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Weak Solutions for Boundary-Value Problems with Nonlinear Fractional Differential Inclusions

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Abstract: This paper deals with the existence of solutions, under the Pettis integrability assumption, for a class of boundary value problems for fractional differential inclusions involving nonlinear integral conditions. Our results are based on the technique of measures of weak noncompactness and a fixed point theorem of Mönch type.

Keywords: *boundary value problem; differential inclusion; Caputo fractional derivative; measure of weak noncompactness; Pettis integrals; weak solution.*

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1 Introduction

This note is concerned with the existence of solutions of the boundary value problem with fractional order differential inclusions and nonlinear integral conditions of the form

$$^{c}D^{\alpha}x(t) \in F(t, x(t)), \quad \text{for a.e.} \quad t \in J = [0, T], \ 1 < \alpha \leq 2, \tag{1}$$

$$x(0) - x'(0) = \int_0^T g(s, x(s))ds, \tag{2}$$

$$x(T) + x'(T) = \int_0^T h(s, x(s))ds, \tag{3}$$

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where ${}^cD^\alpha$, $1 < \alpha \leq 2$, is the Caputo fractional derivative, $F : J \times E \to P(E)$ is a multivalued map, E is a Banach space with the norm $\|\cdot\|$, $P(E)$ is the family of all nonempty subsets of E, and g, $h : J \times E \to E$ are given functions satisfying some assumptions that will be specified later.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control theory, porous media, electromagnetism, etc. (see [18, 24, 30]). There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monographs of Kilbas *et al.* [21], Lakshmikantham *et al.* [23], Podlubny [30], and the papers [2, 3, 11, 17, 28].

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics [13] and cellular systems [1]. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors such as Arara and Benchohra [4], Benchohra *et al.* [10], Infante [20], and the references therein.

In our investigation we apply the method associated with the technique of measures of weak noncompactness and a fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Banaŝ and Goebel [6] and subsequently developed and used in many papers; see, for example, Banaŝ *et al.* [7], Guo *et al.* [19], Krzyska and Kubiaczyk [22], Lakshmikantham and Leela [23], Mönch [25], O'Regan [26, 27], Szufla [32], Szufla and Szukala [33], and the references therein. In [8, 12] Benchohra *et al.* considered some classes of boundary value problems for fractional order differential equations in Banach space by means of the strong measure of noncompactness. As far as we know, they are very few results devoted to weak solutions of boundary value problems for nonlinear fractional differential equations [9]. The present results complement and extend those considered with the strong measure of noncompactness [8, 12].

2 Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel. Let E be the real Banach space with norm $\|\cdot\|$ and dual space E^*, and let $(E, w) = (E, \sigma(E, E^*))$ denote the space E with its weak topology. Here, $C(J, E)$ is the Banach space of all continuous functions $x : J \to E$ with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

We let $L^1(J, E)$ denote the Banach space of functions $x : J \to E$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_0^T \|x(t)\| dt,$$

and $L^\infty(J, E)$ denote the Banach space of bounded measurable functions $x : J \to E$ equipped with the norm

$$\|x\|_{L^\infty} = \inf\{c > 0 : \|x(t)\| \leq c \;\; a.e.\; t \in J\}.$$

Also, $AC^1(J,E)$ will denote the space of functions $x : J \to E$ that are absolutely continuous and whose first derivative, x', is absolutely continuous.

Let $(E, \|\cdot\|)$ be a Banach space and let $P_{cl}(E) = \{Y \in P(E) : Y \text{ is closed}\}$, $P_b(E) = \{Y \in P(E) : Y \text{ is bounded}\}$, $P_{cp}(E) = \{Y \in P(E) : Y \text{ is compact}\}$, and $P_{cp,cv}(E) = \{Y \in P(E) : Y \text{ is compact and convex}\}$. A multivalued map $F : E \to P(E)$ is *convex* (*closed*) valued if $F(x)$ is convex (closed) for all $x \in E$. We say that F is *bounded on bounded sets* if $F(B) = \cup_{x\in B} F(x)$ is bounded in E for all $B \in P_b(E)$ (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$). The mapping F is called *upper semi-continuous* (u.s.c.) on E if for each $x_0 \in E$, the set $F(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. The mapping F has a *fixed point* if there is $x \in E$ such that $x \in F(x)$.

For more details on multivalued maps see the books of Aubin and Frankowska [5] and Deimling [15]. We will need the following definitions in the sequel.

Definition 2.1 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $(x_n)_n$ in E with $x_n(t) \to x(t)$ in (E, w) for each $t \in J$, we have $h(x_n(t)) \to h(x(t))$ in (E, w) for each $t \in J$).

Definition 2.2 A function $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph if for any sequence $(x_n, y_n)_1^\infty \in Q \times Q$, $y_n \in F(x_n)$ for $n \in \{1, 2, ...\}$ with $x_n(t) \to x(t)$ in (E, ω) for each $t \in J$ and $y_n(t) \to y(t)$ in (E, ω) for each $t \in J$, then $y \in F(x)$.

Definition 2.3 [29] The function $x : J \to E$ is said to be Pettis integrable on J if and only if there is an element $x_I \in E$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J,E)$ be the space of all E-valued Pettis integrable functions in the interval J.

Proposition 2.1 [16, 29] *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$ is Pettis integrable.*

Definition 2.4 [14] Let E be a Banach space, Ω_E be the bounded subsets of E, and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta : \Omega_E \to [0, \infty]$ defined by

$$\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E \text{ such that } X \subset \epsilon B_1 + \Omega\}.$$

Properties: The De Blasi measure of noncompactness satisfies the following properties:

(a) $A \subset B \implies \beta(A) \le \beta(B)$;

(b) $\beta(A) = 0 \iff A$ is relatively compact;

(c) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(d) $\beta(\overline{A}^\omega) = \beta(A)$, where $\overline{A}^\omega$ denotes the weak closure of A;

(e) $\beta(A + B) \le \beta(A) + \beta(B)$;

(f) $\beta(\lambda A) = |\lambda|\beta(A)$;

(g) $\beta(conv(A)) = \beta(A)$;

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The following result follows directly from the Hahn–Banach theorem.

Proposition 2.2 *Let E be a normed space with $x_0 \neq 0$. Then there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For completeness, we recall the definitions of the Pettis-integral and the Caputo derivative of fractional order.

Definition 2.5 ([31]) Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where the sign "$\int$" denotes the Pettis integral and Γ is the Gamma function.

Definition 2.6 ([21]) For a function $h : I \to E$, the Caputo fractional-order derivative of h is defined by

$$^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}(s)ds}{(t-s)^{1-n+\alpha}},$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The following theorem will be used to prove our main result.

Theorem 2.1 *Let E be a Banach space with Q a nonempty, bounded, closed, convex, equicontinuous subset of $C([0,T], E)$. Suppose $F : Q \to P_{cl,cv}(Q)$ has a weakly sequentially closed graph. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup F(V)) \Longrightarrow V \quad \textit{is relatively weakly compact} \tag{4}$$

holds for every subset $V \subset Q$, then the operator inclusion $x \in F(x)$ has a solution in Q.

3 Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)–(3).

Definition 3.1 A function $x \in AC^1(J, E)$ is said to be a solution of (1)–(3), if there exists a function $v \in L^1(J, E)$ with $v(t) \in F(t, x(t))$ for a.e. $t \in J$, such that

$$^cD^\alpha x(t) = v(t) \text{ a.e. } t \in J, \quad 1 < \alpha \leq 2,$$

and the function x satisfies the boundary conditions (2) and (3).

For any $x \in C(J, E)$, we define the set

$$S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in J\}.$$

This is known as the set of *selection functions.*

For the existence of solutions to the problem (1)–(3), we need the following auxiliary lemmas.

Lemma 3.1 [34] *Let $\alpha > 0$; then the differential equation ${}^cD^\alpha h(t) = 0$ has the solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, and $n = [\alpha] + 1$.*

Lemma 3.2 [34] *Let $\alpha > 0$; then $I^{\alpha c} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where $n = [\alpha] + 1$.*

As a consequence of Lemmas 3.1 and 3.2, we have the following result which will be useful in the remainder of the paper.

Lemma 3.3 *Let $1 < \alpha \leq 2$ and let σ, σ_1, $\sigma_2 : J \to E$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t) = P(t) + \int_0^T G(t,s)\sigma(s)ds \tag{5}$$

with

$$P(t) = \frac{(T+1-t)}{T+2} \int_0^T \sigma_1(s)ds + \frac{(t+1)}{T+2} \int_0^T \sigma_2(s)ds \tag{6}$$

and

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(T-s)^{\alpha-1}}{(T+2)\Gamma(\alpha)} - \frac{(1+t)(T-s)^{\alpha-2}}{(T+2)\Gamma(\alpha-1)}, & t \leq s < T, \end{cases} \tag{7}$$

if and only if x is a solution of the fractional boundary value problem

$${}^cD^\alpha x(t) = \sigma(t), \quad t \in J,$$

$$x(0) - x'(0) = \int_0^T \sigma_1(s)ds, \quad x(T) + x'(T) = \int_0^T \sigma_2(s)ds.$$

Let

$$\tilde{G} = \sup\left\{ \int_0^T |G(t,s)|ds, \ t \in J \right\}.$$

We are now in a position to state and prove our existence result for the problem (1)–(3). We first list the following hypotheses:

(H1) $F : J \times E \to P_{cp,cl,cv}(E)$ has weakly sequentially closed graph.

(H2) For each $t \in J$, $g(t,\cdot)$ and $h(t,\cdot)$ are weakly sequentially continuous.

(H3) For each continuous $x : J \to E$, there exists a scalarly measurable function $v : J \to E$ with $v(t) \in F(t, x(t))$ a.e. on J and v is Pettis integrable on J.

(H4) For each $x \in C(J,E)$, $g(\cdot, x(\cdot))$ and $h(\cdot, x(\cdot))$ are Pettis integrable on J.

(H5) There exist $p \in L^\infty(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq p(t)\psi(\|x\|).$$

(H6) There exist $\phi_g \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi^* : [0,\infty) \to [0,\infty)$ such that

$$\|g(t,x)\| \leq \phi_g(t)\psi^*(\|x\|).$$

(H7) There exist $\phi_h \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\bar{\psi} : [0,\infty) \to [0,\infty)$ such that

$$\|h(t,x)\| \leq \phi_h(t)\bar{\psi}(\|x\|).$$

(H8) there exists a number $R > 0$ such that

$$\frac{R}{a\psi^*(R) + b\bar{\psi}(R) + c\tilde{G}\psi(R)} > 1, \tag{8}$$

where

$$a = \frac{T+1}{T+2}\int_0^T \phi_g(s)ds, \quad b = \frac{T+1}{T+2}\int_0^T \phi_h(s)ds, \quad \text{and} \quad c = \|p\|_{L^\infty}.$$

(H9) For each bounded set $Q \subset E$ and each $t \in J$,

$$\beta(F(t,Q)) \leq p(t)\beta(Q), \tag{9}$$

$$\beta(g(t,Q)) \leq \phi_g(t)\beta(Q), \tag{10}$$

$$\beta(h(t,Q)) \leq \phi_h(t)\beta(Q). \tag{11}$$

Theorem 3.1 *Let E be a Banach space. Assume that hypotheses (H1)–(H9) hold. If*

$$\frac{T+1}{T+2}\int_0^T [\phi_g(s) + \phi_h(s)]ds + \tilde{G}\|p\|_{L^\infty} < 1, \tag{12}$$

then the problem (1)–(3) has at least one solution.

Proof We transform the problem (1)–(3) into fixed point problem by considering the multivalued operator $N : C(J,E) \to P_{cl,cv}(C(J,E))$ defined by

$$N(y) = \left\{ h \in C(J,E) : h(t) = P_x(t) + \int_0^T G(t,s)v(s)ds, v \in S_{F,x} \right\}, \tag{13}$$

where

$$P_x(t) = \frac{T+1-t}{T+2}\int_0^T g(s,x(s))ds + \frac{t+1}{T+2}\int_0^T h(s,x(s))ds$$

and the function $G(\cdot,\cdot)$ is given by (7). Clearly, from Lemma 3.3, the fixed points of N are solutions of the problem (1)–(3). We first show that (13) makes sense. To see this, let $x \in C(J,E)$; by (H3) there exists a Pettis integrable function $v : J \to E$ such that $v(t) \in F(t,x(t))$ for a.e. $t \in J$. Since $G(t,\cdot) \in L^\infty(J)$, then $G(t,\cdot)v(\cdot)$ is Pettis integrable and thus N is well defined.

Let $R \in \mathbb{R}_+^*$, and consider the set

$$Q = \Bigg\{ x \in C(J,E) : \|x\|_\infty \leq R \ \text{ and } \|x(t_1) - x(t_2)\| \leq \frac{|t_1 - t_2|}{T+2}\psi^*(R)\int_0^T \phi_g(s)ds$$

$$+ \frac{|t_1 - t_2|}{T+2}\bar{\psi}(R)\int_0^T \phi_h(s)ds + \|p\|_{L^\infty}\psi(R)\int_0^T \|G(t_2,s) - G(t_1,s)\|ds \ \text{ for } \ t_1, t_2 \in J \Bigg\}.$$

Notice that Q is a closed, convex, bounded and equicontinuous subset of $C(J,E)$. We shall show that N satisfies the assumptions of Theorem 2.1.

Step 1: $N(x)$ is convex for each $x \in Q$.

Indeed, if h_1 and h_2 belong to $N(x)$, then there exists Pettis integrable functions $\upsilon_1(t)$, $\upsilon_2(t) \in F(t,x(t))$ such that, for all $t \in J$, we have:

$$h_i(t) = P_x(t) + \int_0^T G(t,s)\upsilon_i(s)ds, \ i = 1,2.$$

Let $0 \leq \lambda \leq 1$; then, for each $t \in J$, we have:

$$(\lambda h_1 + (1-\lambda)h_2)(t) = P_y(t) + \int_0^T G(t,s)[\lambda\upsilon_1(s) - (1-\lambda)\upsilon_2(s)]ds.$$

Since F has convex values, $(\lambda\upsilon_1 + (1-\lambda)\upsilon_2)(t) \in F(t,x(t))$, and we have $\lambda h_1 + (1-\lambda)h_2 \in N(x)$.

Step 2: N maps Q into Q.

To see this, take $u \in NQ$. Then there exists $x \in Q$ with $u \in N(x)$ and there exists a Pettis integrable function $\upsilon : J \to E$ with $\upsilon(t) \in F(t,x(t))$ for a.e. $t \in J$. Without loss of generality, we assume $u(s) \neq 0$ for all $s \in J$. Then, there exists $\varphi_s \in E^*$ with $\|\varphi_s\| = 1$ and $\varphi_s(u(s)) = \|u(s)\|$. Hence, for each fixed $t \in J$, we have:

$$\begin{aligned}
\|u(t)\| &= \varphi_t(u(t)) \\
&= \varphi_t\left(P_x(t) + \int_0^T G(t,s)\upsilon(s)ds\right) \\
&\leq \varphi_t(P_x(t)) + \varphi_t\left(\int_0^T G(t,s)\upsilon(s)ds\right) \\
&\leq \|P_x(t)\| + \int_0^T \|G(t,s)\|\varphi_t(\upsilon(s))ds \\
&\leq \frac{T+1}{T+2}\psi^*(\|x\|_\infty)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\bar{\psi}(\|x\|_\infty)\int_0^T \phi_h(s)ds \\
&\quad + \tilde{G}\psi(\|x\|_\infty)\|p\|_{L^\infty}.
\end{aligned}$$

Therefore, by (H8),

$$\|u\|_\infty \leq \frac{T+1}{T+2}\psi^*(R)\int_0^T \phi_g(s)ds + \frac{T+1}{T+2}\bar{\psi}(R)\int_0^T \phi_h(s)ds + \tilde{G}\psi(R)\|p\|_{L^\infty} \leq R.$$

Next suppose $u \in NQ$ and t_1, $t_2 \in J$ with $t_1 < t_2$ so that $u(t_2) - u(t_1) \neq 0$. Then, there exist $\varphi \in E^*$ such that $\|u(t_2) - u(t_1)\| = \varphi(u(t_2) - u(t_1))$. Thus,

$$\begin{aligned} \|u(t_2) - u(t_1)\| &= \varphi\left(P_x(t_2) - P_x(t_1) + \int_0^T (G(t_2, s) - G(t_1, s))\upsilon(s)ds\right) \\ &\leq \varphi(P_x(t_2) - P_x(t_1)) + \varphi\left(\int_0^T (G(t_2, s) - G(t_1, s))\upsilon(s)ds\right) \\ &\leq \|P_x(t_2) - P_x(t_1)\| + \int_0^T \|G(t_2, s) - G(t_1, s)\|\|\upsilon(s)\|ds \\ &\leq \frac{(t_2 - t_1)}{T + 2}\psi^*(R)\int_0^T \phi_g(s)ds + \frac{(t_2 - t_1)}{T + 2}\bar{\psi}(R)\int_0^T \phi_h(s)ds \\ &\quad + \psi(R)\|p\|_{L^\infty}\int_0^T \|G(t_2, s) - G(t_1, s)\|ds. \end{aligned}$$

Therefore, $u \in Q$.

Step 3: N has a weakly sequentially closed graph.

Let $(x_n, y_n)_1^\infty$ be a sequence in $Q \times Q$ with $x_n(t) \to x(t)$ in (E, ω) for each $t \in J$, $y_n(t) \to y(t)$ in (E, ω) for each $t \in J$, and $y_n \in N(x_n)$ for $n \in \{1, 2, ...\}$. We shall show that $y \in Nx$. By the relation $y_n \in N(x_n)$, we mean that there exists $\upsilon_n \in S_{F,x_n}$ such that

$$y_n(t) = P_{x_n}(t) + \int_0^T G(t, s)\upsilon_n(s)ds.$$

We must show that there exists $\upsilon \in S_{F,x}$ such that, for each $t \in J$,

$$y(t) = P_x(t) + \int_0^T G(t, s)\upsilon(s)ds.$$

Since $F(\cdot, \cdot)$ has compact values, there exists a subsequence υ_{n_m} such that

$$\upsilon_{n_m}(\cdot) \to \upsilon(\cdot) \text{ in } (E, \omega) \text{ as } m \to \infty$$

and

$$\upsilon_{n_m}(t) \in F(t, x_n(t)) \text{ a.e. } t \in J.$$

Since $F(t, \cdot)$ has a weakly sequentially closed graph, $\upsilon \in F(t, x)$. The Lebesgue Dominated Convergence Theorem for the Pettis integral then implies that for each $\varphi \in E^*$,

$$\varphi(y_n(t)) = \varphi\left(P_{x_n}(t) + \int_0^T G(t, s)\upsilon_n(s)ds\right) \to \varphi\left(P_x(t) + \int_0^T G(t, s)\upsilon(s)ds\right)$$

i.e., $y_n(t) \to Nx(t)$ in (E, ω). We can repeat this for each $t \in J$, so $y(t) \in Nx(t)$.

Step 4: The implication (4) holds.

Now let V be a subset of Q such that $V = \overline{conv}(N(V) \cup \{0\})$. Clearly, $V(t) \subset \overline{conv}(N(V) \cup \{0\})$ for all $t \in J$. Also, $NV(t) \subset NQ(t)$, for each $t \in J$, and is bounded in

$P(E)$. By $(H9)$ and the properties of the measure β, we have

$$
\begin{aligned}
\beta(NV(t)) &= \beta\left\{P_x(t) + \int_0^T G(t,s)v(s)ds : v \in S_{F,x},\ x \in V,\ t \in J\right\} \\
&\leq \beta\left\{P_x(t) :\ x \in V,\ t \in J\right\} \\
&\quad + \beta\left\{\int_0^T G(t,s)v(s)ds : v \in S_{F,x},\ x \in V,\ t \in J\right\} \\
&\leq \beta\left\{\frac{T+1-t}{T+2}\int_0^T g(s,x(s))ds + \frac{t+1}{T+2}\int_0^T h(s,x(s))ds : x \in V\right\} \\
&\quad + \beta\left\{\int_0^T G(t,s)v(s)ds : v(t) \in F(t,x(t)),\ x \in V,\ t \in J\right\} \\
&\leq \int_0^T \frac{T+1-t}{T+2}\phi_g(s)\beta(V(s))ds + \int_0^T \frac{t+1}{T+2}\phi_h(s)\beta(V(s))ds \\
&\quad + \int_0^T \|G(t,s)\|p(s)\beta(V(s))ds \\
&\leq \frac{T+1}{T+2}\int_0^T \phi_g(s)\beta(V(s))ds + \frac{T+1}{T+2}\int_0^T \phi_h(s)\beta(V(s))ds \\
&\quad + \int_0^T \|G(t,s)\|p(s)\beta(V(s))ds
\end{aligned}
$$

for each $t \in J$. This means that

$$
\|v\|_\infty \leq \|v\|_\infty\left[\frac{T+1}{T+2}\int_0^T(\phi_g(s)+\phi_h(s))ds + \tilde{G}\int_0^T p(s)ds\right]
$$

i.e.,

$$
\|v\|_\infty\left[1 - \frac{T+1}{T+2}\int_0^T(\phi_g(s)+\phi_h(s))ds + \tilde{G}\|p\|_\infty\right] \leq 0.
$$

By (12) it follows that $\|v\|_\infty = 0$. Thus, V is weakly relatively compact. Applying Theorem 2.1, we conclude that N has a fixed point that is a solution of the problem (1)–(3). □

References

[1] Adomian, G. and Adomian, G. E. Cellular systems and aging models, *Comput. Math. Appl.* **11** (1985) 283–291.

[2] Agarwal, R. P., Benchohra, M. and Hamani, S. A survey on existence result of boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Appl. Math.* **109** (2010) 973–1033.

[3] Ahmad, B. and Nieto, J. J. Existence results of nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions. *Bound. Value. Probl.* (2009) Article ID708576, 11 pages.

[4] Arara, A. and Benchohra, M. Fuzzy solutions for boundary value problems with integral boundary conditions. *Acta Math. Univ. Comenianae* **LXXV** (2006) 119–126.

[5] Aubin, J. P. and Frankowska, H. *Set-Valued Analysis.* Birkhauser, Boston, 1990.

[6] Banaš, J. and Goebel, K. *Measures of Noncompactness in Banach Spaces.* Marcel Dekker, New York, 1980.

[7] Banaš, J. and Sadarangani, K. On some measures of noncompactness in the space of continuous functions. *Nonlinear Anal.* **68** (2008) 377–383.

[8] Benchohra, M., Cabada, A. and Seba, D. An existence result for nonlinear fractional differential equations on Banach spaces. *Bound. Value Probl.* **2009**, Art. ID 628916, 11 pages.

[9] Benchohra, M., Graef, J. R. and Mostefai, F. Weak solutions for nonlinear fractional differential equations on reflexive Banach spaces. *Electron. J. Qual. Theory Differ. Equ.* **2010** (54) 10 pages.

[10] Benchohra, M., Hamani, S. and Henderson, J. Functional differential inclusions with integral boundary conditions. *Eletron. J. Qual. Theory Differ. Equ.* **2007** (15) 13 pages.

[11] Benchohra, M., Henderson, J., Ntouyas, S. K. and Ouahab, A. Existence results for fractional functional differential inclusions with infinite delay and applications to control theory. *Fract. Calc. Appl. Anal.* **11** (2008) 35–56.

[12] Benchohra, M., Henderson, J. and Seba, D. Measure of noncompactness and fractional differential equations in Banach spaces. *Commun. Appl. Anal.* **12** (2008) 419–427.

[13] Blayneh, K. W. Analysis of age structured host-parasitoid model. *Far East J. Dyn. Syst.* **4** (2002) 125–145.

[14] De Blasi, F. S. On the property of the unit sphere in a Banach space. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **21** (1977) 259–262.

[15] Deimling, K. *Multivalued Differential Equations.* De Gruyter, Berlin-New York, 1992.

[16] Diestel, J. and Uhl Jr., J. J. Vector Measures. In: *Math. Surveys*, vol. 15, Amer. Math. Soc., Providence, R.I., 1977.

[17] El-Sayed, A. M. A. and Ibrahim, A. G. Multivalued fractional differential equations, Appl. Math. Comput. **68** (1995) 15–25.

[18] Glockle, W. G. and Nonnenmacher, T. F. A fractional calculus approach of self-similar protein dynamics. *Biophys. J.* **68** (1995) 46–53.

[19] Guo,D., Lakshmikantham, V. and Liu, X. *Nonlinear Integral Equations in Abstract Spaces.* Mathematics and its Applications. Kluwer, Dordrecht, 1996.

[20] Infante, G. Eigenvalues and positive solutions of ODEs involving integral boundary conditions. *Discrete Contin. Dyn. Syst.* (2005) suppl. 436–442.

[21] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. *Theory and Applications of Fractional Differential Equations.* North Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[22] Krzyska, S. and Kubiaczyk, I. On bounded pseudo and weak solutions of a nonlinear differential equation in Banach spaces. *Demonstratio Math.* **32** (1999) 323–330.

[23] Lakshmikantham, V., Leela, S. and Vasundhara, J. *Theory of Fractional Dynamic Systems.* Cambridge Scientific Publishers, Cambridge, 2009.

[24] Metzler, F., Schick, W., Kilian, H. G. and Nonnenmacher, T. F. Relaxation in filled polymers: A fractional calculus approach. *J. Chem. Phys.* **103** (1995) 7180–7186.

[25] Mönch, H. Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. *Nonlinear Anal.* **4** (1980) 985–999.

[26] O'Regan, D. Fixed point theory for weakly sequentially continuous mapping. *Math. Comput. Model.* **27** (1998) 1–14.

[27] O'Regan, D. Weak solutions of ordinary differential equations in Banach spaces, *Appl. Math. Lett.* **12** (1999) 101–105.

[28] Ouahab, A. Some results for fractional boundary value problems of differential inclusions, *Nonlinear Anal.* **69** (2008) 3877–3896.

[29] Pettis, B. J. On integration in vector spaces. *Trans. Amer. Math. Soc.* **44** (1938) 277–304.

[30] Podlubny, I. *Fractional Differential Equations.* Academics Press, San Diego, 1999.

[31] Salem, H. A. H., El-Sayed, A. M. A. and Moustafa, O. L. A note on the fractional calculus in Banach spaces. *Studia Sci. Math. Hungar.* **42** (2005) 115–130.

[32] Szufla, S. On the application of measure of noncompactness to existence theorems. *Rend. Sem. Mat. Univ. Padova* **75** (1986) 1–14.

[33] Szufla, S. and Szukala, A. Existence theorems for weak solutions of nth order differential equations in Banach spaces. *Funct. Approx. Comment. Math.* **26** (1998) 313–319.

[34] Zhang, S. Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electron. J. Differential Equations* (2006) (36) 12 pages.