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Quasilinearization Method Via Lower and Upper Solutions for Riemann–Liouville Fractional Differential Equations

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Abstract: Existence and comparison results of the linear and nonlinear Riemann–Liouville fractional differential equations of order q, $0 < q < 1$, are recalled and modified where necessary. Generalized quasilinearization method is developed for nonlinear fractional differential equations of order q, using upper and lower solutions. Quadratic convergence to the unique solution is proved via weighted sequences.

Keywords: *fractional differential equations; lower and upper solutions; quasilinearization method.*

Mathematics Subject Classification (2000): 34A34, 34A45.

1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as in engineering [9], chemistry [10, 17, 18], physics [3, 4, 11], and others [12, 13]. In the majority of the literature existence results for Riemann–Liouville fractional differential equations are proven by a fixed point method. Initially we will recall existence by lower and upper solution method, which is more comparable to our main results. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a solution to a needed solution. In this paper we construct such a method.

The iterative technique we manufacture is the method of quasilinearization for nonlinear Riemann–Liouville fractional differential equations of order q, $0 < q < 1$. This

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method was first developed in [1, 2, 16], the method we construct is more closely related to those found in [15], that is a generalized quasilinearization method via lower and upper solutions. This particular method is much like the monotone method in that we construct monotone sequences from lower and upper solutions of the original equation. Further, each iterate is the solution of the linear fractional differential equation, but unlike in the monotone method, these iterates are not of the form with constant coefficients. In the case of the Riemann–Liouville fractional derivative, the variable coefficient case complicates our method. Therefore, we will recall existence, comparison, and inequality results for this case, including a generalized Gronwall type inequality, which will be paramount to our main result. Further, we will present modifications to these results where pertinent to our work.

Further, in the construction of the quasilinearization method we require a much stronger hypothesis than the monotone iterative technique. We still require the existence of lower and upper solutions v, w such that $v \leq w$, but specifically we require the nonlinear function $f(t, x)$ to be convex (or concave) in x. Though this requirement may initially seem superfluous, with its application we are able to prove that the sequences we construct converge quadratically. Therefore, the sequences we construct may be more unwieldy, and the requirements more strict, than with the monotone method, but with this method the convergence is far faster. Further, with the assumption that f is convex automatically ensures that our solution is unique, which is not necessarily the case with the monotone method.

We note that this method has been studied in [8], but the authors have considered differential equations of the Caputo case. However the Caputo derivative only exists for C^1 functions. We do not make this assumption with the Riemann–Liouville derivative. In fact, the functions we consider generally have a singularity at the left-most endpoint, therefore they are only C^0 on a half open interval, with a special C_p property we will define below. One consequence of using the Riemann–Liouville derivative is that, in general, the sequences we construct, $\{\alpha_n\}, \{\beta_n\}$ do not converge uniformly to the unique solution, but the weighted sequences $\{t^p\alpha_n\}$, $\{t^p\beta_n\}$ converge uniformly and quadratically to $t^p x$, where x is the unique solution of the original equation and $p = 1 - q$.

Finally, we consider the case when f is not convex (nor concave), but there exists a function ϕ such that $f + \phi$ is convex. We construct the quasilinearization for this case and note that a function ϕ will always exist, therefore extending this method to any nonlinear fractional differential equation, provided f is C^2 in x. For more information on the method of quasilinearization via lower and upper solutions as it relates to ordinary differential equations, see [15].

2 Preliminary Results

In this section we consider results regarding the Riemann-Liouville (R-L) differential equations of order q, $0 < q < 1$. Specifically we recall existence and comparison results which will be used in our main result. In the next section we will apply these preliminary results to developing quasilinearization method for R-L fractional differential equations of order q. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

Definition 2.1 Let $p = 1 - q$, a function $\phi(t) \in C(J, \mathbb{R})$ is a C_p function if $t^p\phi(t) \in C(J_0, \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Remark 2.1 By the definition of C_p continuity and the properties of continuous functions it can be shown that the uniform limit of C_p functions is C_p, also $C_p(J, \mathbb{R})$ has a completeness property in that any uniformly Cauchy sequence of C_p functions converges uniformly to a C_p function. Further $C_p(J, \mathbb{R})$ is closed under continuous products, that is, if $x \in C_p(J, \mathbb{R})$ and $y \in C(J_0, \mathbb{R})$ then $xy \in C_p(J, \mathbb{R})$.

Now we define the R-L integral and derivative of order q on the interval J.

Definition 2.2 Let $\phi \in C_p(J, \mathbb{R})$, then $D_t^q \phi(t)$ is the q-th R-L derivative of ϕ with respect to $t \in J$ defined as

$$D_t^q \phi(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_0^t (t-s)^{-q} \phi(s) ds,$$

and $I_t^q \phi(t)$ is the q-th R-L integral of ϕ with respect to $t \in J$ defined as

$$I_t^q \phi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \phi(s) ds.$$

Note that in cases where the initial value may be different, or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equation and is also of great importance in the study of the R-L derivative.

Definition 2.3 The Mittag–Leffler function with parameters $\alpha, \beta \in \mathbb{R}$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

which is entire for $\alpha, \beta > 0$.

Remark 2.2 We note that the C_p weighted Mittag–Leffler function

$$t^{q-1} E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{kq+q-1}}{\Gamma(kq+q)},$$

where λ is a constant, converges uniformly on J. This can be shown by using the fact that $E_{q,q}$ is entire and noting that there exists an $N > 0$ such that $nq + q - 1 > 0$ for all $n \geq N$. From here one can show that the sequence of partial sums of the above series is uniformly Cauchy.

The next result gives us that the q-th R-L integral of a C_p continuous function is also a C_p continuous function. This result will give us that the solutions of R-L differential equations are also C_p continuous.

Lemma 2.1 *Let $f \in C_p(J, \mathbb{R})$, then $I_t^q f(t) \in C_p(J, \mathbb{R})$, i.e. the q-th integral of a C_p continuous function is C_p continuous.*

Note the proof of this theorem for $q \in \mathbb{R}^+$ can be found in [7]. Now we consider results for the nonhomogeneous linear R-L differential equation

$$D_t^q x(t) = y(t)x(t) + z(t) \tag{1}$$

with initial condition $\Gamma(q) t^p x(t)\big|_{t=0} = x^0$, where x^0 is a constant, $y \in C(J_0, \mathbb{R})$, and $z \in C_p(J, \mathbb{R})$.

Theorem 2.1 *If $y \in C(J_0, \mathbb{R})$ and $z \in C_p(J, \mathbb{R})$ then equation (1) has a unique solution $x \in C_p(J, \mathbb{R})$, given explicitly by*

$$x(t) = \sum_{k=0}^{\infty} \frac{x^0}{\Gamma(q)} T_y^k \left[t^{q-1}\right] + T_y^k \left[I_t^q z(t)\right],$$

which converges uniformly on J and where T_y is the operator defined by

$$T_y \phi(t) = I_t^q y(t) \phi(t).$$

Proof The proof of the homogeneous case, and that $t^p x(t)$ converges uniformly on J_0 can be found in [6], the refinement that $x(t)$ converges uniformly on J can be found in [5]. Note the nonhomogeneous case follows in exactly the same way as in [6]. Further in [5] it was assumed that $z \in C_p(J, \mathbb{R})$ such that $I_t^q z \in C(J_0, \mathbb{R})$, here we have relaxed this condition. The proof follows along the same lines as in [5] with appropriate modifications. That is, using that $z \in Cp$, and the fact that $E_{q,q}$ is entire, we can show the partial sums of the series x are uniformly Cauchy on J. That $x \in C_p(J, \mathbb{R})$ follows from applying Remark 2.1 and Lemma 2.1. Note that if $z(t) = 0$ for all $t \in J$ then we get that

$$x(t) = \frac{x^0}{\Gamma(q)} \sum_{k=0}^{\infty} T_y^k \left[t^{q-1}\right].$$

In many cases we may have an explicit form of y that may prove too unwieldy to place in a subscript. In this case we will use the following notation

$$\mathcal{E}(y, f) = \sum_{k=0}^{\infty} T_y^k \left[f\right],$$

and since the case where $f = t^{q-1}$ occurs so often we will define $\mathcal{E}$ with a single parameter to be this case. That is $\mathcal{E}(y) = \mathcal{E}(y, t^{q-1})$. Therefore the solution of (1) can be written as

$$x(t) = \frac{x^0}{\Gamma(q)} \mathcal{E}(y) + \mathcal{E}(y, I_t^q z). \tag{2}$$

Further, if y is identically a constant, say λ, it can be shown that (2) can be expressed as

$$x(t) = x^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

This is the result discussed in [14], hence Theorem 2.1 generalizes the constant coefficient case, as expected.

Next we recall a comparison result we will utilize in our following results. Note this result is similar to the well known comparison result found in literature, as in [14], but we do not require the function to be Hölder continuous of order $\lambda > q$. We weaken this requirement because in our main result we will construct sequences from the solutions of linear R-L differential equations. As previously mentioned the solution to the linear equation with constant coefficient can be rewritten as

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + x^0 \sum_{k=1}^{\infty} \frac{\lambda^k t^{qk+q-1}}{\Gamma(qk+q)} + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) ds,$$

which is not Hölder continuous of any order due to the term containing t^{q-1}. Therefore we utilize the following result which weakens the Hölder continuity requirement, so that we can incorporate it in our main results.

Lemma 2.2 *Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t \in (0, t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \geq 0.$$

The proof of this lemma can be found in [7], along with further discussion as to why and how we weaken the Hölder continuous requirement of this known comparison result. We use this Lemma in the proof of the later main comparison result which will be paramount in the construction of the quasilinearization method. First we recall the nonlinear R-L fractional differential equation.

$$\begin{aligned} &D_t^q x = f(t, x), \\ &\Gamma(q) t^p x(t)\big|_{t=0} = x^0, \end{aligned} \tag{3}$$

where $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$. Note that a solution $x \in C_p(J, \mathbb{R})$ of (3) also satisfies the equivalent R-L integral equation

$$x(t) = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds. \tag{4}$$

Thus if $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ then (3) is equivalent to (4). See [12, 14] for details. Now we will recall a Peano type existence theorem for equation (3).

Theorem 2.2 *Suppose $f \in C(R_0, \mathbb{R})$ and $|f(t, x)| \leq M$ on R_0, where*

$$R_0 = \{(t, x) \ : \ |t^p x(t) - x^0| \leq \eta, t \in J_0\}.$$

Then the solution of (3) exists on J.

This result is presented in [14], and in [7] it was proven that the solution can be extended to all of J, and the set R_0 was modified for our succeeding results regarding existence by method of upper and lower solutions. In the direction of this result we will consider the following comparison result, which will in turn yield a general Gronwall type inequality.

Theorem 2.3 *Let $f \in C(J_0 \times \mathbb{R}, \mathbb{R})$ and let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of (3), i.e.*

$$\begin{aligned} &D_t^q v \leq f(t, v), \\ &\Gamma(q) t^p v(t)\big|_{t=0} = v^0 \leq x^0, \end{aligned}$$

and

$$\begin{aligned} &D_t^q w \geq f(t, w), \\ &\Gamma(q) t^p w(t)\big|_{t=0} = w^0 \geq x^0. \end{aligned}$$

If f satisfies the following Lipschitz condition

$$f(t, x) - f(t, y) \leq L(x - y), \quad \text{when } x \geq y,$$

where $L > 0$, then $v(t) \leq w(t)$ on J.

The proof follows as in [14] with appropriate modifications, specifically we use Lemma 2.2 and do not require local Hölder continuity of order $\lambda > q$. Next we present a Gronwall type inequality for R-L fractional differential equations. A similar result in terms of fractional integral equations can be found in [6].

Theorem 2.4 *Let $v, z \in C_p(J, \mathbb{R})$ and $y \in C(J_0, \mathbb{R}^+)$, and suppose that*

$$D_t^q v \leq y(t)v(t) + z(t).$$

Then

$$v(t) \leq \frac{v^0}{\Gamma(q)}\mathcal{E}(y) + \mathcal{E}(y, I_t^q z).$$

The proof follows directly from Theorem 2.1 and Theorem 2.3. That is, since $y \geq 0$, $f(t,x) = yx + z$ satisfies the Lipschitz condition of Theorem 2.3 and letting x be the solution of (1) with $x^0 = v^0$ we obtain $v \leq x$. When y is identically a constant $\lambda \geq 0$, then we get the following Corollary.

Corollary 2.1 *Let $v, z \in C_p(J, \mathbb{R})$ and let $\lambda \geq 0$ be a constant, and suppose that*

$$D_t^q v \leq \lambda v(t) + z(t).$$

Then

$$v(t) \leq v^0 t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

Now we will recall a result that gives us existence of a solution to (3) via lower and upper solutions.

Theorem 2.5 *Let $v, w \in C_p(J, \mathbb{R})$ be lower and upper solutions of (3) such that $v(t) \leq w(t)$ on J and let $f \in C(\Omega, \mathbb{R})$, where Ω is defined as*

$$\Omega = \{(t, y) \,:\, t^p v(t) \leq y \leq t^p w(t), t \in J_0\}.$$

Then there exists a solution $x \in C_p(J, \mathbb{R})$ of (3) such that $v(t) \leq x(t) \leq w(t)$ on J.

The proof of this theorem can be found in [7]. We also note a final uniqueness result which is comparable to the analogous result for ordinary differential equations. As one might expect, if f satisfies the Lipschitz condition found in Theorem 2.3, then the solution x of (3) is unique. Further this result is proved in much the same way as in the case of ordinary differential equations, see [14] for more details. We mention this result here since it will be necessary in the construction of the quasilinearization method.

3 Method of Quasilinearization

In this section we develop the method of quasilinearization via lower and upper solutions. We consider three different cases, when the forcing function f is convex, concave in x, and can be made convex by the addition function ϕ. We construct monotone sequences such that the sequences of continuous extensions converge uniformly and monotonically to the continuous extension of the unique solution x of (3). Further, the rate convergence is quadratic.

Theorem 3.1 *Assume that*

(A1) $\alpha_0, \beta_0 \in C_p(J, \mathbb{R})$ *are lower and upper solutions of (3) respectively such that* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(A2) $f \in C(\Omega, \mathbb{R})$, $f_x(t,x) \geq 0$, $f_{xx}(t,x) \geq 0$ *exist and are continuous on* Ω, *where*

$$\Omega = \{(t,y) \,:\, \alpha_0(t) \leq y \leq \beta_0(t), t \in J_0\}.$$

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p\alpha_n$, *and* $t^p\beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

Proof First, by (A2) we have that f and f_x are nondecreasing in x on J_0, Lipschitz with respect to x on J_0, and

$$f(t,x) \geq f(t,y) + f_x(t,y)(x-y)$$

for any $(t,y) \in \Omega$. Further the function

$$g(t,x,y) = f(t,y) + f_x(t,y)(x-y)$$

is linear in x on J_0. Now we will construct the sequences $\{\alpha_n\}$ and $\{\beta_n\}$. Let α_{n+1} be the unique solution of the Riemann–Liouville differential equation

$$\begin{aligned} &D_t^q \alpha_{n+1} = f(t,\alpha_n) + f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n), \\ &\Gamma(q) t^p \alpha_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{5}$$

for all $n \geq 0$, and where α_0 is the lower solution of (3) given in the hypothesis. Note that the above equation is of the form (1), therefore it has a unique solution by Theorem 2.1 provided $(t, \alpha_n) \in \Omega$, and therefore our sequence is well defined. Similarly, let β_{n+1} be the unique solution of

$$\begin{aligned} &D_t^q \beta_{n+1} = f(t,\beta_n) + f_x(t,\alpha_n)(\beta_{n+1} - \beta_n), \\ &\Gamma(q) t^p \beta_{n+1}(t)\big|_{t=0} = x^0. \end{aligned} \tag{6}$$

Now we will show that $\alpha_n \leq \beta_n$ for all $n \geq 0$. To do this first note that by hypothesis we have that $\alpha_0 \leq \beta_0$ on J, so letting this be our basis step, suppose that $\alpha_k \leq \beta_k$ is true up to some $k \geq 0$. Then we have

$$D_t^q \alpha_{k+1} = f(t,\alpha_k) + f_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k),$$

and by the consequences of (A2) we have that

$$D_t^q \beta_{k+1} \geq f(t,\alpha_k) + f_x(t,\alpha_k)(\beta_{k+1} - \alpha_k),$$

which by Theorem 2.3 gives us that $\alpha_{k+1} \leq \beta_{k+1}$ on J and thus by induction proves the claim.

Now we wish to show that that $\{\beta_n\}$ is monotone. To do so consider that

$$D_t^q \beta_1 \leq f(t,\beta_0) + f_x(t,\beta_0)(\beta_1 - \beta_0) \leq f(t,\beta_1),$$

which again, by Theorem 2.3 gives us that $\beta_1 \leq \beta_0$ on J. Now suppose $\beta_k \leq \beta_{k-1}$ up to some $k \geq 1$, then letting $\omega = \beta_{k+1} - \beta_k$, with $\omega^0 = 0$, by the consequences of (A2) and that $\alpha_n \leq \beta_n$ for all $n \geq 0$, we obtain

$$D_t^q \omega \leq [f_x(t,\beta_k) - f_x(t,\alpha_{k-1})](\beta_k - \beta_{k-1}) + f_x(t,\alpha_k)\omega \leq f_x(t,\alpha_k)\omega.$$

This implies by Theorem 2.4 that

$$\beta_{k+1} - \beta_k \leq \frac{\omega^0}{\Gamma(q)} \mathcal{E}(f_x(t,\alpha_k)) = 0,$$

thus proving, by induction, that $\{\beta_n\}$ is monotone. The proof that $\{\alpha_n\}$ is monotone follows by arguments similar to either of the previous induction proofs.

We now prove that

$$t^p\alpha_n \to t^p x \quad \text{and} \quad t^p\beta_n \to t^p x,$$

uniformly on J_0, and where x is the unique solution of (3). This result follows from an application of the Arzelà–Ascoli Theorem since for all $n \geq 0$ we have that

$$|t^p\alpha_n| \leq t^p|\alpha_n - \alpha_0| + t^p|\alpha_0| \leq t^p|\beta_0 - \alpha_0| + t^p|\alpha_0|,$$

implying that $\{t^p\alpha_n\}$ is uniformly bounded on J_0. That this sequence is equicontinuous is proved in a similar fashion to that found in [19]. We can prove a similar result for $\{t^p\beta_n\}$ as well. To show that both sequences converge to $t^p x$, suppose that $t^p\alpha_n$ instead converges uniformly to $t^p\alpha$, which gives us that α_n converges to α pointwise on J. Now consider the continuous extension of the integral form of α_{n+1},

$$t^p\alpha_{n+1} = \frac{x^0}{\Gamma(q)} + \frac{t^p}{\Gamma(q)} \int_0^t (t-s)^{q-1}\big(f(s,\alpha_n) + f_x(s,\alpha_n)(\alpha_{n+1} - \alpha_n)\big)ds.$$

Applying the convergence properties outlined above we can show that the limit α satisfies

$$\alpha = \frac{x^0}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,\alpha) ds$$

on J. Implying that $\alpha = x$, since x is the unique solution of (3). We note that $\{t^p\beta_n\}$ satisfies an analogous property.

Now we will prove that the sequences of continuous extensions $\{t^p\alpha_n\}$ and $\{t^p\beta_n\}$ converge quadratically. First we note that, since f is continuous on J_0, there exists a function F such that $f(t,x) = F(t,t^p x)$. Then we have that $f_{xx}(t,x) = t^{2p}F_{xx}(t,t^p x)$. Using this result, along with the mean value theorem we obtain

$$\begin{aligned} D_t^q(x - \alpha_{n+1}) &= f(t,x) - f(t,\alpha_n) - f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &= f_x(t,\xi)(x - \alpha_n) - f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &\leq f_x(t,x)(x - \alpha_n) - f_x(t,\alpha_n)(\alpha_{n+1} - \alpha_n) \\ &= [f_x(t,x) - f_x(t,\alpha_n)](x - \alpha_n) + f_x(t,\alpha_n)(x - \alpha_{n+1}) \\ &= f_{xx}(t,\eta)(x - \alpha_n)^2 + f_x(t,\alpha_n)(x - \alpha_{n+1}) \\ &= F_{xx}(t,t^p\eta)t^{2p}(x - \alpha_n)^2 + f_x(t,\alpha_n)(x - \alpha_{n+1}) \\ &\leq Nt^{2p}(x - \alpha_n)^2 + M(x - \alpha_{n+1}). \end{aligned}$$

Here $\alpha_n \le \xi, \eta \le x$ on J, and N and M are bounds of F_{xx} and f_x respectively. Now by Corollary 2.1 and Remark 2.2 we have that

$$\begin{aligned} t^p(x-\alpha_{n+1}) &\le t^p \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) N s^{2p} (x-\alpha_n)^2 ds \\ &\le t^p N \big\| t^p(x-\alpha_n) \big\|^2 \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) ds \\ &= t^p N \big\| t^p(x-\alpha_n) \big\|^2 \int_0^t \sum_{k=0}^{\infty} \frac{M^k (t-s)^{kq+q-1}}{\Gamma(qk+q)} ds \\ &= t^p N \big\| t^p(x-\alpha_n) \big\|^2 \sum_{k=0}^{\infty} \frac{M^k t^{kq+q}}{\Gamma(qk+q+1)} \\ &\le \frac{t^p N}{M} E_{q,1}(Mt^q) \big\| t^p(x-\alpha_n) \big\|^2. \end{aligned}$$

Here $\|\cdot\|$ is the uniform norm on $C(J_0, \mathbb{R})$. Giving us that

$$\big\| t^p(x-\alpha_{n+1}) \big\| \le K \big\| t^p(x-\alpha_n) \big\|^2,$$

where $K = \frac{T^p N}{M} E_{q,1}(MT^q)$.

Now, letting $\rho_n = x - \alpha_n$ and $\omega_n = \beta_n - x$, showing that $\{t^p \beta_n\}$ converges quadratically follows with a similar argument, but in this case we get

$$D_t^q \omega_{n+1} \le F_{xx}(t,\sigma) t^{2p} [\omega_n + \rho_n] \omega_n + f_x(t,\alpha_n)(\omega_{n+1}) \le (N/2) t^{2p} (3\omega_n^2 + \rho_n^2) + M\omega_{n+1}.$$

Then from Corollary 2.1 we get

$$t^p \omega_{n+1} \le \frac{N t^p}{2M} E_{q,1}(Mt^q) \big\| t^{2p}(3\omega_n^2 + \rho_n^2) \big\|,$$

which finally implies that

$$\big\| \beta_{n+1} - x \big\| \le \frac{3K}{2} \big\| t^p(\beta_n - x) \big\|^2 + \frac{K}{2} \big\| t^p(x-\alpha_n) \big\|^2.$$

This concludes the proof.

A natural query is whether the results of Theorem 3.1 will still hold if f is concave as opposed to convex. The answer is affirmative, and we state the result below without the details of the proof.

Theorem 3.2 *Suppose (A1) of Theorem 3.1 holds. Further suppose that* $f \in C(\Omega, \mathbb{R})$, $f_x(t,x) \le 0$, $f_{xx}(t,x) \le 0$ *exist and are continuous on* Ω. *Then there exist monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p \alpha_n$, *and* $t^p \beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

We note that the proof of this theorem follows in the same lines as that of Theorem 3.1. The next case we consider is whether it is possible to construct the quasilinearization method when $f \in C^{0,2}(\Omega, \mathbb{R})$ is neither convex nor concave. As we will show, it is indeed possible provided we can find a function $\phi \in C^{0,2}(\Omega, \mathbb{R})$ such that $f + \phi$ is convex. We present this case as our final theorem.

Theorem 3.3 *Assume that*

(B1) $\alpha_0, \beta_0 \in C_p(J, \mathbb{R})$ *are lower and upper solutions of (3) respectively, such that* $\alpha_0 \le \beta_0$ *on* J.

(B2) $f, \phi \in C^{0,2}(\Omega, \mathbb{R})$, $f_{xx} + \phi_{xx} \ge 0$ *and* $\phi_{xx} > 0$ *on* Ω, *where* Ω *is defined as in Theorem 3.1.*

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $C_p(J, \mathbb{R})$ *such that* $t^p\alpha_n$ *and* $t^p\beta_n$ *both converge uniformly and quadratically to* $t^p x$ *on* J_0, *where* x *is the unique solution of (3) on* J.

Proof Firstly, by consequences of (B2) we have that f is Lipschitz with respect to x. Further, since $f + \phi$ is convex we have that

$$F(t, x) \ge F(t, y) + F_x(t, y)(x - y), \tag{7}$$

where $F(t, x) = f(t, x) + \phi(t, x)$.

We construct the monotone sequences by letting α_{n+1} and β_{n+1} be the unique solutions of the linear R-L fractional differential equations,

$$\begin{aligned} &D_t^q \alpha_{n+1} = f(t, \alpha_n) + \big(F_x(t, \alpha_n) - \phi_x(t, \beta_n)\big)(\alpha_{n+1} - \alpha_n), \\ &\Gamma(q) t^p \alpha_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} &D_t^q \beta_{n+1} = f(t, \beta_n) + \big(F_x(t, \alpha_n) - \phi_x(t, \beta_n)\big)(\beta_{n+1} - \beta_n), \\ &\Gamma(q) t^p \beta_{n+1}(t)\big|_{t=0} = x^0, \end{aligned} \tag{9}$$

for all $n \ge 0$ and for $(t, \alpha_n), (t, \beta_n) \in \Omega$. Now we wish to show that $\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n$ for all $n \ge 0$. First we will show that $\alpha_0 \le \alpha_1$, to do so notice that

$$D_t^q \alpha_0 \le f(t, \alpha_0) + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\alpha_0 - \alpha_0).$$

Therefore by Theorem 2.3 we have that $\alpha_0 \le \alpha_1$ on J since $\alpha_0^0 \le x^0$, and by a similar argument we also have that $\beta_1 \le \beta_0$. Now we will show that $\alpha_1 \le \beta_1$ on J. Note by consequences of (B2), that is (7), that ϕ_x is increasing in x, and by the application of the mean value theorem we can show that

$$\begin{aligned} D_t^q \beta_1 &\ge f(t, \alpha_0) + F_x(t, \alpha_0)(\beta_0 - \alpha_0) - [\phi(t, \beta_0) - \phi(t, \alpha_0)] \\ &\quad + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \beta_0) \\ &= f(t, \alpha_0) + F_x(\alpha_0)(\beta_0 - \alpha_0) - \phi_x(t, \xi)(\beta_0 - \alpha_0) \\ &\quad + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \beta_0) \\ &\ge f(t, \alpha_0) + \big(F_x(t, \alpha_0) - \phi_x(t, \beta_0)\big)(\beta_1 - \alpha_0), \end{aligned}$$

where $\alpha_0 \le \xi \le \beta_0$. Therefore by Theorem 2.3 we have $\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0$ on J. Letting this be our basis step suppose $\alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1}$ on J up to some $k \ge 1$, then by a similar process as when showing $\alpha_1 \le \beta_1$ we have that,

$$\begin{aligned} D_t^q \alpha_{k+1} &\ge f(t,\alpha_k) + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &\ge f(t,\alpha_{k-1}) - [\phi(t,\alpha_k) - \phi(t,\alpha_{k-1})] + F_x(t,\alpha_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &= f(t,\alpha_{k-1}) - \phi_x(t,\xi)(\alpha_k - \alpha_{k-1}) + F_x(t,\alpha_{k-1})(\alpha_k - \alpha_{k-1}) \\ &\quad + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_k) \\ &\ge f(t,\alpha_{k-1}) + \big(F_x(t,\alpha_{k-1}) - \phi_x(t,\beta_{k-1})\big)(\alpha_{k+1}-\alpha_{k-1}). \end{aligned}$$

Therefore by Theorem 2.3 we have that $\alpha_k \le \alpha_{k+1}$ on J, and by similar arguments we can show that $\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k$, which by induction implies that $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and $\alpha_n \le \beta_n$ for all $n \ge 0$. That $t^p\alpha_n$ and $t^p\beta_n$ converge uniformly to $t^p x$, where x is the unique solution of (3), is done in the same way as in Theorem 3.1. Now we will show that the sequences of continuous extensions converge quadratically on J_0. To do so, first note, as in Theorem 3.1 that there exist functions $G, \Phi \in C^{0,2}(\Omega,\mathbb{R})$ such that $G(t,t^p x) = F(t,x)$, and $\Phi(t,t^p x) = \phi(t,x)$, thus giving us that

$$F_{xx}(t,x) = t^{2p} G_{xx}(t,t^p x) \quad \text{and} \quad \phi_{xx}(t,x) = t^{2p}\Phi_{xx}(t,t^p x).$$

Now letting $\rho_{n+1} = x - \alpha_{n+1}$ and $\omega_{n+1} = \beta_{n+1} - x$, we have that

$$\begin{aligned} D_t^q \rho_{n+1} &= f(t,x) - [f(t,\alpha_n) + \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n)] \\ &= F(t,x) - F(t,\alpha_n) - \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n) \\ &\quad - [\phi(t,x) - \phi(t,\alpha_n)] \\ &= F_x(t,\xi_1)\rho_n - \big(F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)(\alpha_{n+1}-\alpha_n)) - \phi_x(t,\xi_2)\rho_n \\ &\le [F_x(t,x) - F_x(t,\alpha_n)]\rho_n + (F_x(t,\alpha_n) - \phi_x(t,\beta_n)\big)\rho_{n+1} \\ &\quad + [\phi_x(t,\beta_n) - \phi_x(t,\alpha_n)]\rho_n \\ &\le F_{xx}(t,\eta_1)\rho_n^2 + f_x(t,\alpha_n)\rho_{n+1} + \phi_{xx}(t,\eta_2)(\omega_n + \rho_n)\rho_n \\ &= G_{xx}(t,t^p\eta_1)t^{2p}\rho_n^2 + f_x(t,\alpha_n)\rho_{n+1} + \Phi_{xx}(t,t^p\eta_2)t^{2p}(\omega_n+\rho_n)\rho_n \\ &\le N t^{2p}\rho_n^2 + M\rho_{n+1} + (L/2)t^{2p}(3\rho_n^2 + \omega_n^2), \end{aligned}$$

Where $\alpha_n \le \xi_1, \xi_2, \eta_1 \le x$, $\alpha_n \le \eta_2 \le x$, and where N, M, and L are bounds on G_{xx}, f_x, and Φ_{xx} respectively. Then by Corollary 2.1 and Remark 2.2 we have that

$$\begin{aligned} t^p\rho_{n+1} &\le t^p \int_0^t (t-s)^{q-1} E_{q,q}(M(t-s)^q) s^{2p}\big[(N + 3L/2)\rho_n^2 + (L/2)\omega_n^2\big] ds \\ &\le \frac{t^p}{M} E_{q,1}(Mt^q)\Big[(N+3L/2)\big\|t^p\rho_n\big\|^2 + (L/2)\big\|t^p\omega_n\big\|^2\Big]. \end{aligned}$$

Which finally gives us that

$$\big\|t^p(x-\alpha_{n+1})\big\| \le \frac{K}{2}(2N+3L)\big\|t^p(x-\alpha_n)\big\|^2 + \frac{KL}{2}\big\|t^p(\beta_n - x)\big\|^2,$$

where $K = \frac{T^p}{M}E_{q,1}(MT^q)$. Similarly, we can show that

$$\big\|t^p(\beta_{n+1}-x)\big\| \le \frac{K}{2}(3N+2L)\big\|t^p(\beta_n - x)\big\|^2 + \frac{KN}{2}\big\|t^p(x-\alpha_n)\big\|^2,$$

which finishes the proof.

This final case greatly extends the potential of the quasilinearization method. This is because for any function $f \in C^{0,2}(\Omega, \mathbb{R})$ we can always find a function $\phi \in C^{0,2}(\Omega, \mathbb{R})$ such that $f_{xx} + \phi_{xx} \geq 0$, and $\phi_{xx} > 0$. To show why this is true, suppose that f is not convex, then we can choose $A > 0$ such that

$$\min_{\Omega} \left\{ f_{xx}(t,x) \right\} = -A < 0.$$

Then we need only choose $\phi(t,x) = At^{2p}x^2$, to satisfy (B2). Further, since we can always find such a function we need not consider the case where f can be made concave by the sum of another function.

Remark 3.1 If we use lower and upper solutions one can extend the method of quasilinearization to forcing functions which are the sum of convex and concave functions as in [15]. This generalization will include all our results as special cases. However, this involves the study of linear fractional systems with variable coefficients. We will investigate this result elsewhere.

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References

[1] Bellman, R. *Methods of Nonlinear Analysis*, volume II. Academic Press, New York, 1973.

[2] Bellman, R. and Kalaba, R. *Quasilinearization and Nonlinear Boundary Value Problems.* American Elsevier, New York, 1965.

[3] Caputo, M. Linear models of dissipation whose Q is almost independent, II. *Geophy. J. Roy. Astronom* **13** (1967) 529–539.

[4] Chowdhury, A. and Christov, C.I. Memory effects for the heat conductivity of random suspensions of spheres. *Proc. R. Soc. A* **466** (2010) 3253–3273.

[5] Denton, Z. and Vatsala, A.S. Fractional differential equations and numerical approximations. In: *Proceedings of Neural, Parallel, and Scientific Computations* (G.S. Ladde, N.G. Medhin, C. Peng and M. Sambandham, eds.). Atlanta, GA, 2010. Dynamic Publishers, **4** 119–123.

[6] Denton, Z. and Vatsala, A.S. Fractional integral inequalities and applications. *Computers and Mathematics with Applications* **59** (2010) 1087–1094.

[7] Denton, Z. and Vatsala, A.S. Monotone iterative technique for finite systems of nonlinear Riemann–Liouville fractional differential equations. *Opuscula Mathematica* **31** (3) (2011) 327-339.

[8] Devi, J.V. and Suseela, C.H. Quasilinearization for fractional differential equations. *Communications in Applied Analysis* **12** (4) (2008) 407–418.

[9] Diethelm, K. and Freed, A.D. On the solution of nonlinear fractional differential equations used in the modeling of viscoplasticity. In: *Scientific Computing in Chemical Engineering II: Computational Fluid Dynamics, Reaction Engineering, and Molecular Properties* (F. Keil, W. Mackens, H. Vob and J. Werther, eds.). Heidelberg, Springer, 1999, 217–224.

[10] Glöckle, W.G. and Nonnenmacher, T.F. A fractional calculus approach to self similar protein dynamics. *Biophy. J.* **68** (1995) 46–53.

[11] Hilfer, R. (editor). *Applications of Fractional Calculus in Physics*. World Scientific Publishing, Germany, 2000.

[12] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*. Elsevier, North Holland, 2006.

[13] Kiryakova, V. *Generalized fractional calculus and applications*. Pitman Res. Notes Math. Ser., vol. 301. Longman-Wiley, New York, 1994.

[14] Lakshmikantham, V., Leela, S. and Vasundhara, D.J. *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publishers, 2009.

[15] Lakshmikantham, V. and Vatsala, A.S. *Generalized Quasilinearization for Nonlinear Problems*. Kluwer Academic Publishers, Netherlands, 2010.

[16] Lee, E.S. *Quasilinearization and Invariant Imbedding*. Academic Press, New York, 1968.

[17] Metzler, R., Schick, W., Kilian, H.G. and Nonnenmacher, T.F. Relaxation in filled polymers: A fractional calculus approach. *J. Chem. Phy.* **103** (1995) 7180–7186.

[18] Oldham, B. andSpanier, J. *The Fractional Calculus*. Academic Press, New York–London, 2002.

[19] Ramirez, J.D. and Vatsala, A.S. Monotone iterative technique for fractional differential equations with periodic boundary conditions. *Opuscula Mathematica* **29** (3) (2009) 289-304.