quad t \in [a,b]$$

$$P(a) = 0 \quad \text{and} \quad \textstyle\sum_{i=1}^m \alpha_i P(\xi_i) = P(\sigma(b)).$$

It must be true that $P(\sigma(b)) \geq 0$. To see this, assume to the contrary that $P(\sigma(b)) < 0$. Since $P(a) = 0$ and $P(t)$ is concave downward, we have

$$P(t) \geq \frac{t-a}{\sigma(b)-a}P(\sigma(b)), \quad t \in [a,\sigma(b)].$$

Therefore,

$$\sum_{i=1}^{m}\alpha_i P(\xi_i) - P(\sigma(b)) \geq \sum_{i=1}^{m}\alpha_i \frac{\xi_i - a}{\sigma(b)-a}P(\sigma(b)) - P(\sigma(b))$$

$$> \sum_{i=1}^{m}\alpha_i P(\sigma(b)) - P(\sigma(b))$$

$$> P(\sigma(b)) - P(\sigma(b)) = 0,$$

which is a contradiction.

Now, $P(a) = 0, \quad P(\sigma(b)) \geq 0$, and $P(t)$ is concave downward, so we have

$$P(t) = y^{\Delta^{n-2}}(t) \geq 0, \quad t \in [a, \sigma(b)].$$

This completes the proof of the lemma. □

Let $\mathbb{B}$ be the Banach space defined by

$$\mathbb{B} = \{y : y^{\Delta^n} \text{ is continuous on } [a,b],\ y^{\Delta^i}(a) = 0 \quad 0 \leq i \leq n-3\},$$

with the norm $\|y\| = \max_{t\in[a,\sigma(b)]} |y^{\Delta^{n-2}}(t)|$ and let

$$\mathcal{P} = \{y \in \mathbb{B} : y^{\Delta^{n-2}}(t) \geq 0, \quad \min_{t\in[\xi_1,\sigma(b)]} y^{\Delta^{n-2}}(t) \geq k\|y\|\ \},$$

where k is as in (2.3).

Solving the BVP (1.1) is equivalent to finding fixed points of the operator $L_\lambda : \mathcal{B} \to \mathcal{B}$ defined by

$$L_\lambda y(t) = \lambda \int_a^{\sigma(b)} G_n(t,s) f(y^\sigma(s))\Delta s, \quad t \in [a, \sigma(b)]. \tag{2.4}$$

It can be verified that

$$G_2(t,s) = G_n^{\Delta^{n-2}}(t,s). \tag{2.5}$$

From (2.5), it follows that

$$(L_\lambda y)^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s. \tag{2.6}$$

Solving the BVP (1.1) in $\mathbb{B}$ is equivalent to finding fixed points of the operator $L_\lambda^{\Delta^{n-2}}$ defined by (2.6).

Lemma 2.3 *The operator L_λ is completely continuous such that $L_\lambda(\mathcal{P}) \subset \mathcal{P}$.*

Proof From the continuity of $G_2(t,s)$ and $f(t)$ it follows that the operator L_λ defined by (2.4) is completely continuous in $\mathbb{B}$. By Lemma 2.1, Lemma 2.2, and definition of $\mathcal{P}$, we get $L_\lambda\mathcal{P} \subset \mathcal{P}$. □

3 Existence of Positive Solutions

Now we are ready to establish a few sufficient conditions for the existence of at least one or two positive solutions and the nonexistence of positive solutions of (1.1).

Now we define

$$l^0 = \lim_{\|u\|\to 0} \frac{f(u)}{\|u\|}, \quad l^\infty = \lim_{\|u\|\to\infty} \frac{f(u)}{\|u\|}.$$

Theorem 3.1 *For each* λ, *satisfying*

$$\frac{1}{kl^\infty \int_a^{\sigma(b)} \psi(s)\Delta s} < \lambda < \frac{1}{l^0 \int_a^{\sigma(b)} \psi(s)\Delta s}, \tag{3.1}$$

there exists at least one positive solution of (1.1).

Proof Let λ be given as in (3.1). Now, let $\epsilon > 0$ be chosen such that

$$\frac{1}{k(l^\infty - \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s} \le \lambda \le \frac{1}{(l^0 + \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s}.$$

Now, turning to l^0, there exists an $p > 0$ such that $f(y) \le (l^0 + \epsilon)\|y\|$ for $0 < \|y\| \le p$. So, for $y \in \mathcal{P}$ with $\|y\| = p$, we have from the fact that $0 \le G_2(t,s) \le \psi(s)$ for $t \in [a, \sigma(b)], s \in [a,b]$,

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\le \lambda \int_a^{\sigma(b)} \psi(s) f(y^\sigma(s))\Delta s \\ &\le \lambda(l^0 + \epsilon)\int_a^{\sigma(b)} \psi(s)\Delta s \|y\| \\ &\le \|y\| = p.\end{aligned}$$

Next, considering l^∞, there exists $\hat{q} > 0$ such that $f(y) \ge (l^\infty - \epsilon)\|y\|$ for $\|y\| \ge \hat{q}$. Let $q = \max\{2p, \hat{q}\}$. Then for $y \in \mathcal{P}$ with $\|y\| = q$, and $t \in [\xi_1, \sigma(b)]$ we get

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\ &\ge \lambda k \int_a^{\sigma(b)} \psi(s)\Delta s (l^\infty - \epsilon)\|y\| \\ &\ge \|y\| = q.\end{aligned}$$

By Theorem 1.1, L_λ has a fixed point y such that $p \leq \|y\| \leq q$. The proof is complete. □

Theorem 3.2 *For each* λ *satisfying*

$$\frac{1}{kl^0 \int_a^{\sigma(b)} \psi(s)\Delta s} < \lambda < \frac{1}{l^\infty \int_a^{\sigma(b)} \psi(s)\Delta s}, \tag{3.2}$$

there exists at least one positive solution of (1.1).

Proof Let λ be given as in (3.2), and choose let $\epsilon > 0$ such that

$$\frac{1}{k(l^0 - \epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s} \leq \lambda \leq \frac{1}{(l^\infty + \epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s}.$$

Beginning with l^0, there exists an $p > 0$ such that $f(y) \geq (l^0 - \epsilon)\|y\|$ for $0 < \|y\| \leq p$. So, for $y \in \mathcal{P}$ with $\|y\| = p$, and $t \in [\xi_1, \sigma(b)]$ we have

$$\begin{aligned}
(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\
&\geq \lambda k \int_a^{\sigma(b)} \psi(s) f(y^\sigma(s))\Delta s \\
&\geq \lambda k (l^0 - \epsilon) \int_a^{\sigma(b)} \psi(s)\Delta s \|y\| \\
&\geq \|y\| = p.
\end{aligned}$$

It remains to consider l^∞. There exists $\hat{q} > 0$ such that $f(y) \leq (l^\infty + \epsilon)\|y\|$ for $\|y\| \geq \hat{q}$. There are two cases:

For case (a), suppose $N > 0$ is such that $f(y) \leq N$, for all $0 \leq y < \infty$. Let $q = \max\{2p, \lambda N \int_a^{\sigma(b)} \psi(s)\Delta s\}$. Then $y \in \mathcal{P}$ and $\|y\| = q$, we have

$$\begin{aligned}
(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\
&\leq \lambda N \int_a^{\sigma(b)} \psi(s)\Delta s \\
&\leq \|y\| = q.
\end{aligned}$$

For case (b), let $g(h) := \max\{f(y) : 0 \leq y^{\Delta^{n-2}} \leq h\}$. The function g is nondecreasing and $\lim_{h\to\infty} g(h) = \infty$. Choose $q = \max\{2p, \hat{q}\}$ so that $g(q) \geq g(h)$ for $0 \leq h \leq q$. For $y \in \mathcal{P}$ and $\|y\| = q$, we have

$$\begin{aligned}
(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \\
&\leq \lambda g(q) \int_a^{\sigma(b)} \psi(s)\Delta s
\end{aligned}$$

$$\leq \lambda(l^\infty + \epsilon)q \int_a^{\sigma(b)} \psi(s)\Delta s$$

$$\leq \|y\| = q.$$

By Theorem 1.1, L_λ has a fixed point y such that $p \leq \|y\| \leq q$. The proof is complete. □

In the rest of the paper we assume that $f(y) > 0$ on $\mathbb{R}^+$. Set

$$A = \int_a^{\sigma(b)} \psi(s)\Delta s.$$

Theorem 3.3 *If either* $l^0 = \infty$ *or* $l^\infty = \infty$, *then for all* $0 < \lambda \leq \lambda_0$, *where*

$$\lambda_0 := \tfrac{1}{A} \sup_{r>0} \tfrac{r}{\max_{0<\|u\|\leq r} f(u)}, \tag{3.3}$$

(1.1) *has at least one positive solution.*

(*b*) *If either* $l^0 = 0$ *or* $l^\infty = 0$, *then for all* $\lambda \geq \lambda_0$, *where*

$$\lambda_0 := \frac{1}{A} \inf_{r>0} \frac{r}{\min_{0<\|u\|\leq r} f(u)},$$

(1.1) *has at least one positive solution.*

Proof We now prove the part (*a*) of Theorem 3.3. By (3.3), there exists $r > 0$ such that

$$\lambda_0 = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0<\|u\|\leq r} f(u)}.$$

If $\|y\| = r$, it follows that

$$\|L_\lambda y\| = \max_{t\in[a,\sigma(b)]} |(L_\lambda y)^{\Delta^{n-2}}(t)| \leq \lambda_0 \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \leq r.$$

So for all $0 < \lambda \leq \lambda_0$ we have

$$\|L_\lambda y\| \leq \|y\|.$$

Fix $\lambda \leq \lambda_0$. Choose $R > 0$ sufficiently large so that

$$\lambda R k \int_a^{\sigma(b)} \psi(s)\Delta s \geq 1. \tag{3.4}$$

Since $l^0 = \infty$, there is $p > 0$ such that

$$\frac{f(y)}{\|y\|} \geq R$$

for $t \in [a, \sigma(b)],\ 0 < \|y\| \leq p$. Hence we have that

$$f(y) \geq R\|y\|$$

for $t \in [a, \sigma(b)],\ 0 < \|y\| \leq p$. For $y \in \mathcal{P},\ \|y\| = p$ and $t \in [\xi_1, \sigma(b)]$, we get

$$(L_\lambda y)^{\Delta^{n-2}}(t) \geq \lambda R k \int_a^{\sigma(b)} \psi(s)\Delta s\|y\| \geq \|y\| = p$$

by (3.4). By Theorem 1.1, L_λ has a fixed point y such that $\min\{p, r\} \leq \|y\| \leq \max\{p, r\}$.

Next, we use the assumption that $l^\infty = \infty$. Since $l^\infty = \infty$ there is a $q > 0$ such that

$$\frac{f(y)}{\|y\|} \geq R$$

for $\|y\| \geq q$ and R is chosen so that (3.4) holds. It follows that

$$f(y) \geq R\|y\|$$

for $\|y\| \geq q$.

For $y \in \mathcal{P},\ \|y\| = q$ and $t \in [\xi_1, \sigma(b)]$, we have

$$\begin{aligned}(L_\lambda y)^{\Delta^{n-2}}(t) &= \lambda \int_a^{\sigma(b)} G_2(t, s) f(y^\sigma(s))\Delta s \\ &\geq \lambda R k \int_a^{\sigma(b)} \psi(s)\Delta s\|y\| \\ &\geq q = \|y\|\end{aligned}$$

by (3.4). By Theorem 3.1, then L_λ has a fixed point y such that $\min\{q, r\} \leq \|y\| \leq \max\{q, r\}$. This completes the proof of part (a). Part (b) holds in an analogous way. □

Theorem 3.4 *a) If* $l^0 = l^\infty = \infty$, *then there is a* $\lambda_0 > 0$ *such that for all* $0 < \lambda \leq \lambda_0$, (1.1) *has two positive solutions.*
b) If $l^0 = l^\infty = 0$, *then there is a* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$, (1.1) *has two positive solutions.*

Now, we give a nonexistence result as follows.

Theorem 3.5 (a) *If there is a constant* $c > 0$ *such that* $f(y) \geq c\|y\|$, *then there is a* $\lambda_0 > 0$ *such that* (1.1) *has no positive solutions for* $\lambda \geq \lambda_0$.
(b) *If there is a constant* $c > 0$ *such that* $f(y) \leq c\|y\|$, *then there is a* $\lambda_0 > 0$ *such that* (1.1) *has no positive solutions for* $0 < \lambda \leq \lambda_0$.

Proof We now prove the part (a) of this theorem. Assume there is a constant $c > 0$ such that $f(y) \geq c\|y\|$. Assume $y(t)$ is a solution of the BVP (1.1). We will show that for λ sufficiently large this leads to a contradiction. We have for $t \in [\xi_1, \sigma(b)]$,

$$y^{\Delta^{n-2}}(t) = \lambda \int_a^{\sigma(b)} G_2(t,s) f(y^\sigma(s))\Delta s \geq ck\lambda_0 \int_a^{\sigma(b)} \psi(s)\Delta s \|y\|.$$

If we pick λ_0 sufficiently large so that $ck\lambda_0 \int_a^{\sigma(b)} \psi(s)\Delta s > 1$ for all $\lambda \geq \lambda_0$, then we have $y^{\Delta^{n-2}} > \|y\|$ which is a contradiction. The proof of part (b) is similar. □

Example 3.1 We illustrate Theorem 3.2 with specific time scale $\mathbb{T} = \{\frac{1}{2^n} : n \in \mathbb{N}_0\} \cup \{0\} \cup [1,5]$.

Consider the system:

$$\begin{cases} y^{\Delta^n}(t) + \lambda f(y^\sigma(t)) = 0, & t \in [0,1/2] \subset \mathbb{T}, \\ y^{\Delta^i}(0) = 0, \quad 0 \leq i \leq n-2, & \\ 1/3y(1/4) + 1/5y(1/8) + 1/10y(1/64) = y(5), & \end{cases} \tag{3.5}$$

where $f = 1 + \sqrt{y},\ \alpha_1 = 1/3,\ \alpha_2 = 1/5, \alpha_3 = 1/10,\ a = 0,\ b = 1/2,\ \xi_1 = 1/4,\ \xi_2 = 1/8,\ \xi_3 = 1/64,\ n \geq 3$.

Since $f = 1 + \sqrt{y}$, we have

$$l^0 = \infty \qquad l^\infty = 0.$$

We get $\psi(s) = \frac{4096}{1967} s(1-2s)$, $\int_0^1 \psi(s)\Delta s = \frac{4096}{41307}$. Therefore the assumptions of Theorem 3.2 are satisfied. By Theorem 3.2, for all $\lambda \in (0,\infty)$, (3.5) has at least one positive solution.

Example 3.2 We illustrate Theorem 3.3 with specific time scale $\mathbb{T} = \{\frac{n}{3} : n \in \mathbb{N}\} \cup [7/3, 5]$.

Consider the system:

$$\begin{cases} y^{\Delta^3}(t) + \lambda f(y^\sigma(t)) = 0, & t \in [1,2], \\ y(1) = y^\Delta(1) = 0, & \\ 1/2y(4/3) + 1/3y(5/3) = y(7/3), & \end{cases} \tag{3.6}$$

where $f = e^y,\ \alpha_1 = 1/2,\ \alpha_2 = 1/3,\ a = 1,\ b = 2,\ \xi_1 = 4/3,\ \xi_2 = 5/3$.

Hence $l^\infty = \infty$. Since

$$A = \int_1^{7/3} \psi(s)\Delta s = \frac{60}{153}, \quad \sup_{r>0} \frac{r}{\max_{0<\|y\|\leq r} e^y} = \sup_{r>0} \frac{r}{e^r} = \frac{1}{e},$$

we have

$$\lambda_0 = \frac{1}{A} \sup_{r>0} \frac{r}{\max_{0<\|y\|\leq r} f(y)} = \frac{153}{60} e^{-1}.$$

So, by Theorem 3.3, for all $\lambda \in (0, \frac{153}{60} e^{-1}]$, (3.6) has one positive solution.

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